EECS501
Probability and Random Processes
Exam 1
Feb 16, 2009

This 2 hour exam is open book and open notes. Each of the following problems is worth equal credit. Please answer questions as completely as possible to obtain full credit. If you feel that additional assumptions need to be made to answer any part of a question state your assumption explicitly. Please make sure that your name and student id number are on your exam, and if not using a blue book, make sure all of your pages are stapled in the correct order before handing in.

1 Let X and Y be random variables. As you know, we say that X less than Y with probability one (w.p.1) when $P(X < Y) = 1$. On the other hand, we say that X is stochastically less than Y when $P(X > x) \leq P(Y > x)$ for any value of x. In this problem you will prove the following

Thm. $X < Y$ w.p.1 implies that X is stochastically less than Y.

For each of (a)-(c) you are expected to invoke an elementary axiom or property covered in class that is satisfied by a probability measure P. Failure to do so will result in significant loss of credit.

(a) Show that $X < Y$ w.p.1. implies that $P(X \geq x, Y \leq x) = 0$ for any value of x. (Hint: consider the intersection of two regions in the X, Y plane).

`Soln:`

`As shown in class for any events` A `and` B `with` $P(B) = 1$: $P(A, B) = P(A)$ `and` $P(A|B) = P(A)$. `Define the two events` $A = \{X \geq x\} \cup \{Y \leq x\}$ `and` $B = \{X < Y\}$. `These two events are disjoint, i.e.,` $A \cap B = \phi$, `the null event. We conclude from these two facts` $P(A) = P(A|B) = P(A \cap B)/P(B) = 0$, `since, by Axiom i (p. 22 of Gubner)` $P(\phi) = 0$.

(b) Show that (a) implies $P(X < x, Y \leq x) = P(Y \leq x)$ for any value of x.

`Soln:`

`Use the partition property` $P(Y \leq x) = P(X < x, Y \leq x) + P(X \geq x, Y \leq X)$. `But by part (a) the second term on the right is equal to zero which establishes (b).`

(c) Show that (b) implies $P(X > x) \leq P(Y > x)$ for any value of x.

`Soln:`

`From (b) and monotonicity:` $P(Y \leq x) = P(X < x, Y \leq x) \leq P(X < x) \leq P(X \leq x)$. `Since` $P(Y > x) = 1 - P(Y \leq x)$ `and` $P(X > x) = 1 - P(X \leq x)$ `this establishes part (c).`

2 A coin is flipped twice producing possible outcomes of the form HH, HT, TH, TT. The coin has probability p of coming out heads and probability $1 - p$ of coming out tails on any one toss and the coin tosses are statistically independent. A certain observer of this coin flip experiment only has access to partial information about the outcome.

(a) The information available is the total number of heads occurring in the two tosses. Find the conditional probability that the result of the first coin toss was a head given this information. Your answer must treat each case, i.e., no heads, only one head or two heads.

`Soln:`

`We define Bernouilli random variable` X_i `as 1 or 0 depending on whether the outcome of the` i`-th coin toss is a H or a T, respectively. Define` $Y = X_1 + X_2$. `Then` $Y = 0$ `if the outcome is TT,` $Y = 1$ `if the outcome is either TH or HT, and` $Y = 2$ `if the outcome is HH. Since the tosses are statistically independent:` $P(TT) = P(X_1 = 0)P(X_2 = 0) = (1-p)^2$, $P(TH) = P(X_1 = 0)P(X_2 = 1) - (1-p)p$, $P(HT) = P(X_1 =$

$1)P(X_2 = 0) - p(1-p)$, $P(HH) = P(X_1 = 1)P(X_2 = 1) - p^2$. Furthermore, $P(Y = 0) = (1-p)^2$, $P(Y = 1) = 2p(1-p)$, $P(Y = 2) = p^2$ and

$$P(Y = 0|X_1 = x) = \begin{cases} 0, & x = 1 \\ 1-p, & x = 0 \end{cases}$$

$$P(Y = 1|X_1 = x) = \begin{cases} 1-p, & x = 1 \\ p, & x = 0 \end{cases}$$

$$P(Y = 2|X_1 = x) = \begin{cases} p, & x = 1 \\ 0, & x = 0 \end{cases}$$

By Bayes' rule we have the conditional probability

$$P(X_1 = 1|Y = y) = \frac{P(Y = y|X_1 = 1)P(X_1 = 1)}{P(Y = y)}$$

which is

$$P(X_1 = 1|Y = y) = \begin{cases} 0, & y = 0 \\ \frac{(1-p)p}{2p(1-p)} = \frac{1}{2}, & y = 1 \\ \frac{p^2}{p^2} = 1, & y = 2 \end{cases}$$

(b) The information available is that the total number of heads occurring in the two tosses is even. Find the conditional probability that the first coin toss was a head given this information. Repeat when the information available is that the number of heads is odd.

Soln:

Define the Bernouilli random variable Z as '1' if the number of heads is even and '0' otherwise. $Z = 1$ iff the outcome is either HH or TT and $Z = 0$ otherwise. Thus we have $P(Z = 1) = p^2 + (1-p)^2 = 1 - 2p(1-p)$, $P(Z = 0) = 1 - P(Z = 1)$. Furthermore,

$$P(Z = 1|X_1 = x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$

Therefore the conditional probability that the first toss was a head given the total number of heads was even is

$$P(X_1 = 1|Z = 1) = \frac{p^2}{1 - 2p(1-p)}.$$

On the other hand, the conditional probability that the first toss was a head given the total number of heads was odd is

$$P(X_1 = 1|Z = 0) = \frac{(1-p)p}{2(1-p)p} = \frac{1}{2}.$$

(c) Repeat part (a) for the case of n coin flips, i.e. find the conditional probability that the first coin toss is a head given that a total of k heads occured, $0 \le k \le n$. (Hint: define Bernoulli random variables). Comment on the dependence of your expression on p.

Soln:

Define the Bernouilli random variable X_i as above. As shown in class, the sum of n i.i.d. Bernouilli random variables $\sum_{i=1}^n X_i$ is binomial distributed, i.e., with the definition $Y_n = \sum_{i=1}^n X_i$: $P(Y_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$. Now, given

$X_1 = 1$, $Y_n = 1 + \sum_{i=2}^n X_i$ and, by the substitution property and independence of the X_i's, the conditional probability $P(Y_n = k|X_1 = 1)$ is

$$P(Y_n = k|X_1 = 1) = P(\sum_{i=2}^n X_i = k-1|X_1 = 1) = P(\sum_{i=2}^n X_i = k-1) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}.$$

Hence, by Bayes' formula

$$P(X_1 = 1|Y_n = k) = \frac{P(Y_n = k|X_1 = 1)P(X_1 = 1)}{P(Y_n = k)} = \frac{\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}p}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

This probability does not depend on p.

3 Let X and Y be two discrete random variables that have the relationship $Y = X^2/(1+X^2)$.

(a) Give an example of a probability mass function $p_X(x)$ for which the correlation correlation $E[XY]$ is equal to zero but X and Y are dependent random variables.

Soln:

This exercise is virtually identical to exercise 44 in ch 2 of Gubner that we covered in class. As explained in class the key is to choose p_X so that we obtain an even times an odd function in the expectation. Let X have equally likely outcomes over the three values $\{-1, 0, 1\}$, i.e., $p_X(x) = 1/3$. Then

$$E[XY] = E[X^3/(1+X^2)] = (-1/2)\frac{1}{3} + (+1/2)\frac{1}{3} = 0.$$

However, $P(X = 1, Y = 0) = 0$ which is not equal to $P(X = 1)P(Y = 0) = \left(\frac{1}{3}\right)^2$.

(b) Give an example of a probability mass function $p_X(x)$ for which the correlation $E[XY]$ is equal to zero and X and Y are independent random variables (Hint: a probability one event is independent of any other event).

Soln:

Define X as equally likly over the two values $\{-1, 1\}$. Then again $E[XY] = 0$. However, in this case $Y = 1/2$ for all outcomes. Thus $P(X = x, Y = 1/2) = P(X = x) = P(X = x)P(Y = 1/2)$ while $(X = x, Y = 0) = 0 = P(X = x)P(Y = 0)$ so X and Y are in fact independent.

4 A random variable Y is defined as a "randomly censored" mixture of Poisson random variables if it has the representation

$$Y = \sum_{i=1}^n a_i X_i$$

where X_i are independent identically distributed (i.i.d.) Poisson random variables with mean $E[X_i] = \lambda > 0$ and a_i are i.i.d. Bernouilli random variables with $E[a_i] = p$, $p \in [0, 1]$. This model is commonly used to represent the total flux generated by a diffuse field of photons incident on a charge-coupled photodetector surface.

(a) Compute the mean and variance of Y under the assumption that a_i and X_i are independent random variables.

Soln:

Use the linearity property of expectation and the property that $E[g(U)h(V)] = E[g(U)]E[h(V)]$ for independent r.v.s U and V and functions g and h, to obtain

$$E[Y] = \sum_{i=1}^n E[a_i X_i] = \sum_{i=1}^n E[a_i]E[X_i] = np\lambda.$$

Next use the independence over i of the summand to obtain:

$$\text{var}(Y) = \sum_{i=1}^{n} \text{var}(a_i X_i) = n\text{var}(a_i X_i)$$

and

$$\text{var}(a_i X_i) = E[(a_i X_i)^2] - E^2[a_i X_i] = E[a_i^2]E[X_i^2] - E^2[a_i]E^2[X_i] = pE[X_i^2] - (p)^2(\lambda)^2$$

since $E[a_i^2] = E[a_i] = p$. With $E[X_i^2] = \lambda^2 + \lambda$ (we did this in class for a Poisson rv X) we obtain

$$\text{var}(Y) = n[p\lambda^2 + p\lambda - (p\lambda)^2]$$

(b) Prove the following inequality

$$E\left[\frac{1}{1+Y}\right] \geq (1 + np\lambda)^{-1}.$$

Soln:

Since the second derivative of the function $g(y) = 1/(1+y)$ is positive for all $y \geq 0$, $g(y)$ is convex for $y \geq 0$. Therefore, invoking Jensen's inequality $E[g(Y)] \geq g(E[Y])$, the answer follws directly from part (a)..

(c) Find an expression for the probability generating function (PGF) of Y.

Soln:

The PGF is defined as $G_Y(z) = E[z^Y]$ so, by the fact that the a_i and X_i are i.i.d., $i = 1, \ldots, n$

$$G_Y(z) = E\left[\prod_{i=1}^{n} z^{a_i X_i}\right] = \prod_{i=1}^{n} E\left[z^{a_i X_i}\right] = \left(E\left[z^{a_i X_i}\right]\right)^n$$

Now by definition of the expectation, $E\left[z^{a_i X_i}\right]$ is

$$E\left[z^{a_i X_i}\right] = \sum_{a,x} z^{ax} p_{X,a}(x, a)$$

where $p_{X,a}(x, a)$ is the joint pmf of X_i and a_i. Therefore,

$$E\left[z^{a_i X_i}\right] = P(a = 0)\sum_x p_{X|a}(x|0) + P(a = 1)\sum_x z^x p_{X|a}(x|1).$$

which, again by independence of X_i and a_i, is

$$E\left[z^{a_i X_i}\right] = P(a = 0)\sum_x p_X(x) + P(a = 1)\sum_x z^x p_X(x)$$

The first term on the right is simply $P(a = 1) = 1-p$ since p_X is a pmf. The second term is $P(a = 1)G_X(z) = pe^{\lambda(z-1)}$. We conclude that

$$G_Y(z) = \left((1 - p) + pe^{\lambda(z-1)}\right)^n.$$

(d) Using the PGF that you have derived, find a series expression for the probability mass function $p_Y(y)$. Verify that your expression is a valid pmf.

Soln:

First represent G_Y using the Binomial expansion

$$G_Y(z) = \sum_{m=0}^{n} \binom{n}{m} \left(pe^{\lambda(z-1)}\right)^m (1 - p)^{n-m}$$

Next use the inversion property $p_Y(k) = \frac{1}{k!}G_Y^{(k)}(0)$ and term-by-term differentiation wrt z to obtain

$$G_Y^{(k)}(z) = \sum_{m=0}^{n} \binom{n}{m} (\lambda m)^k \left(pe^{\lambda(z-1)}\right)^m (1-p)^{n-m}$$

so that

$$p_Y(k) = \frac{1}{k!}G_Y^{(k)}(0) = \frac{1}{k!}\sum_{m=0}^{n} \binom{n}{m} (\lambda m)^k p^m e^{-\lambda m}(1-p)^{n-m}$$

Next verify that this sums to one

$$\begin{aligned}\sum_{k=0}^{\infty} p_Y(k) &= \sum_{m=0}^{n} \binom{n}{m} \left[\sum_{k=0}^{\infty} \frac{(\lambda m)^k}{k!}\right] p^m e^{-\lambda m}(1-p)^{n-m} \\ &= \sum_{m=0}^{n} \binom{n}{m} p^m (1-p)^{n-m} = (p+1-p)^n = 1\end{aligned}$$

where we use the fact that

$$\left[\sum_{k=0}^{\infty} \frac{(\lambda m)^k}{k!}\right] = e^{\lambda m}$$