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INTERPRETING SECOND-ORDER LOGIC IN THE MONADIC THEORY OF ORDER[1]

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Abstract. Under a weak set-theoretic assumption we interpret second-order logic in the monadic theory of order.

§0. Introduction. The monadic (second-order) theory of a chain (i.e. a linearly ordered set) C can be defined as the first-order theory of the two-sorted structure: The universe of C, the power-set of C; the order relation $<$ on elements of C, and the containment relation $\in$ between elements and subsets of C.

In a similar way we can define the monadic theory of any other structure.

The monadic second-order logic appears to be an appropriate logic to handle linear order. It gives several natural, expressive and manageable theories. See a discussion on this subject in Gurevich [2].

The decision problem for the monadic theory of (linear) order was a long-standing open problem. (Recall that Rabin [5] proved decidability of the monadic theory of countable chains.) Assuming the Continuum Hypothesis, Shelah [6] interpreted the true first-order arithmetic in the monadic theory of the real line (which is easily interpretable in the monadic theory of order). Confirming Shelah's conjecture and assuming the Gödel constructibility axiom $V = L$, Gurevich [1] interpreted the second-order theory of continuum in the monadic theory of the real line. The monadic theory of the real line (and therefore the monadic theory of order) was proved undecidable (without using any extra set-theoretic assumptions) in Gurevich and Shelah [3].

Here we assume that for every cardinal λ there is a regular cardinal $\kappa > \lambda$ such that $2^{<\kappa}$, i.e. $\Sigma\{2^{\mu}: \mu < \kappa\}$, is equal to κ. Under this assumption we interpret second-order logic in the monadic theory of order. In other words we assign effectively a sentence ϕ' in the monadic language of order to arbitrary second-order sentence ϕ in such a way that every nonempty set satisfies ϕ iff every chain satisfies ϕ'.

Our proof is based on the technique developed in the mentioned papers Shelah [6], Gurevich [1] and Gurevich and Shelah [3]. This paper is self-contained nevertheless.

We think that the second-order logic can be interpreted in the monadic theory

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of order without using any extra set-theoretic assumptions. It needs more complicated constructions however.

Speaking about intervals of a chain we mean nonempty open intervals. Intervals $\{x: x < a\}$ and $\{x: x > a\}$ are denoted $(-\infty, a)$ and (a, ∞), respectively. Speaking about topological properties of a chain we mean the interval topology.

§1. A suitable chain. Given a regular cardinal $\kappa > \aleph_1$ with $2^{<\kappa} = \kappa$ we define a chain U whose monadic theory is especially convenient for interpreting the second-order theory of nonempty sets of cardinality less than κ.

Elements of U are functions $x: \alpha \to \omega_1$ such that either $\alpha < \kappa$ or $\alpha = \kappa$ and $\{\beta: \beta < \kappa \text{ and } x(\beta) \neq 0\}$ is cofinal in κ. In other words x is a sequence of at most countable ordinals, the length of x is at most κ, and if the length of x is equal to κ then x does not have a tail of zeroes. In this section x, y, z range over U.

Note that the inclusion relation on U gives a tree of height κ. The αth level of the tree consists of sequences of length α. If $\alpha = \operatorname{dom}(x) < \kappa$ and $\beta < \omega_1$ then $x \hat{} \beta$, i.e. $x \cup \{(\alpha, \beta)\}$, is a successor of x. We say that x is *limit* or *successor* if the length, i.e. the domain of x, is so. The *meet* $x \wedge y$ of sequences x, y is the maximal common initial segment of x and y, it is the greatest lower bound of x and y in the tree. We say that x and y are *tree compatible* if either $x \subseteq y$ or $y \subseteq x$.

Now we describe the linear order of U. Given $x \neq y$ consider $z = x \wedge y$ and $\alpha = \operatorname{dom}(z)$. If z is a proper initial segment of both x and y and $x(\alpha) < y(\alpha)$ then $x < y$. If $z = x$ then $y < x$. In other words the order is lexicographic on tree incompatible pairs and if x is a proper initial segment of y then $y < x$. Note that the empty sequence Λ is the last element in U and there is no first element in U. Speaking about intervals of U we always mean nonempty open intervals. Speaking about topological properties of U we always mean the interval topology.

Let $D = \{x \in U: \operatorname{dom}(x) < \kappa\}$. If $x \in D$ then $x = \sup\{x \wedge \alpha: \alpha < \omega_1\}$ and cofinality of the interval $(-\infty, x)$ is ω_1. If $x \in U - D$ then $x = \sup\{(x|\alpha) \hat{} 0: x(\alpha) > 0\}$ and the cofinality of $(-\infty, x)$ is κ.

Let $D^l = \{x \in D: x \text{ is limit}\}$. If $x \in U - D$ then $x = \inf\{x|\alpha: \alpha < \kappa\}$ and coinitiality of the interval $(x, \Lambda]$ is κ. If $x \in D^l$ then $x = \inf\{x|\alpha: \alpha < \operatorname{dom}(x)\}$ and coinitiality of $(x, \Lambda]$ is $\operatorname{dom}(x)$. If x is successor then the coinitiality of $(x, \Lambda]$ is κ. For, suppose $x = y \hat{} \alpha$ and for every $\beta < \kappa$ let y_β be obtained from $y \hat{} (\alpha + 1)$ by attaching a tail of β zeroes. Then $x = \inf\{y_\beta: \beta < \kappa\}$.

For $a \in D$ let $\operatorname{Cone}(a) = \{x: a \subset x\}$. Then $a \notin \operatorname{Cone}(a)$, and $\operatorname{Cone}(a)$ is an interval, and $a = \sup \operatorname{Cone}(a)$. Every interval includes some $\operatorname{Cone}(a)$. For, choose $x < y$ in the given interval. If x, y are tree incompatible, $z = x \wedge y$ and $\alpha = \operatorname{dom}(z)$ take

$$a = z \hat{} x(\alpha) \hat{} (x(\alpha + 1) + 1).$$

And if $y \subset x$ and $\alpha = \operatorname{dom}(y)$ take

$$a = y \hat{} (x(\alpha) + 1).$$

Claim 1. *The union of less than κ nowhere dense subsets of U is nowhere dense in U.*

PROOF. Suppose $\lambda < \kappa$ is an infinite cardinal and $\{X_\alpha: \alpha < \lambda\}$ is a family of nowhere dense sets. Given an interval I select a sequence $\langle a_\alpha: \alpha \leq \lambda \rangle$ of elements of D such that $\mathrm{Cone}(a_0) \subseteq I$ and $\mathrm{Cone}(a_{\alpha+1}) \subseteq \mathrm{Cone}(a_\alpha) - X_\alpha$ for $\alpha < \lambda$ and $a_\alpha = \bigcup\{a_\beta: \beta < \alpha\}$ for limit α. Evidently $\langle a_\alpha: \alpha \leq \lambda\rangle$ increases by inclusion hence $\langle \mathrm{Cone}(a_\alpha): \alpha \leq \lambda\rangle$ decreases by inclusion. Thus $\mathrm{Cone}(a_\lambda)$ is included into I and avoids any X_α. □

A nonempty subset A of D will be called *auxiliary* if (i) A is cofinal in the interval $(-\infty, a)$ for every $a \in A$, and (ii) if A is coinitial in the interval $(d, \Lambda]$ for some $d \in D^l$ then $d \in A$. Note that this definition is expressible in the monadic theory of chain U with a parameter D^l.

Claim 2. *Let E be a function from D to D. Suppose that for every $a \in D$:*

(1) *If $\alpha < \beta < \omega_1$ then there are $\alpha' < \beta' < \omega_1$ such that $E(a)\,\hat{}\,\alpha' \subseteq E(a\,\hat{}\,\alpha)$ and $E(a)\,\hat{}\,\beta' \subseteq E(a\,\hat{}\,\beta)$; and*

(2) *If* $\mathrm{dom}(a)$ *is limit then* $E(a) = \bigcup\{E(a|\alpha): \alpha < \mathrm{dom}(a)\}$.

Then for every a, b in D:

(A) $a \subset b$ *iff* $E(a) \subset E(b)$,

(B) $a < b$ *iff* $E(a) < E(b)$,

(C) $E(a \wedge b) = E(a) \wedge E(b)$

and the range $E(D)$ of E is auxiliary.

PROOF. An easy induction on b proves an implication $a \subset b \to E(a) \subset E(b)$. By (1) E preserves tree incompatibility. That takes care of (A). If a, b are tree compatible then (A) implies (B) and (C). Suppose that $a < b$ are tree incompatible and $c = a \wedge b$. Then there are $\alpha < \beta < \omega_1$ with $c\,\hat{}\,\alpha \subseteq a$, $c\,\hat{}\,\beta \subseteq b$. By (1) there are $\alpha' < \beta' < \omega_1$ with $E(c)\,\hat{}\,\alpha' \subseteq E(c\,\hat{}\,\alpha) \subseteq E(a)$ and $E(c)\,\hat{}\,\beta' \subseteq E(c\,\hat{}\,\beta) \subseteq E(b)$. Hence $E(a) < E(b)$ and $E(c) = E(a) \wedge E(b)$.

It remains to prove that $E(D)$ satisfies conditions (i) and (ii) in the definition of auxiliary sets. The first is easy: by (1) $E(a) = \sup\{E(a\,\hat{}\,\alpha): \alpha < \omega_1\}$ for every $a \in D$. To prove the second, suppose that $E(D)$ is coinitial in $(d, \Lambda]$ for some $d \in D^l$.

For every $\alpha < \mathrm{dom}(d)$ there is $\alpha < \beta < \mathrm{dom}(d)$ with $d|\beta \in E(D)$. For, choose consecutively $a \in D$, $\gamma < \mathrm{dom}(d)$ and $b \in D$ such that $d|\alpha > E(a) > d|\gamma > E(b) > d$. Then

$$E(a \wedge b) = E(a) \wedge E(b) = E(a) \wedge d|\gamma \subseteq d$$

and

$$d|\alpha \subseteq E(a) \wedge E(b) = E(a \wedge b);$$

hence $E(a \wedge b) = d|\beta$ for some $\beta > \alpha$.

By (A) the E-preimages of elements $d|\alpha \in E(D)$ are tree compatible. Form $a = \bigcup\{E^{-1}(d|\alpha): d|\alpha \in E(D)\}$. Using (2) it is easy to check that $E(a) = d$. □

For every $X \subseteq U$ we define the *tree closure* of X as follows:

$TC(X) = \{y \in U$: either $y \in X$ or y is limit and for every $\alpha < \mathrm{dom}(y)$ there is $\alpha < \beta < \mathrm{dom}(y)$ with $y|\beta \in X\}$.

Evidently $TC(X)$ is a part of $\bar{X}$ which is the closure of X in the interval topology of U.

Claim 3. *If $A \subseteq D$ is auxiliary then $|A| = \kappa$ and $|TC(A)| = |\bar{A}| = 2^\kappa$.*

PROOF. It is easy to construct $E: D \to A$ satisfying the conditions of Claim 2. Extend E on U by $E(x) = \bigcup\{E(x|\alpha): x|\alpha \in A\}$ for $x \in U - D$. Then

$$\kappa = 2^{<\kappa} = |D| \leq |A| \leq |E(D)| = |D| = 2^{<\kappa}$$

and

$$2^{\kappa} = |U| \leq |TC(A)| \leq |E(U)| = |U| = 2^{\kappa}. \quad \square$$

§2. Coding. We work in the chain U of §1. Given a set of cardinality less than κ we would like to interpret its second-order theory in the monadic theory of U. The main step of the desired interpretation is made in this section. Elements of the given set will be coded by everywhere dense subsets of U.

Every ordinal α is uniquely represented as $\omega\beta + n$ for some ordinal β and natural number n. We say that α is *even* or *odd* if n is so. Let Odd be the set of odd ordinals less than κ, and

$$D^0 = \{a \in D: \operatorname{dom}(a) \in \text{Odd}\}.$$

THEOREM 1. *Let F be a family of subsets of D such that $2 \leq |F| < \kappa$ and $\bigcup F = D - D^l$. Suppose that for every $C \in F$ there is a set* $\operatorname{Ord}(C)$ *of successor ordinals such that*

$C = \{a \in D: \operatorname{dom}(a) \in \operatorname{Ord}(C)\}$, *and*

Odd $\cap$ Ord(C) *is cofinal in κ, and*

Odd $\cap$ Ord(C_0) $\cap$ Ord$(C_1) = \varnothing$ *for different $C_0, C_1 \in F$.*

Then there is $W \subseteq U - D$ such that for every interval I of U and for every subset X of D with $D^0 \cap X$ dense in I the following statements are equivalent:

(A) *For every interval $I_0 \subseteq I$ there are $C \in F$ and a subinterval $J \subseteq I_0$ with $J \cap X \subseteq C$.*

(B) *For every interval $I_1 \subseteq I$, every $X_0 \subseteq D^0 \cap X$ and every $X_1 \subseteq X$, if X_0, X_1 are dense in I_1 then there is an auxiliary $A \subseteq D$ such that X_0, X_1 are dense in A and $|\bar{A} \cap W| \leq 1$.*

Note that (B) abbreviates a monadic formula with parameters D, D^l, D^0. Theorem 1 states that this monadic formula expresses (A).

PROOF OF THEOREM 1. We adapt the following terminology. Members of F are *colors.* A subset X of U *varies* at $\alpha < \kappa$ if $\{x(\alpha): x \in X\}$ contains at least two ordinals. X is *of color C* if C contains every successor ordinal α such that X varies at α. X is *mono* if there is a unique color C such that X is of color C. X is *motley* if it has a pair of tree incompatible elements and for every pair x, y of tree incompatible elements of X there are colors C_0, C_1 such that $\{x, y\}$ varies at some $\alpha \in \operatorname{Ord}(C_0) - \operatorname{Ord}(C_1)$ and at some $\beta \in \operatorname{Ord}(C_1) - \operatorname{Ord}(C_0)$. Note that a pair $\{x, y\}$ is motley if there are colors C_0, C_1 such that $\{x, y\}$ varies at some $\alpha \in \text{Odd} \cap \operatorname{Ord}(C_0)$ and at some $\beta \in \operatorname{Ord}(C_1) - \operatorname{Ord}(C_0)$.

LEMMA 2. *Suppose that A is an auxiliary set and C_0, C_1 are colors such that $C_0 \cap D^0$ and $C_1 - C_0$ are dense in A. Then there is an auxiliary motley subset of A.*

PROOF. It suffices to construct a map $E: D \to A$ satisfying the conditions of Claim 2 in §1 and such that for every $a \in D$ and every $\alpha < \beta < \omega_1$ the pair $\{E(a \frown \alpha), E(a \frown \beta)\}$ is motley. We construct E by induction. Choose $E(\Lambda)$ arbitrary. If $a \in D$ is limit set $E(a) = \bigcup\{E(a|\alpha): \alpha < \operatorname{dom}(a)\}$.

Now, given $E(a) \in A$ we want to select $E(a \frown \alpha)$ for $\alpha < \omega_1$. The set $M = \{\alpha < \omega_1: \operatorname{Cone}(E(a) \frown \alpha) \text{ meets } A\}$ is cofinal in ω_1. For every $\alpha \in M$ choose a_α in

$A \cap \text{Cone}(E(a)\,\hat{}\,\alpha) \cap C_0 \cap D^0$, the set $M_\alpha = \{\beta < \omega_1\colon \text{Cone}(a_\alpha\,\hat{}\,\beta) \text{ meets } A\}$ is cofinal in ω_1. For every $\alpha \in M$ and $\beta \in M_\alpha$ choose $b_{\alpha\beta}$ in $A \cap \text{Cone}(a_\alpha \wedge \beta) \cap (C_1 - C_0)$, let $M_{\alpha\beta} = \{\gamma < \omega_1\colon \text{Cone}(b_{\alpha\beta}\hat{}\gamma) \text{ meets } A\}$. Choose $\delta < \kappa$ exceeding every $\text{dom}(b_{\alpha\beta})$. For every $\alpha \in M$, $\beta \in M_\alpha$, $\gamma \in M_{\alpha\beta}$ choose $c_{\alpha\beta\gamma}$ of length at least δ in $A \cap \text{Cone}(b_{\alpha\beta}\hat{}\gamma)$.

Let f be an order-preserving function from ω_1 onto M. Suppose that $\alpha < \omega_1$ and for every $\alpha' < \alpha$, $E(a\,\hat{}\,\alpha')$ is chosen to be some $c_{f\alpha',\beta,\gamma}$. Choose β in

$$M_{f\alpha} - \{E(a\,\hat{}\,\alpha')(\text{dom}(a_{f\alpha}))\colon \alpha' < \alpha\}.$$

Then choose γ in

$$M_{f\alpha,\beta} - \{E(a\,\hat{}\,\alpha')(\text{dom}(b_{f\alpha,\beta}))\colon \alpha' < \alpha\}.$$

Set $E(a\,\hat{}\,\alpha) = c_{f\alpha,\beta,\gamma}$. For every $\alpha' < \alpha$ the pair $\{E(a\,\hat{}\,\alpha'), E(a\,\hat{}\,\alpha)\}$ varies at $\text{dom}(a_{f\alpha})$ and $\text{dom}(b_{f\alpha,\beta})$ which belong to $\text{Odd} \cap \text{Ord}(C_0)$ and $\text{Ord}(C_1) - \text{Ord}(C_0)$, respectively. Lemma 2 is proved.

Next we construct $W \subseteq U - D$. It will be a motley set meeting the tree closure of every motley auxiliary set.

Since $|D| = 2^{<\kappa} = \kappa$ and every auxiliary set is a subset of D the motley auxiliary sets can be arranged into a sequence $\langle A_\alpha\colon \alpha < 2^\kappa\rangle$. By induction on $\alpha < 2^\kappa$ we choose x_α in $TC(A_\alpha) - D$. Suppose that elements x_β, $\beta < \alpha$, are chosen. If $\beta < \alpha$ and $C \in F$ then there is at most one element $x \in TC(A_\alpha) - D$ such that the pair $\{x, x_\beta\}$ is mono and of color C. (For, suppose that x, y are different elements of $TC(A_\alpha) - D$ and the pairs $\{x, x_\beta\}$, $\{y, x_\beta\}$ are mono and of color C. Then $\{x, y\}$ is mono and of color C. Let $z = x \wedge y$. There are a, b in A_α with $z \subset a \subset x$, $z \subset b \subset y$. Then $\{a, b\}$ is mono which is impossible.) By Claim 3 in §1 we can find $x_\alpha \in TC(A_\alpha) - D$ such that $\{x_\alpha, x_\beta\}$ is motley for every $\beta < \alpha$. Set $W = \{x_\alpha\colon \alpha < 2^\kappa\}$.

It is easy to see that $|\bar{A} \cap W| \geq 2$ for every motley auxiliary A. If A is mono then $|\bar{A} \cap W| \leq 1$. For, suppose A is of color C, and x, y are different elements of $\bar{A} \cap W$. Then $\{x, y\}$ is motley, hence it varies at some successor $\alpha \notin \text{Ord}(C)$. A meets $\text{Cone}(x|(\alpha + 1))$ and $\text{Cone}(y|(\alpha + 1))$, hence A varies at α which is impossible.

Given I and X as in Theorem 1 we prove that (A) is equivalent to (B). First suppose (A). Given I_1, X_0, X_1 as in (B) it suffices to build a map $E\colon D \to D \cap I_1$ such that E satisfies the conditions of Claim 2 in §1 and X_0, X_1 are dense in the range $E(D)$ and $E(D)$ is mono.

Arrange all elements of D into a sequence $\langle d_\alpha\colon \alpha < \kappa\rangle$ such that $\text{dom}(d_\alpha) \subseteq \alpha$ and for every successor $a \in D$ there are an even α and an odd β with $a = d_\alpha = d_\beta$. By (A) we can suppose that $I_1 \cap X \subseteq C$ for some color C. Choose $E(\Lambda)$ in $D \cap I_1$ such that $\text{Cone}(E(\Lambda) \subseteq I_1$. Suppose that $\alpha < \kappa$ and $E(a)$ is chosen already for every $a \in D$ of length less than α. Suppose also that for every $\beta < \alpha$ all sequences $E(a)$ with $\text{dom}(a) = \beta$ have the same length.

If α is limit and a is an element of D of length α set $E(a) = \bigcup\{E(a|\beta)\colon \beta < \alpha\}$. Suppose α is successor. There is a sequence $a \in D$ of length $\alpha - 1$ such that $d_{\alpha-1} \subseteq a$. If $\alpha - 1$ is even choose $x \in \text{Cone}(Ea) \cap X_0$, if $\alpha - 1$ is odd choose $x \in \text{Cone}\,(Ea) \cap X_1$. If $b \in D$, $\beta < \omega_1$ and $\text{dom}(b) = \alpha - 1$ choose $y = E(b\,\hat{}\,\beta)$ such that

$$\mathrm{dom}(y) = \mathrm{dom}(x), \qquad E(b)\frown\beta \subset y, \quad \text{and}$$

$$y(\gamma) = x(\gamma) \quad \text{for } \mathrm{dom}(Eb) < \gamma < \mathrm{dom}(x).$$

Let A be the range of E. By Claim 2 in §1 A is auxiliary. It is easy to see that X_0, X_1 are dense in A. It is easy to see that A is mono (and of color C).

Now suppose that (A) fails, i.e. there is an interval $I_0 \subseteq I$ such that $J \cap X - C \neq 0$ for any color C and any subinterval $J \subseteq I_0$. By Claim 1 in §1 there is a color C_0 such that $C_0 \cap D^0 \cap X$ is dense in some interval $I_0' \subseteq I_0$, and there is a color C_1 such that $C_1 \cap X - C_0$ is dense in some interval $I_1 \subseteq I_0'$. We check that (B) fails for this I_1 and

$$X_0 = C_0 \cap D^0 \cap X, \qquad X_1 = C_1 \cap X - C_0.$$

If A is an auxiliary set and X_0, X_1 are dense in A then, by Lemma 2, A has an auxiliary motley subset B. Then

$$|\bar{A} \cap W| \geq |\bar{B} \cap W| \geq 2.$$

Theorem 1 is proved.

Note that clause (B) of Theorem 1 abbreviates a certain formula

$$\phi(X, D, D^l, D^0, W, I)$$

in the monadic language of order. Let Storey(X, D, D^l, D^0, W) be a formula in the monadic language or order saying the following:

$X \subseteq D$, and $D^0 \cap X$ is everywhere dense, and

$$\phi(X, D, D^l, D^0, W, \text{the whole chain}),$$

and there are are no I, Y such that I is an interval, $Y \subseteq D - X$, Y is dense in I and $\phi(X \cup Y, D, D^l, D^0, W, I)$.

THEOREM 3. *Let F be as in Theorem 1. There is $W \subseteq U - D$ such that* Storey(X, D, D^l, D^0, W) *holds in U iff $X \subseteq D$ and for every interval I there are $C \in F$ and an interval $J \subseteq I$ with $C \cap J = X \cap J$.*

PROOF. Construct W as above. Now use Theorem 1. □

§3. **Distributivity.** In this section we work in a T_3 topological space U. Recall that an open set is called *regular* if it is the interior of some closed set. It is well known and easy to check that the regular open sets form a complete Boolean algebra with 0 being the empty set, $1 = U$, $G \leq H$ meaning $G \subseteq H$, $G \cdot H = G \cap H$, $G + H$ being the interior of the closure of $G \cup H$, and $-G$ being the interior of $U - G$. This Boolean algebra will be denoted $RO(U)$. If $S \subseteq RO(U)$ the infimum and the supremum of S will be denoted ΠS and ΣS respectively. (If S is empty then $\Pi S = 1$ and $\Sigma S = 0$.)

Let κ be a cardinal. A complete Boolean algebra B is called *κ-distributive* if it satisfies a certain distributive law equivalent (see Lemma 17.7 in Jech [4]) to the following property: every collection of κ partitions of B has a common refinement. We shall say that U is *κ-distributive* if $RO(U)$ is so.

Claim 1. *Suppose that U is κ-distributive. Then the union of every collection of κ nowhere dense sets is nowhere dense.*

PROOF. We use the T_3 property to prove that every nonempty open set V includes a nonempty regular open subset. There are $x \in V$ and disjoint open neighborhoods M, N of x and $U - V$ respectively. Evidently $x \in$ (the interior of $\bar{M}$) $\subseteq V$.

For any nowhere dense set X there is a partition $\langle G_\alpha: \alpha < \pi\rangle$ of $RO(U)$ such that every G_α avoids X. Suppose that $\langle G_\beta: \beta < \alpha\rangle$ is already built and $H_\alpha = \Sigma\{G_\beta: \beta < \alpha\}$. If $H_\alpha \neq 1$ let G_α be a nonempty regular open subset of $(U - \bar{H}_\alpha) - \bar{X}_\alpha$.

Given nowhere dense sets X_α, $\alpha < \kappa$, build partitions $P_\alpha, \alpha < \kappa$, such that every $\bigcup P_\alpha$ avoids X_α. There is a partition P refining all partitions P_α. Then $U - \bigcup P$ is nowhere dense and includes all X_α. □

It is easy to see that U is κ-distributive for every κ if isolated points are dense in U. If U has no isolated points then it is not $|U|$-distributive.

We will say that a cardinal Δ is the *distributivity* of U if U is κ-distributive for $\kappa < \Delta$ but U is not Δ-distributive. Recall that U is called *orderable* if its topology is the interval topology of some linear order on the points of U.

Claim 2. *Suppose that U is orderable and without isolated points. Let Δ be the distributivity of U and $\langle A_\alpha: \alpha < \Delta\rangle$ be a sequence of everywhere dense subsets of U. There is a sequence $\langle X_\alpha: \alpha < \Delta\rangle$ of nowhere dense subsets of U such that $X_\alpha \subseteq A_\alpha$ and $\bigcup\{X_\alpha: \alpha < \Delta\}$ is dense in some nonempty open subset of U.*

PROOF. Fix an order on U whose interval topology is the topology of U. There is a sequence $\langle P_\alpha: \alpha < \Delta\rangle$ of partitions of $RO(U)$ such that (i) if $\alpha < \beta < \Delta$ then P_β refines P_α, and (ii) no partition of $RO(U)$ refines all partitions P_α. Without loss of generality every P_α is composed from intervals. There is an interval G_0 such that every interval $G \subseteq G_0$ meets at least two different members of some P_α.

For all $\alpha < \Delta$ and $I \in P_\alpha$ choose a point $x(\alpha, I) \in A_\alpha \cap I$. Let $X_\alpha = \{x(\alpha, I): I \in P_\alpha\}$ and $X = \bigcup\{X_\alpha: \alpha < \Delta\}$. Each X_α is nowhere dense and included into A_α. We check that X is dense in G_0.

Let $G \subseteq G_0$ be an interval. There are $\alpha < \Delta$ such that G meets some different members I_0, I_1 of P_α. Without loss of generality $x(\alpha, I_0) < x(\alpha, I_1)$. There is $\alpha < \beta < \Delta$ such that $G \cap I_1$ meets different members I_2, I_3 of P_β. Without loss of generality $x(\beta, I_2) < x(\beta, I_3)$. Then $I_2 \subset G$ and $x(\beta, I_2) \in G \cap X$. □

Claim 3. *Let C be a chain without isolated points. Suppose that every interval of C has a subchain of type ω_1 or ω_1^*. Then C is $\aleph_0$-distributive.*

PROOF. Without loss of generality every nonempty subset of C has the supremum and the infimum in C. By contradiction suppose that sets X_n, $n < \omega$, are nowhere dense but their union X is dense in some interval I. Without loss of generality $I = C$ and every X_n is closed.

If $a \in C - X$ and the interval $(-\infty, a)$ does not have the last point then its cofinality is equal to ω. For, let I_n be the maximal interval containing a and avoiding sets X_m, $m < n$. Then $a = \sup\{\inf I_n: n < \omega\}$. Similarly, if $a \in C - X$ and the interval (a, ∞) does not have the first point then its coinitiality is equal to ω.

Note that $C - X$ is everywhere dense. By Claim 2 (with $A_n = C - X$ for $n < \omega$) there are nowhere dense sets $Y_n \subseteq C - X$, $n < \omega$, whose union Y is dense in some interval J. Without loss of generality $J = C$ and every Y_n is closed.

Check as above that for every $a \in C - Y$, if the interval $(-\infty, a)$ does not have the last element then its cofinality is equal to ω, and if the interval (a, ∞) does not have the first element then its coinitiality is equal to ω. Without loss of generality C has a subchain C' isomorphic to ω_1; let $a = \sup C'$. The cofinality of the interval $(-\infty, a)$ is equal to ω_1 which is impossible. □

§4. A short pairing tower. We work in a $\aleph_0$-distributive chain U without isolated points. We define and study towers on U.

Letters G, H with or without indices will denote nonempty regular open subsets of U. Note that every G forms a subchain of U whose interval topology coincides with the topology inherited from U. If $\phi(V_1, \ldots, V_n)$ is a formula in the monadic language of U and the only free variables of ϕ are the shown set variables and $X_1, \ldots, X_n$ are subsets of U then the sentence $\phi(X_1, \ldots, X_n)$ will be called a U-sentence. Define

$$\operatorname{dom}(\phi(X_1, \ldots, X_n)) = \Sigma\{G: \phi(X_1 \cap G, \ldots, X_n \cap G) \text{ holds in the subchain } G\}.$$

Let $t = (D, D^l, D^0, D^1, D^2, W)$ be a sequence of subsets of U. A subset X of D will be called a *storey* (of t) if dom(Storey(X, t)) $= 1$. Here Storey is the monadic formula described in §2. If $X \subseteq D$ then $D^i \cap X$ will be denoted X^i for $i = 0, 1, 2$. We will say that t is a *tower* if it satisfies the following conditions:

(T1) D^0, D^1, D^2 are disjoint and everywhere dense subsets of D.

(T2) There is at least one storey and for every storey A, B:

$$\begin{aligned}
&\operatorname{dom}(A^0 = B^0) = \operatorname{dom}(A^1 = B^1) = \operatorname{dom}(A = B),\\
&\operatorname{dom}(A^0 \cap B^0 = 0) = -\operatorname{dom}(A^0 = B^0),\\
&\operatorname{dom}(A^1 \subseteq B^1) + \operatorname{dom}(B^1 \subseteq A^1) = 1.
\end{aligned}$$

(T3) There are no G and $X \subseteq D^0$ such that $G \subseteq \operatorname{dom}(A^0 \subseteq X)$ for some storey A, and for every storey A with $G \subseteq \operatorname{dom}(A^0 \subseteq X)$ there is a storey B such that

$$G \subseteq \operatorname{dom}(B^0 \subseteq X - A^0 \;\&\; B^1 \subseteq A^1).$$

Claim 1. *Suppose that* $t = (D, D^l, D^0, D^1, D^2, W)$ *is a tower in* U.

(i) *If* $X \subseteq D$ *and for every* G *there are a storey* A *of* t *and* $H \subseteq G$ *such that* $A \cap H = X \cap H$ *then* X *is a storey of* t.

(ii) *Let* $t|G = (D \cap G, D^l \cap G, D^0 \cap G, D^1 \cap G, D^2 \cap G, W \cap G)$. *The* $t|G$ *is a tower in the subchain* G. *Moreover if* A *is a storey of* t *then* $A \cap G$ *is a storey of* $t|G$.

PROOF. (i) A straightforward analysis of the formula Storey.

(ii) is obvious. □

In the rest of this section t is a tower and letters A, B, C with or without indices denote storeys of t. We say that $A \leq B$ on G if $G \subseteq \operatorname{dom}(A^1 \subseteq B^1)$. We say that $A < B$ on G if $A \leq B$ on G and $B^1 - A^1$ is dense in G. By induction on ordinal α we define relations $A = \alpha$ modulo t on G. Suppose that relations $A = \beta$ modulo t on G are defined for $\beta < \alpha$. We say that $A = \alpha$ modulo t on G if

(i) there are no $\beta < \alpha$ and $H \subseteq G$ such that $A = \beta$ modulo t on H, and

(ii) $A \leq B$ on G for every B such that there are no $\beta < \alpha$ and $H \leq G$ with $B = \beta$ modulo t on H.

Claim 2. $A = 0$ *modulo* t *on* G *iff* $A \leq B$ *on* G *for every* B. *If* $A = \alpha$ *modulo* t *on*

G then $A = \alpha$ modulo t on every $H \subseteq G$. If $A = \alpha$ modulo t on G and $B = \alpha$ modulo t on G then $G \subseteq$ dom$(A = B)$. *If $A = \alpha$ and $B = \beta$ modulo t on G and $\alpha < \beta$ then $A < B$ on G.*

Proof is clear.

The minimal ordinal α such that there is no A with $A = \alpha$ modulo t will be called the *height* of t. Let τ be the height of t.

The set $\Sigma\{G$: there is no A such that $A = \tau$ modulo $t\}$ will be called the *arena* of t.

Claim 3. *The arena of t is not empty.*

PROOF. Suppose the contrary. Then for every G there are A and $H \subseteq G$ such that $A = \tau$ modulo t on H. Construct a maximal family $\{(A_i, H_i): i \in I\}$ such that $\{H_i: i \in I\}$ is disjoint and every $A_i = \tau$ modulo t on H_i. Then $\bigcup\{A_i \cap H_i: i \in I\}$ is a storey and it is equal to τ modulo t on U, which is impossible. □

In the rest of this section we suppose that the arena of t is equal to U.

THEOREM 4. *For all A, G there are $\alpha < \tau$, $H \subseteq G$ such that $A = \alpha$ modulo t on H.*

PROOF. By contradiction suppose that for some G the collection $K = \{A$: there are no $\alpha < \tau, H \subseteq G$ with $A = \alpha$ modulo t on $H\}$ is not empty. If $A \in K$ and $A \leq B$ on G for every $B \in K$ then $A = \tau$ modulo t on G, which is impossible. Hence for all $A \in K$, $G_1 \subseteq G$ there are $B \in K$, $H \subseteq G_1$ with $B < A$ on H. Moreover for every $A \in K$ there is $B \in K$ with $B < A$ on G. For, construct a maximal family $\{(B_i, H_i): i \in I\}$ such that $B_i \in K$, $H_i \subseteq G$, $B_i < A$ on H_i and $\{H_i: i \in I\}$ is disjoint. Then $(\bigcup\{B_i \cap H_i: i \in I\}) \cup (A - G)$ is some storey B and $B < A$ on G.

Therefore there is a sequence $\langle A_n: n < \omega\rangle$ of members of K such that $A_{n+1} < A_n$ on G. Let

$$X = \bigcup\{A_n^0: n < \omega\}.$$

We show that G, X violate (T3) in the definition of towers. Evidently $G \subseteq$ dom$(A_0^0 \subseteq X)$. Suppose that $G \subseteq$ dom$(A^0 \subseteq X)$ for some storey A. Since U is $\aleph_0$-distributive, for every $H \subseteq G$ there is n such that dom$(A = A_n)$ meets H. For, otherwise $A^0 \cap A_n^0 \cap H$ is nowhere dense, hence $A^0 \cap X \cap H$ is nowhere dense, which contradicts $G \subseteq$ dom$(A^0 \subseteq X)$. Thus $(\bigcup\{A_{n+1} \cap$ dom$(A = A_n) \cap G: n < \omega\}) \cup (A_0 - G)$ is some storey B and $B < A$ on G. □

A tower t will be called *short* if it satisfies the following two conditions.

(ST1) For every A there is $E \subseteq D^0$ such that dom$(B^0 \subseteq E) = 1$ if $B < A$ on U and dom$(B^0 \subseteq E) = 0$ if $A \leq B$ on U.

(ST2) If $X \subseteq D^0$ and every $A \cap X$ is nowhere dense then X is nowhere dense.

Claim 5. *Suppose that t is short and Δ is the distributivity of U. Then $\tau < \Delta$.*

PROOF. By contradiction suppose $\Delta \leq \tau$. For $\alpha < \tau$ choose A_α such that $A_\alpha = \alpha$ modulo t on U. Without loss of generality $\{A_\alpha: \alpha < \Delta\}$ is disjoint because A_α can be replaced by $A_\alpha - \bigcup\{A_\beta: \beta < \alpha\}$ for $\alpha < \Delta$.

There is $E \subseteq D^0$ such that dom$(A_\alpha^0 \subseteq E) = 1$ for $\alpha < \Delta$ and dom$(A_\alpha^0 \subseteq E) = 0$ for $\alpha \geq \Delta$. If $\Delta > \tau$ use (ST1) with $A = A_\Delta$ to find an appropriate set E. If $\Delta = \tau$ set $E = D^0$.

By Claim 2 in §3 there is a sequence $\langle X_\alpha: \alpha < \Delta\rangle$ of nowhere dense sets such that $X_\alpha \subseteq A_\alpha \cap E$ for $\alpha < \Delta$ and the union $X = \bigcup\{X_\alpha: \alpha < \Delta\}$ is somewhere dense. If $\alpha < \Delta$ then $A_\alpha \cap X = X_\alpha$. If $\alpha \geq \Delta$ then $A_\alpha \cap X \subseteq A_\alpha \cap E$. In either case $A_\alpha \cap X$ is nowhere dense. By Theorem 4 $A \cap X$ is nowhere dense for every A. By (ST2) X is nowhere dense which gives a contradiction. □

A storey A will be called *zero* if $A \leq B$ on U for any B. We will say that $B = A + 1$ if $A < B$ on U and there are no C, G with $A < C$ and $C < B$ on G. We will say that A is *limit* if for every $B < A$ on U there is C with $B < C < A$ on U. We will say that $C = \max(A, B)$ if $A \leq C$ on U and $B \leq C$ on U and $\mathrm{dom}(A = C) + \mathrm{dom}(B = C) = 1$. Below, $A + 1$ denotes any B with $B = A + 1$, and $A + 2$ denotes $(A + 1) + 1$, and $\max(A, B)$ denotes any C with $C = \max(A, B)$.

Claim 6. *If* $B = A + 1$ *and* $C = A + 1$ *then* $\mathrm{dom}(B = C) = 1$. *If* $A = \alpha$ *modulo* t *and* $B = A + 1$ *then* $B = \alpha + 1$ *modulo* t. *If* $A = \alpha$ *modulo* t *then*: A *is limit iff* $\alpha = 0$ *or* α *is limit*. τ *is limit iff for every* A *there is* $A + 1$. *For every* A, B *there is* $\max(A, B)$. *If* $C_0 = \max(A, B)$ *and* $C_1 = \max(A, B)$ *then* $\mathrm{dom}(C_0 = C_1) = 1$. *If* $A = \alpha$ *modulo* t, $B = \beta$ *modulo* t *and* $C = \max(A, B)$ *then* $C = \max(\alpha, \beta)$ *modulo* t.

Proof is easy.

Recall a well-known ordering of pairs of ordinals: $(\alpha_0, \beta_0) < (\alpha_1, \beta_1)$ if either $\max(\alpha_0, \beta_0) < \max(\alpha_1, \beta_1)$ or the maximums are equal and (α_0, β_0) precedes (α_1, β_1) lexicographically. If (α, β) is the γth pair in that order we write $nu(\alpha, \beta) = \gamma$. We are interested in towers coding a portion of this pairing function.

A tower t will be called *pairing* if it satisfies the following conditions.

(PT1) For every A there is a limit B with $A < B$ on U.

(PT2) For every limit A, B there is a limit C such that

$$\mathrm{dom}(A^2 \cup (B + 1)^2 = (C + 2)^2) = 1,$$

and for every limit C there are limit A, B such that

$$\mathrm{dom}(A^2 \cup (B + 1)^2 = (C + 2)^2) = 1.$$

(PT3) Suppose that A_i, B_i, C_i are limit and

$$\mathrm{dom}(A_i^2 \cup (B_i + 1)^2 = (C_i + 1)^2) = 1 \quad \text{for } i = 0, 1.$$

If $\max(A_i, B_i) < \max(A_{1-i}, B_{1-i})$ on G, or $G \leq \mathrm{dom}(\max(A_0, B_0) = \max(A_1, B_1))$ and $A_i < A_{1-i}$ on G or $G \leq \mathrm{dom}(A_0 = A_1)$ and $B_i < B_{1-i}$ on G, then $C_i < C_{2-i}$ on G for $i = 0, 1$.

Claim 7. *Suppose that* t *is a pairing tower*.

(i) τ *is limit. Moreover* $\tau = nu(0, \tau)$.

(ii) *If* $A = \omega\alpha$, $B = \omega\beta$, $C = \omega\gamma$ *modulo* t *and*

$$\mathrm{dom}(A^2 \cup (B + 1)^2 = (C + 2)^2) = 1$$

then $\gamma = nu(\alpha, \beta)$.

Proof is clear.

§5. Interpretation. In this section second-order logic is interpreted in the monadic theory of order. First we reduce second-order logic to a certain monadic theory.

Order pairs of ordinals as follows:

$(\alpha, \beta) < (\gamma, \delta)$ if either $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ or $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ and (α, β) precedes (γ, δ) lexicographically.

We write $\gamma = nu(\alpha, \beta)$ if (α, β) is the γth pair in this order. It is easy to see that $\omega = nu(0, \omega)$ and moreover $\kappa = nu(0, \kappa)$ for every infinite cardinal κ. An ordinal δ will be called *pairing* if $\delta = nu(0, \delta)$. For every pairing ordinal $\delta > 0$ let

$$P_\delta = \{(\alpha, \beta, \gamma): \gamma = nu(\alpha, \beta) < \delta\},$$

and let M_δ be the structure $\langle \delta, P_\delta \rangle$ (δ is the universe of M_δ and P_δ is the only relation of M_δ).

Note that P_δ gives a 1-1 function from $\delta \times \delta$ onto δ.

Claim 1. *Second-order logic is interpretable in the monadic theory of structures M_δ.*

PROOF. Since P_δ gives a pairing function on δ it is easy to interpret the second-order theory of δ in the monadic theory of M_δ. For example, an arbitrary binary relation R on δ can be coded by $\{nu(\alpha, \beta): (\alpha, \beta) \in R\}$. Moreover the straightforward interpretation of the second-order theory of δ in the monadic theory of M_δ is uniform in δ, i.e. to each second-order sentence ϕ we assign effectively (by induction on ϕ) a sentence ϕ' in the monadic theory of a ternary predicate in such a way that for every pairing ordinal δ, δ satisfies ϕ iff M_δ satisfies ϕ'.

Given a second-order sentence ψ write a second-order sentence ϕ saying that every nonempty subset of elements satisfies ψ. Translate ϕ into a sentence ϕ' in the monadic language of a ternary predicate as above. It is easy to see that ψ is true in all nonempty sets iff ϕ' is true in all structures M_δ. $\square$

Given a formula $\phi(v_1, \ldots, v_m, V_1, \ldots, V_n)$ in the monadic language of a ternary predicate P, a chain U without isolated points, a tower $t = (D, D^l, D^0, D^1, D^2, W)$ in U, storeys $A_1, \ldots, A_m$ of t, and subsets $X_1, \ldots, X_n$ of D^0, we define (by induction on ϕ) a regular open subset $\phi_t(A_1, \ldots, A_m, X_1, \ldots, X_n)$ of U:

$(P(A_i, A_j, A_k))_t = \mathrm{dom}(A_i^2 \cup (A_j + 1)^2 = (A_k + 2)^2)$,

$(A_i \in X_j)_t = \mathrm{dom}(A_i^0 \subseteq X_j)$,

$(\sim\phi)_t = 1 - \phi_t$, $(\phi \text{ or } \psi)_t = (\phi_t \text{ or } \psi_t)$,

$(\exists v\, \phi(v))_t = \Sigma\{\phi_t(A): A \text{ is a zero or limit storey of } t\}$,

$(\exists V\, \phi(V))_t = \Sigma\{\phi_t(X): X \subseteq D^0\}$.

We are especially interested in the case when ϕ is a sentence, i.e. ϕ has no free variables. In this case $(\phi_t = 1)$ can be considered as a formula (with free variables D, D^l, D^0, D^1, D^2, W) in the monadic language of order. It is a specific formula, its construction does not depend on the choice of U, t.

THEOREM 2. *Suppose that U is a chain without isolated points, and the distributivity Δ of U is uncountable. Suppose that $t = (D, D^l, D^0, D^1, D^2, W)$ is a short pairing tower in U of height $\tau = \omega\delta$, and the arena of t is equal to U. Then M_δ satisfies a monadic sentence ϕ iff $\phi_t = 1$ in U.*

PROOF. For $\alpha < \delta$ let A_α be a storey of t such that $A_\alpha = \omega\alpha$ modulo t. By Claim 5 in §4, $\tau < \Delta$. Hence we can suppose that the collection $\{A_\alpha^0: \alpha < \delta\}$ is disjoint. (Just change A_α for $A_\alpha - \bigcup\{A_\beta^0: \beta < \alpha\}$ if necessary.) For every subset I of δ let $S(I) = \bigcup\{A_\alpha^0: \alpha \in I\}$.

LEMMA 3. *Suppose that G is a nonempty regular open subset of U, and*

$$\phi(u_1, \ldots, u_m, V_1, \ldots, V_n)$$

is a formula in the monadic language of a ternary predicate, and $B_1, \ldots, B_m$, $C_1, \ldots, C_m$ are storeys of t with $G \subseteq \mathrm{dom}(B_i = C_i)$ for $1 \le i \le m$, and

$X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are subsets of D^0 such that $G \cap \mathrm{dom}(B^0 \subseteq X_j) = G \cap \mathrm{dom}(B^0 \subseteq Y_j)$ for $1 \leq j \leq n$ and any storey B of t. Then

$$G \cap \phi_t(B_1, \ldots, B_m, X_1, \ldots, X_n) = G \cap \phi_t(C_1, \ldots, C_m, Y_1, \ldots, Y_n).$$

PROOF. An easy induction on ϕ. □

By induction on a formula ϕ in the monadic language of a ternary predicate P we prove the following:

If $\phi(\alpha_1, \ldots, \alpha_m, I_1, \ldots, I_n)$ holds (respectively fails) in M_δ then

$$\phi_t(A_{\alpha_1}, \ldots, A_{\alpha_m}, S(I_1), \ldots, S(I_n))$$

is equal to 1 (respectively to 0) in U.

In the case $\phi = P(\alpha_1, \alpha_2, \alpha_3)$ use the fact that t is a pairing tower. Cases $\phi = (\alpha \in I)$, $\phi = \sim\phi_1$, $\phi = (\phi_1 \text{ or } \phi_2)$ are easy.

Suppose $\phi = \exists v \psi(v)$. If ϕ holds in M_δ then some $\psi(\alpha)$ holds in M_δ, hence $\psi_t(\alpha) = 1$ and $\phi_t = 1$. If $\phi_t \neq 0$ then there is a zero or limit storey A with $\psi_t(A) \neq 0$. By Theorem 4 in §4 some $\psi_t(A_\alpha) \neq 0$. Hence $\psi(\alpha)$ holds in M_δ and ϕ holds in M_δ.

Suppose $\phi = \exists V \psi(V)$. If ϕ holds in M_δ then there is $I \subseteq \delta$ such that $\psi(I)$ holds in M_δ, hence $\psi_t(S(I)) = 1$ and $\phi_t = 1$. Suppose $\phi_t \neq 0$. Then there is $X \subseteq D^0$ with $\psi_t(X) \neq 0$. Since $\delta < \Delta$ there is a partition of $RO(U)$ refining all partitions

$$\mathrm{dom}(A^0_\alpha \subseteq X) + (1 - \mathrm{dom}(A^0_\alpha \subseteq X)) = 1.$$

Hence there is a nonempty regular open set G such that $G \subseteq \psi_t(X)$ and for every $\alpha < \delta$, either G is included into $\mathrm{dom}(A^0_\alpha \subseteq X)$ or G avoids it. Let

$$I = \{\alpha: G \subseteq \mathrm{dom}(A^0_\alpha \subseteq X)\}.$$

By Lemma 3, $G = G \cap \psi_t(X) = G \cap \psi_t(S(I))$, hence $\psi_t(S(I)) \neq 0$, and $\psi(I)$ holds in M_δ, and ϕ holds in M_δ. Theorem 2 is proved.

Claim 4. *Suppose that $\phi(v_1, \ldots, v_m, V_1, \ldots, V_n)$ is a formula in the monadic language of a ternary predicate, and U is a chain without isolated points, and $t = (D, D^l, D^0, D^1, D^2, W)$ is a tower in U, and $A_1, \ldots, A_m$ are storeys of t, and $X_1, \ldots, X_n$ are subsets of D^0. Then $\phi_t = 1$ (respectively $\phi_t = 0$) in U iff every interval I of U has a subinterval J such that $\phi_{t|J} = 1$ (respectively $\phi_{t|J} = 0$) in J. (About $t|J$ see Claim 1 in §4.)*

PROOF. Easy induction on ϕ.

Given a sentence ϕ in the monadic language of a ternary predicate write down a sentence ϕ^* in the monadic language of order saying the following:

If there are no isolated points and every interval embeds either ω_1 or ω_1^* then for every short pairing tower t, $\phi_t = 1$.

Claim 5. ϕ holds in all structures M_δ iff ϕ^* holds in every chain.

PROOF. First suppose that ϕ holds in all structures M_δ, and U is a chain without isolated points, and every interval of U embeds either ω_1 or ω_1^* and t is a short pairing tower in U.

We build a partition $\langle G_\alpha: \alpha < \pi \rangle$ or $RO(U)$ such that the arena of every $t|G_\alpha$ is equal to G_α. Let G_0 be the arena of t itself. Suppose that $\langle G_\beta: \beta < \alpha \rangle$ is constructed; if $H = \Sigma\{G_\beta: \beta < \alpha \neq 1\}$ let G_α be the arena of $t|(-H)$.

By Claim 3 in §3 U is $\aleph_0$-distributive. By Theorem 2 $\phi_{t|G} = 1$ for all $\alpha < \pi$. By Claim 4 $\phi_t = 1$.

Now suppose that ϕ fails in some M_δ. Using our set-theoretic assumption find a cardinal κ such that $\kappa > \aleph_1$, $\kappa > \delta$ and $2^{<\kappa} = \kappa$.

Let U, D, D^l, D^0 be as in §§1 and 2. It is easy to construct subsets D^1, D^2 of D and a family $F = \{A_\alpha : \alpha < \omega\delta\}$ of subsets of D such that:

(i) For $i = 1, 2$ there is a cofinal subset $\mathrm{Ord}(D^i)$ of κ with $D^i = \{a \in D : \mathrm{dom}(a) \in \mathrm{Ord}(D^i)\}$.

(ii) D^1, D^2 partition $D - (D^l \cup D^0)$.

(iii) F satisfies the conditions of Theorem 1 in §2.

(iv) If $\alpha < \beta < \omega\delta$ then $A_\alpha \cap D^1 \subset A_\beta$ and $A_\beta \cap D^1 - A_\alpha$ is everywhere dense.

(v) For all $\alpha, \beta, \gamma < \omega\delta$, $\gamma = nu(\alpha, \beta)$ iff $(A_{\omega\alpha} \cup A_{\omega\beta+1}) \cap D^2 = A_{\omega\gamma+2} \cap D^2$.

Let W be as in Theorem 1 of §2. It is easy to see that $t = (D, D^l, D^0, D^1, D^2, W)$ is a short pairing tower of height $\omega\delta$. Evidently the arena of t is equal to U. By Claim 1 in §1 the distributivity of U exceeds $\omega\delta$. By Theorem 2 $\phi_t = 0$ in U. Thus ϕ^* fails in U.

Claims 1 and 5 give:

COROLLARY 6. *Second-order logic is interpretable in the monadic theory of order.*

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