Infinite Games *

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Infinite games are widely used in mathematical logic [2, 8, 12]. In particular, infinite games proved to be a useful tool in dealing with the monadic second-order theories of infinite strings and infinite trees [3, 4, 7]. Recently [1, 15, 13], infinite games were used in connection to concurrent computational processes that do not necessarily terminate. For example, an operating system may be seen as playing a game "against" the disruptive forces of users. The classical question of the existence of winning strategies turns out to be of importance to practice. Here we attempt to explain basics of infinite game theory.

- *Quisani:* What is an infinite game?
- *Author:* In the simplest form, there are two players, Player 1 and Player 2. Player 1 starts by choosing a binary bit a_1, then Player 2 chooses a binary bit a_2, then Player 1 chooses a binary bit a_3, then Player 2 chooses a binary bit a_4, and so on *ad infinitum*. If the resulting infinite string X (i.e. the function $X(i) = a_i$ on positive integers, the *play*) belongs to an *a priori* fixed set W of infinite strings then Player 2 wins the game; otherwise Player 1 does.
- Q: Do you mean that different sets W give different games?
- A: Yes. The set W defines the goals of the players. Let W_1 be the complement of W and $W_2 = W$. Then the goal of Player ε is to ensure that the play belongs to his winning set W_ε.
- Q: I guess, your simplest form is a little too simple. As one of those disruptive users, I want more than 2 keys on my keyboard.
- A: You are right. Let me describe a more general setting. We are given an infinite countable tree A called the *arena* and a set W of branches of A. Nodes of A are possible positions of the game $\Gamma(A, W)$; the root is the initial position. Player 1 begins by choosing a child x_1 of the root, then Player 2 chooses a child x_2 of x_1, and so on. If

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the resulting branch (the play) belongs to W, then Player 2 wins, otherwise Player 1 wins.

- Q: Can a branch be finite? In other words, are there dead-end positions?

- A: Dead-end positions may be natural in applications and useful in theory [14], but for simplicity let us stick here to arenas without dead-end nodes.

- Q: Let me return to the game of an operating system against the users. If there are n users then in effect there are $n+1$ players.

- A: It is easier to analyze 2 player games. We can view all users together as one force against which the operating system has to play.

- Q: Can you give me an example how the winning set of the operating system specifies its goals?

- A: Suppose that the users compete for some resource, say, a printer. A computation is called *fair* if, whenever a user asks for a printer, she or he eventually gets it. You may want to require that the winning set of the operating system comprises only fair plays. If the operating system plays a winning strategy then fairness will be ensured.

- Q: What is a winning strategy exactly? Even simpler, what is a strategy?

- A: A (possibly nondeterministic) strategy for Player ε in a game $\Gamma(A, W)$ is a function F that, given a position x where Player ε makes a move, produces a nonempty set $F(x)$ of children of x. F is *deterministic* if every $F(x)$ is singleton. F is *winning* if Player ε wins every play consistent with F.

- Q: I have forgot the exact definition of games with complete information, but I remember that finite games with complete information are always determinate, i.e., one of the players has a winning strategy. Isn't any $\Gamma(A, W)$ a game with complete information? Is determinacy a problem in the infinite case?

- A: Yes, each $\Gamma(A, W)$ is a game with complete information, but do not worry about the definition of games with complete information: we will not use it. And yes, determinacy is problematic in the infinite case. An example of an indeterminate infinite game appeared already in the paper by Gale and Stewart [6] where they introduced infinite games of interest to us here. (Actually, Gale and Stewart rediscovered infinite games in the context of emerging game theory. Similar infinite games were studied by East European mathematicians much earlier [11].) Here is their example.

  Let A be the arena of binary strings. Index the deterministic strategies (the notion of strategy does not depend on winning sets) of Player 1 (resp. Player 2) as F_α (resp. G_α) where α ranges over ordinals of cardinality less than continuum. By induction on α, define branches X_α and Y_α such that X_α is consistent with G_α and different from any

Y_β with $\beta < \alpha$, and Y_α is consistent with F_α and different from any X_β with $\beta \leq \alpha$. This is possible because there are continuum many branches consistent with any G_α (resp. F_α) whereas the set $\{Y_\beta : \beta < \alpha\}$ (resp. $\{X_\beta : \beta \leq \alpha\}$) contains less than continuum many members.

Let W comprise all branches Y_α. The game $\Gamma(A, W)$ is indeterminate. For, any strategy F_α of Player 1 is defeated if Player 2 plays Y_α, and any strategy G_α of Player 2 is defeated if Player 1 plays X_α.

- Q: Is determinacy a big issue in the infinite game theory?

- A: Yes. It plays also an important role in mathematical applications. The so-called Axiom of Determinacy is an exciting alternative to Axiom of Choice. Much of infinite game theory is related to determinacy, and it is my intention to speak today mainly about determinacy. Have you noticed anything special about the indeterminate game above?

- Q: The arena was natural, but the winning set was weird. I can see an essential use of Axiom of Choice. Do you want to say that games with nice winning sets are determinate?

- A: Yes. And we need topology to define appropriate nice sets. Call a set U of branches of an arena A *open* if every branch $X \in U$ has a finite prefix x such that the *cone* $[x] = \{Y :\ Y \text{ is a branch with prefix } x\}$ is included into U; this is the well-known Cantor topology. A game $\Gamma(A, W)$ is called *open, closed, Borel*, etc. if W is so. Tony Martin proved that every Borel game is determinate [9, 10].

- Q: Are open games of interest? Can you prove that open games are determinate?

- A: Open games are of great interest. In the example with users competing for a printer, consider a particular request R of a printer. It is easy to see that the collection of branches, i.e. computations, where R is satisfied is open and gives rise to an open game. And I am happy to prove the determinacy of open games for you.

Theorem 1 [6] *Every closed or open game is determinate.*

- Q: Isn't enough to speak about open games only?

- A: Well, there is this little asymmetry: Player 1 begins. Let me start with a few definitions.

  A *support* for an open set U of branches of an arena A is any subset S of A such that $\bigcup_{x \in S}[x] = U$. The collection of all nodes x with $[x] \subseteq U$ is the largest support for U. The *rank function* corresponding to a subset S of A and to Player ε is the unique function ρ from A to the set of countable ordinals augmented with a maximal element ∞ such that for every node x:

- $\rho(x) = 0$ if and only if x has a prefix in S.
- $\rho(x) \le \alpha \ne 0$ if and only if either Player ε makes a move in position x and $\rho(y) < \alpha$ for some child y of x or else the opponent makes a move in position x and $\rho(y) < \alpha$ for every child y of x.

If $\rho(x) = \infty$, I will say that $\rho(x)$ is *extra ordinal.*

Now we are ready to prove Theorem 1. Let Player ε be the "open player" in a closed or open game $\Gamma(A, W)$ (so that W_ε is open) and let $\delta = 3 - \varepsilon$. The desired winning strategy of Player ε is to minimize the rank function for a support for W_ε, and the desired winning strategy for Player δ is to maximize that function.

- Q: Will you elaborate?

- A: Sure. Let S be any support for W_ε, and ρ be the rank function corresponding to S and to Player ε. Player ε wins a play X if and only if X meets S, i.e., X has a prefix that belongs to S. Let F be the "Decrease the rank" strategy for Player ε:

  If $0 < \rho(x) < \infty$ then $F(x)$ comprises all children y of x such that $\rho(y) < \rho(x)$; otherwise $F(x)$ comprises all children of x.

  It is easy to see that F is winning if $\rho(\text{root})$ is ordinal. All the time that the rank is positive, it will decrease with each move. Since the order of ordinals is well founded, the rank will reach 0 in finitely many steps, and therefore Player ε will win.

  Let G be the "Keep the rank extra ordinal" strategy for Player δ:

  If $\rho(x)$ is extra ordinal then $G(x)$ comprises all children y of x such that $\rho(y)$ is extra ordinal; otherwise $G(x)$ comprises all children of x if $\rho(x)$ is ordinal.

  It is easy to see that G is winning if $\rho(\text{root})$ is extra ordinal. That finishes the proof of Theorem 1.

- Q: Why do you speak about countable, rather than finite, ordinals in the definition of rank?

- A: Consider the case when the root of A has countably many children $x_1, x_2, \ldots$. Choose a set $S \subset A$ in such a way that the rank of x_n with respect to S and to Player 2 is n, and consider the game where W_2 is the open set of branches supported by S. Then the rank of the root is the first infinite ordinal ω, and the second player has a winning strategy in the game.

- Q: I thought for some reason that we are talking about arenas of bounded branching. Why do you want to allow infinite branching?

- A: Here is one example. In the game of an operating system against the users, a user may type a great deal before she or he lets the operating system to make a move.

- Q: I would like to see again the definition of the Borel hierarchy.

- A: Let α range over countable ordinals, and let $\overline{S}$ be the complement of a subset S of the given arena A.

  - Σ^0_1 is the collection of open sets.
  - $\Pi^0_\alpha = \{\overline{S} : \ S \in \Sigma^0_\alpha\}$.
  - Every $S \in \Sigma^0_\alpha$ is the union of countable many members of $\bigcup_{\beta<\alpha} \Pi^0_\beta$.

- Q: Do Σ^0_2 or Π^0_2 winning sets appear naturally when an operating system plays against the users? If yes, I would like to see the determinacy proof for Σ^0_2 games.

- A: Recall the fairness condition and consider the game of an operating system against the users where the winning set of the operating system comprises those and only those plays, i.e. computations, that are fair. That winning set is the intersection of countably many open sets $U(R)$ where $U(R)$ is the set of computations where the request R is satisfied. Thus, the winning set is Π^0_2.

  Theorem 2 [16] *Every Σ^0_2 and every Π^0_2 game is determinate.*

  To prove Theorem 2, let Player ε be the Π^0_2 player in a Σ^0_2 or Π^0_2 game $\Gamma(A, W)$ (so that W_ε is Π^0_2) and let $\delta = 3 - \varepsilon$. Fix open sets U_n such that $W_\varepsilon = \bigcap_n U_n$, and let S_n be the largest support for U_n. The goal of Player ε is to hit every S_n. All rank functions in the proof of Theorem 2 correspond to Player ε.

- Q: If the root has the extra ordinal rank with respect to some S_n then the strategy "Keep the rank extra ordinal" of Player δ is obviously winning. Otherwise, Player ε can hit any S_n.

- A: Being able to hit every S_n does not suffice: The winning strategy of Player ε should guarantee hitting all sets S_n. Let P_1 be the collection of positions x such that x has an ordinal rank with respect to every S_n. From a position in P_1, Player ε can reach some position in any S_n, but that position may have the extra ordinal rank with respect to some other S_m.

- Q: I see your point. Let P_2 be the set of positions x such that x has an ordinal rank with respect to every set $P_1 \cap S_n$. From a position in P_2, Player ε can hit any $P_1 \cap S_n$ from where he can hit any S_m; this is again insufficient for a winning strategy. By induction on k, define P_{k+1} to be the set of positions x such that x has an ordinal rank with respect to every $P_k \cap S_n$. Finally, let $P_\omega = \bigcap_k P_k$. I think about positions in P_α as those with potential α to hit sets S_n. It looks like Player ε has a winning strategy if the initial position is in P_ω.

- A: Not quite. Potential ω does not suffice. Given any k and n, where $k < \omega$ of course, and starting in a position of potential ω, Player ε can reach a position of potential k in S_n, but it is not guaranteed that he can reach a position of potential ω in S_n.

- Q: But even uncountable potential does not guarantee the ability to reach a position of potential ω in S_1, then in S_2, etc. because any descending sequence of ordinals is finite. Oh, I see. What we need is that $P_{\mu+1} = P_\mu$ for some μ. Then potential μ guarantees that Player ε can hit a position of potential μ in S_1, from where he can hit a position of potential μ in S_2, and so on. Thus, potential μ guarantees victory for Player ε. And of course, there is an ordinal μ with $P_{\mu+1} = P_\mu$ because, for each ν with $P_{\nu+1} \neq P_\nu$, the difference $P_\nu - P_{\nu+1}$ contains at least one position and there are only countable many positions.

- A: Very good. Let me just repeat your argument a little more formally. By induction on countable ordinal α, define sets P_α:

  - $P_0 = A$.
  - $P_{\alpha+1}$ is the set of positions x such that, for every n, the rank of x with respect to $P_\alpha \cap S_n$ is ordinal.
  - If α is a limit ordinal then $P_\alpha = \bigcap_{\beta<\alpha} P_\beta$.

  It is easy to see that the sequence of P_α decreases; hence indeed $P_\mu = P_{\mu+1}$ for some μ. Fix an appropriate μ and define $P = P_\mu$. Define a strategy F for Player ε:

  > If the set $N(x) = \{n : x \notin S_n\}$ is empty, let $F(x)$ comprise all children of x. Otherwise, let $m = \min N(x)$, ρ be the rank function corresponding to $P \cap S_m$ and to Player ε and let $F(x)$ comprise all children y of x such that $\rho(y) < \rho(x)$.

  It is easy to see that F is winning if the initial position is in P.

- Q: But now it is not obvious that Player δ has a winning strategy whenever the initial position is not in P.

- A: There is only a little work to do. If $x \notin P$, let $\alpha(x)$ be the ordinal such that $x \in P_\alpha - P_{\alpha+1}$ and $n(x)$ be the least number n such that x has the extra ordinal rank with respect to $P_\alpha \cap S_n$. Define a strategy G for Player δ:

  > If $x \notin P$ then $G(x)$ comprises children y of x such that y has the extra ordinal rank with respect to $P_{\alpha(x)} \cap S_{n(x)}$; otherwise it comprises all children of x.

- Q: I do not understand something. Suppose that the initial position x_0 does not belong to P. The idea of G seems to be to prevent Player ε from hitting $S_{n(x_0)}$, but this may be impossible.

- A: The idea is slightly more subtle. It is true that, if $\alpha(x_0) > 0$, then $S_{n(x_0)}$ may be hit at some position x_1; but then $\alpha(x_1) < \alpha(x_0)$. If $\alpha(x_1) > 0$, then $S_{n(x_1)}$ may be hit at some x_2; but then $\alpha(x_2) < \alpha(x_1)$.

- Q: I see. Eventually, some x_k is reached such that $S_{n(x_k)}$ is never hit, and therefore Player δ wins.

- A: That's right, and the proof of Theorem 2 is finished. In connection to Π^0_2 sets, the following observation is of interest: For every Π^0_2 set W of branches of an arena A, there exists a subset S of A such that an arbitrary branch X belongs to W if and only if it hits S infinitely many times. To prove this, suppose that W is the intersection of open sets U_n and let S_n, $n \geq 1$, be a support for U_n. Imagine the tree A growing upward. Let T_1 be the set of those nodes $x \in S_1$ that there is no $y \in S_1$ below x; T_1 is still a support for U_1. Let T_2 be the set of those nodes $x \in S_2$ that there is $y \in T_1$ below x but there is no $y \in S_2$ below x; T_2 supports $U_1 \cap U_2$. Let T_3 be the set of nodes $x \in S_3$ such that there is $y \in T_2$ below x but there is no $y \in S_3$ below x; T_3 supports $U_1 \cap U_2 \cap U_3$. And so on. The desired S is the union of sets T_n.

- Q: How about proving Martin's theorem in full?

- A: I am reluctant to explain Martin's proof here. Let me mention however that it uses a relatively powerful set theory. Namely, the existence of cardinal BETH_α is assumed in the proof of Σ^0_α determinacy. (Recall that $\mathrm{BETH}_0 = \aleph_0$, $\mathrm{BETH}_{\alpha+1} = 2^{\mathrm{BETH}_\alpha}$, and $\mathrm{BETH}_\alpha = \sup\{\mathrm{BETH}_\beta : \beta < \alpha\}$ if α is limit.) Apparently, the assumption is essential. Let ZC be the standard Zermelo-Fraenkel set theory ZFC without Fraenkel's replacement axiom. ZC has the power set axiom and therefore the existence of every BETH_n, $n < \omega$, is provable in ZC. However, if ZFC is consistent, then ZC has a model where BETH_ω does not exist and where a Σ^0_ω game may be indeterminate.

  One class of Borel games is of special importance in connection to the monadic second-order theories of infinite strings and infinite trees. Let $\mathcal{B}$ comprise boolean combinations of Σ^0_2 sets.

  Theorem 3 *Every $\mathcal{B}$ game is determinate.*

- Q: Now the situation is quite symmetric and I wonder which Player will you chose to be Player ε.

- A: We'll break symmetry. Let Γ be a $\mathcal{B}$ game $\Gamma(A, W)$. Fix Π^0_2 sets $V_1, \ldots, V_m$ such that W is a boolean combination of $V_1, \ldots, V_m$. Let Player ε be the player whose winning set includes the intersection of all V_i.

- Q: Why should the intersection of all sets V_i be included into either winning set?

- A: Every boolean combination E of sets V_i either includes the intersection of all sets V_i or else is disjoint from it. This can be checked by induction on E.

We prove Theorem 3 by induction on m. The case $m = 1$ is taken care by Theorem 2. It remains to prove the induction step.

Let D be the set of positions x such that Player δ has a winning strategy in the remainder of the game. If the root of A has an ordinal rank with respect to D and to Player δ then Player δ can ensure that a position in D is reached and therefore that he wins the game. Thus, we may suppose that the rank of the root with respect to D and to Player δ is extra ordinal. Obviously, any winning strategy of Player ε is a refinement of the strategy of keeping the D-rank extra ordinal, and therefore we may restrict attention to the subarena of positions reachable when Player ε keeps the D-rank extra ordinal. In other words, without loss of generality, we may suppose that D is empty. In the rest of the proof, all rank function will correspond to Player ε.

- Q: You were right about breaking symmetry. It remains to prove that Player ε has a winning strategy in Γ.

- A: For each V_i, fix a node set S_i such that an arbitrary branch belongs to V_i if and only if it hits S_i infinitely many time. Player ε wins if the play hits every S_i infinitely many times. For example, Player ε may adapt the strategy of hitting sets S_i in the cyclic order. In the beginning, he is in mode 1 of going after S_1. If Player ε is in mode i and hits S_i, then he changes the mode to $i + 1 \bmod m$.

- Q: I do not see how Player ε can be sure that he will eventually hit S_i when he is in mode i.

- A: Let S_i^x be the set of nodes $y \in S_i$ above x. If the rank of x with respect to S_i^x is ordinal, then Player ε may be sure to hit S_i in the remainder of the game. Suppose that Player ε goes to mode i in a position x such that the S_i^x-rank of x is extra ordinal and that, moreover, Player δ will play a refinement of the strategy to keep the S_i^x-rank extra ordinal. What will happen? The strategy of keeping the S_i^x-rank extra ordinal defines a subarena where all branches avoid V_i; it follows that, over the subarena, the winning sets are boolean combinations of $m-1$ Π_2^0 sets. By the induction hypothesis, one of the players has a winning strategy F_x in the remainder of the game. Who can it be?

- Q: Of course, it is Player ε. Otherwise $x \in D$, but D is empty.

- A: This allows us to complete the description of a winning strategy for Player ε. Suppose he goes to mode i in a position x. If the S_i^x-rank of x is ordinal, he decreases the rank until he hits S_i and goes to a new mode. Otherwise Player ε picks the strategy F_x and sticks to it unless Player δ makes a move that results in a position of an ordinal S_i^x-rank. If and when that happens, Player ε decreases the S_i^x-rank until it hits S_i and goes to a new mode. That finishes the proof of Theorem 3.

- Q: I like the mathematics of infinite games. And I trust you that infinite games are of great importance in logic. But what about those applications to concurrent processing? Is it all about the game of an operating system against the user?
- A: Operating systems are but one example. Others are networks and file systems [1].
- Q: It seems to me that shear existence of a winning strategy does not suffice. In the case of, say, operating systems, the system should be able to implement a winning strategy.
- A: You are raising a very interesting issue. I hope we can address it one day.

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