CHAPTER 4

LEVEL CROSSING REPRESENTATIONS, POISSON ASYMPTOTICS
AND APPLICATIONS TO PASSIVE ARRAYS

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I. INTRODUCTION

In this chapter we present representation theorems on the distribution of level crossing statistics for general non-stationary random processes and then discuss applications to passive arrays in underwater acoustics. Along with some regularity conditions, a consequence of the first representation theorem is an asymptotic result relating the distribution of the level crossings to an inhomogeneous Poisson law. This asymptotic result is then used to model the occurrence of large error in time difference of arrival estimates across a passive sensor array.

The distribution function of the number of crossings of a level by a random process in a given interval is an important, but difficult to obtain, function which has received considerable attention in recent years [1]. Explicit results are known only for a handful of specific random processes, e.g. the Markov class. Here a general representation for the probability of getting one or more level crossings is presented for a wide class of random processes, which allows the deviation of this probability from the probability of getting one or more points from an inhomogeneous Poisson process to be characterized. This representation is then exploited to obtain a generalization of asymptotic results of Leadbetter [12] for stationary random processes to the non-stationary case. The most familiar non-precise statement of Leadbetters' result is as follows. Define a suitable sequence of increasingly high levels $\{l_m\}$. Given a stationary, almost surely continuous random process X which satisfies some regularity conditions, the counting process N^m associated with the crossings by X of the increasingly high level l_m, behaves increasingly like a (homogeneous) Poisson process. An equivalent interpretation: if μ_m and σ_m^2 are the mean and variance of X_m, which is one in a sequence of processes $\{X_m\}$, the Poisson character of its zero upcrossings, N^m, is assured as $\mu_m / \sigma_m \to \pm\infty$.

Here we establish an analogous, but inhomogeneous, Poisson character for non-stationary processes satisfying some additional conditions to those in [12]. Specifically, we assume certain asymptotic conditions on the trajectories of the process such as mixing. It is demonstrated that if the upcrossings of zero are made to become progressively rarer events, in a sense to be made clear later, then a normalized version of the number of upcrossings as a function of time converges in distribution to a Poisson process.

Results of the above type are of interest in connection with maximum likelihood parameter estimation when large errors may be significant. As is well known, the Cramer-Rao-Lower-Bound only characterizes local error; that is, when the estimate is in the immediate vicinity of the true parameter [4]. When the trajectory of the so called "likelihood function" is predisposed to display large multiple local maxima over the parameter space, or peak ambiguities, an additional large error measure is needed. One possible choice is the probability that a section of the trajectory exceeds a threshold, specified by the height of the trajectory at the true parameter. This probability can then be expressed within the framework of level crossing probabilities.

For time delay estimation in a two sensor array, the maximum likelihood estimate is approximately given by the location over time where the (non-stationary) cross-correlation function takes on its global peak. Motivated by the asymptotic result described in the first part of this chapter, a Poisson model for the large errors is applied to derive an approximate expression for the global variance of the correlator estimate. This expression is shown to be an approximate upper bound on the variance, which complements previously derived lower bounds for the time delay estimation problem, e.g. Cramer-Rao [10], Barankin [2], Ziv-Zakai [20] and others [17].

Section II and III present the relevant theory concerning point process representations and asymptotic theory. Section IV contains a discussion of the application of the Poisson model to peak ambiguity in two-sensor arrays. Finally in Section V we compare the Poisson variance approximation to the Ziv-Zakai-Lower-Bound (ZZLB), the Cramer-Rao-Lower-Bound (CRLB) and experimental results.

II: A REPRESENTATION FOR THE PROBABILITY OF LEVEL CROSSINGS

Let (Λ, Φ, P) be a complete probability space and define the nested sequence of σ-fields : Φ_t, $t \in R$, ($\Phi_t \subset \Phi$, $\Phi_s \subset \Phi_u$ for s < u). We assume the Φ_t-measurable random process $X(t)$, $-\infty < t < +\infty$, to have the following properties: separability, almost sure (a.s.) sample function continuity, existence of the bivariate densities, $f_{t,\tau}(y,z)$, of $X(t)$ and $X(\tau)$, for $t \neq \tau$. Additional properties will be imposed shortly. For expository convenience we focus on the relevant level crossing theory for upcrossings. However, the consideration of downcrossings is completely analogous to the presentation to follow.

We define an upcrossing analogously to Leadbetter in [11]. A realization of $X(t)$, $x(t)$, upcrosses zero in the interval $[\sigma,\nu)$ if there exists an open interval centered at some point

$t_u \in (\sigma,\nu)$, $(t_u - \delta, t_u + \delta)$ say, over which $X(t) < 0$ to the left of t_u and $X(t) > 0$ to the right of t_u. We denote this occurrence by the notation $A_{\sigma,\nu}$. Symbolically

$$A_{\sigma,\nu} \equiv \{w \in \Lambda | \exists\ t_u \in (\sigma, \nu), \exists\ \delta > 0\ ;\ s.t.\ X_{t_u - h} < 0 < X_{t_u + h}, \text{for } 0 < h < \delta\} \tag{1}$$

This definition essentially excludes any "non-regular" upcrossings such as tangencies or non-smooth intersections of zero. It is shown in [11] that $A_{\sigma,\nu}$ is Φ_ν-measurable and that non-regular upcrossings are of zero probability. Hence definition (1) is sufficiently general for our purposes. The definition of downcrossings is analogous to (1).

We further define the number of upcrossings by $X(\tau)$ in $[\sigma, \nu)$, denoted $N(\sigma, \nu)$, as the number of distinct points t_u at which upcrossings occur. More specifically we define: $N(\sigma,\nu) = \underset{\Delta \to \infty}{limsup} \sum_{t_i} I(A_{t_i, t_i + \Delta})$. Here $I(A)$ is the indicator of the event $t_u \in A$ and $\{t_i\}$ is an increasingly dense partition of (σ,ν) with inter grid spacing Δ.

By the assumed continuity of the random process $X(t)$ it is reasonable to expect that it can be well approximated by a piecewise linear process tied to $X(t)$ at a sufficiently dense set of points $t = t_o, t_1, \ldots, t_M$. That is, let $\xi_n(t)$ denote a random process defined on an interval $[t_o, t_f]$ for which

$$\xi_n(t) = \begin{cases} X(t) & t = t_k \\ X(t_k) + \dfrac{[X(t_{k+1}) - X(t_k)]}{(t_f - t_o)2^{-n}}\ (t - t_k) & t_k < t < t_{k+1} \end{cases} \tag{2}$$

$$t_k = t_o + k 2^{-n}(t_f - t_o),\ k = 0, 1, 2 \cdots, 2^n$$

If N_n is the number of upcrossings of zero by $\xi_n(t)$ then the following is due to Ylvisaker [21].

Lemma 1.1

Let $N_n(t)$ be the number of upcrossings of zero by ξ_n in the interval $[t_o, t)$. Then $N_n(t)$ is monotonically non-decreasing in n and converges to $N(t)$, the number of up-crossings of zero by X in the same interval, with probability one as $n \to \infty$.

From the above lemma it follows by monotone convergence that

$$P(N(t) \leq k) = \lim_{n \to \infty} P(N_n(t) \leq k),\ \text{for } k = 0,1,2, \tag{3}$$

Hence, as far as the computation of upcrossing probabilities is concerned, $\xi_n(t)$ and $X(t)$ can be used interchangeably in the sense of (3).

The following will be important for the upcoming development and are essentially Theorems (2.1) and (2.2) of [13].

Lemma 1.2

Let $[t_o, t_f]$ have the partition $\{t_i\}_{i=0}^{2^n}$. Then with $N(t)$ the number of upcrossings (downcrossings) of zero by X in $[t_o, t)$ and $N_n(\tau, \sigma)$ the number of upcrossings (downcrossings) by ξ_n in $[\tau, \sigma)$

$$\mathbf{E}[N(t_f)] = \lim_{n \to \infty} \sum_{i=0}^{2^n-1} P(A_{t_i, t_{i+1}}) = \lim_{n \to \infty} \sum_{i=0}^{2^n-1} P(N_n(t_i, t_{i+1}) > 0) \tag{4}$$

Define $g_{t,\tau}(y,z)$ the joint density of $X(t)$ and $[X(t+\tau) - X(t)]/\tau$. Then by elementary transformations

$$g_{t,\tau}(y,z) = \tau f_{t,t+\tau}(y, y + \tau z) \tag{5}$$

The following are essential to the development and are known as Leadbetter's conditions [11, Thm. 2]

$$g_{t,\tau}(y,z) \quad \text{is continuous in } t, y \text{ for each } \tau, z \tag{6}$$

$$g_{t,\tau}(y,z) \to p_t(y,z) \quad \text{as } \tau \to 0 \quad \text{uniformly in } t, y \tag{7}$$

$$g_{t,\tau}(y,z) \le l(z) \quad \text{for all } t, \tau, y, z \tag{8}$$

where

$$\int_{-\infty}^{\infty} |z| \; l(z) dz < \infty$$

If the above three conditions are satisfied then the following representation theorem holds for the probability of getting an upcrossing in $[t_o, t)$, $P(A_{t_o, t})$, here denoted $p(t)$.

Theorem 1.1

Suppose $X(\tau)$ has continuous sample functions with probability one and let the conditions (6) through (8) hold. Then the expected value of the number of upcrossings of zero by $X(\tau)$ in any finite interval $[t_o, t_f)$ is finite and given by

$$\mathbf{E}[N(t_o, t_f)] = \int_{t_o}^{t_f} p(\tau) d\tau \tag{9}$$

$$\rho(\tau) = \int\limits_{0}^{\infty} z \ p_\tau(0,z)dz$$

Furthermore the probability of getting at least one upcrossing of zero by $X(\tau)$ in $[t_o,t)$, $p(t)$, satisfies the relation

$$p(t) = \int\limits_{t_o}^{t} \rho(\tau)(1 - p(\tau))d\tau + Q(t) \tag{10}$$

where

$$Q(t) = \lim_{n \to \infty} \sum_{i=0}^{2^n-1} q(t_i) \tag{11}$$

$$q(t_i) = P(N_n(t_i,t_{i+1})>0, \ N_n(t_i)=0) - P(N_n(t_i,t_{i+1})>0)P(N_n(t_i)=0) \tag{12}$$

Here $\{t_i\}_{i=0}^{2^n}$ is a partition of $[t_o,t]$

The probability of downcrossings satisfies an analogous Theorem, the only difference being the particular form of the intensity ρ in (9). For downcrossings $\rho(\tau) \overset{\Delta}{=} \int\limits_{-\infty}^{0} |z| \ p_\tau(0,z)dz$.

The function $\rho(\tau)$, given as the derivative of (9), is the incremental average number of level crossings per unit time at time τ, $\rho(\tau)d\tau = \mathbf{E}\{dN(\tau)\}$. In the theory of point processes ρ is called the (incomplete) intensity function of the point process N. $Q(\tau)$ in (12) can be interpreted as a measure of the dependency structure of the upcrossing process N over disjoint intervals (for independent increment processes Q would be zero).

Eq. (9) of Theorem 1.1 is obtained directly by modifying the proof of Theorem 2 of Leadbetter for downcrossings [11] to the case of upcrossings. The proof of the rest of Theorem 1.1 depends on a particular decomposition of the event that an upcrossing of zero by ξ_n occurs on $[t_o,t)$, which we denote $B_{t_o,t}$. If $N_n(t)$ is finite we can define $B_{\sigma,\nu}^1$: the event that the first instance of an upcrossing occurs in the subinterval $[\sigma,\nu)$ of $[t_o,t)$. That is

$$B_{\sigma,\nu}^1 \equiv B_{\sigma,\nu} \cap \bar{B}_{t_o,\sigma} \tag{13}$$

where we read this as: ξ_n first upcrosses in $[\sigma,\nu)$ if there is an upcrossing in $[\sigma,\nu)$ but none in $[t_o,\sigma)$.

We note the following two rather obvious properties of $B_{\sigma,\nu}^1$.

For $[\sigma,\nu)$ and $[s,t)$ disjoint

$$B_{\sigma,\nu}^1 \quad \text{and} \quad B_{s,t}^1 \quad \text{are disjoint} \tag{14}$$

and

$$B_{t_o,t}^1 \equiv B_{t_o,t} \quad , \ t \in [t_o,t) \tag{15}$$

Eqs. (14) and (15) follow directly from the definition (13). The following proposition is central to the decomposition alluded to above.

Proposition 1.1

Let $A_{\sigma,\nu}$ denote an upcrossing of zero, and $A_{\sigma,\nu}^1$ the first instance of an upcrossing, by a random process $X(t)$ in $[\sigma,\nu)$, where $X(t)$ has absolutely continuous distributions. Then if the expected number of upcrossings of zero in $[t_o,t)$, $\mathbf{E}\{N(t_o,t)\}$, is finite the following equivalence holds with probability one

$$A_{t_o,t} \equiv \bigcup_{i=0}^{2^n-1} A_{t_i,t_{i+1}}^1 \tag{16}$$

where $\{t_i\}_{i=0}^{2^n}$ is a partition of $[t_o,t]$

■

Proof

Note that the number of upcrossings in $[t_o,t)$ is finite with probability one since

$$P(N(t_o,t) > k) \leq \sum_{i=k+1}^{\infty} P(N(t_o,t) = i) \tag{17}$$

$$\leq \sum_{i=k+1}^{\infty} iP(N(t_o,t) = i)$$

which must converge to zero as $k \to \infty$ by the finiteness of the mean number of upcrossings. Thus a "first instance of an upcrossing" is well defined. The inclusion "$\supset$" in (16) is trivial since any upcrossing in a subinterval of $[t_o,t)$ implies an upcrossing occurred in the entire interval. As for "$\subset$", if there is an upcrossing in $[t_o,t)$ it is either interior to one of the $[t_i,t_{i+1})$ or at one of the endpoints t_i , $i = 0, 1, \ldots, 2^n-1$ However from the absolute continuity of the distribution of $X(t)$, with respect to Lebesgue measure, this latter event has probability zero. Therefore the proposition follows.

■

Proof of Theorem 1.1

Partition $[t_o,t)$ into 2^n-1 subintervals of length $\Delta = (t_f - t_o)2^{-n}$ for $n = 0,1,\cdots$. That is we have intervals $[t_i,t_{i+1})$ with $t_i = t_o + i\Delta$, $i = 0,1,\ldots,2^n$. Define $B_{\sigma,\nu}$

$\sigma,\nu \in \{t_i\}_{i=1}^{2^n}$, the event that the polygonal approximation, ξ_n upcrosses zero in $[\sigma,\nu)$, i.e. $N_n(\sigma,\nu) > 0$. Then from Proposition 1 and Eq. (15) for $k = 2^n$ $(t_k = t)$

$$P(B_{t_o,t_i}) = P\left(\bigcup_{i=0}^{2^n-1} B_{t_i,t_{i+1}} \cap \overline{B}_{t_o,t_i}\right) = \sum_{i=0}^{2^n-1} P(B_{t_i,t_{i+1}} \cap \overline{B}_{t_o,t_i}) \tag{18}$$

Here we have used the disjointness property (14). Now add and subtract the product $P(B_{t_i,t_{i+1}})P(\overline{B}_{t_o,t_i})$ from each term under the sum (18)

$$P(B_{t_o,t}) = \sum_{i=0}^{2^n-1} [P(B_{t_i,t_{i+1}})P(\overline{B}_{t_i,t_i}) + q(t_i)] \tag{19}$$

where

$$q(t_i) = P(B_{t_i,t_{i+1}} \cap \overline{B}_{t_o,t_i}) - P(B_{t_i,t_{i+1}})P(\overline{B}_{t_o,t_i}) \tag{20}$$

as in the statement of the Theorem, Eq. (12).

$B_{t_i,t_{i+1}}$ is equivalent to the event

$$B_{t_i,t_{i+1}} \equiv \{\xi_n(t_i) < 0 < \xi_n(t_{i+1})\} \tag{21}$$

Define

$$\eta_n(t_i) = \frac{\xi_n(t_{i+1}) - \xi_n(t_i)}{\Delta} \tag{22}$$

Combining Eqs. (21) and (22)

$$B_{t_i,t_{i+1}} = \{\xi_n(t_i) < 0 < \xi_n(t_i) + \Delta\eta_n(t_i)\} \tag{23}$$

$$= \{\xi_n(t_i) \in (-\Delta z, 0), \; \eta_n(t_i) = z > 0\}$$

Therefore by the definition of the joint density, $g_{t_i,\delta}$, of $\xi_n(t_i)$ and $\eta_n(t_i)$

$$P(B_{t_i,t_{i+1}}) = \int_0^\infty dz \int_{-\Delta z}^0 g_{t_i,\Delta}(x,z)dx \tag{24}$$

Now make a change of variable in the argument x of (24) and substitute the result back into Eq. (19) to obtain

$$P(B_{t_o,t}) = \sum_{i=0}^{2^n-1} [\Delta \int_0^\infty dz \int_{-z}^0 g_{t_i,\Delta}(\Delta x, z)P(\overline{B}_{t_o,t_i})dx + q(t_i)] \tag{25}$$

By the pointwise continuity and uniform convergence conditions, (6) and (7), for Δ sufficiently

small

$$\int\limits_{-z}^{0} g_{t_i,\Delta}(\Delta x, z)dx = z \ \mathrm{p}_\tau(0,z) \ , \qquad \tau \in [t_i, t_{i+1}] \tag{26}$$

Condition (8) asserts that $zl(z)$ is integrable over the positive real line where $l(z)$ upper bounds $g_{t,\tau}(y,z)$. Therefore the limit of (26) as $\Delta \to 0$ is bounded except possibly on some set of measure zero. From Lemma 1.1 and Eq. (3) $P(\overline{B}_{t_o,t_i})$ converges to $P(\overline{A}_{t_o,t_i}) = 1 - p(t_i)$. Defining

$$a_\Delta(t_i, z) = \int\limits_{-z}^{0} g_{t_i,\Delta}(\Delta x, z)P(\overline{B}_{t_o,t_i})dx \tag{27}$$

we have as Δ goes to zero

$$a_\Delta(t_i, z) \to z \ \mathrm{p}_{t_i}(0,z)(1 - p(t_i)) \qquad a.e. \tag{28}$$

and

$$a_\Delta(t_i, z) \le zl(z) \tag{29}$$

Hence by dominated convergence the first term of the expression (19) becomes in the limit

$$\lim_{n \to \infty} \sum_{i=0}^{2^n-1} P(B_{t_i,t_{i+1}})P(\overline{B}_{t_o,t_i}) \tag{30}$$

$$= \lim_{n \to \infty} \Delta \sum_{i=0}^{2^n-1} \int\limits_{0}^{\infty} dz \int\limits_{-z}^{0} g_{t_i,\Delta}(\Delta x, z)P(\overline{B}_{t_o,t_i})$$

$$= \lim_{n \to \infty} \int\limits_{t_o}^{t} d\tau \int\limits_{0}^{\infty} dz \int\limits_{-z}^{0} dx \ g_{\tau,\Delta}(\Delta x, z)P(\overline{B}_{t_o,\tau})$$

$$= \int\limits_{t_o}^{t} d\tau \int\limits_{0}^{\infty} z \ \mathrm{p}_\tau(0,z)(1 - p(\tau))dz < \infty$$

This is the first additive term in Eq. (10).

From the expression (20)

$$-P(B_{t_i,t_{i+1}})P(\overline{B}_{t_o,t_i}) \le q(t_i) \le P(B_{t_i,t_{i+1}})P(\overline{B}_{t_o,t_i}) \tag{31}$$

so that the $q(t_i)$ are absolutely summable by Lemma 1.2 and the finiteness of $\mathbf{E}[N(t_o,t_f)]$. Finally Theorem 1.1 follows by performing the limiting operation in (16) as $n \to \infty$, taking account of the regularity conditions shown above. ∎

III. ASYMPTOTIC RESULTS

Theorem 1.1 gives an implicit relation for the probability of getting an upcrossing in a bounded interval. Although the intensity function ρ may be known, in general the $Q(t)$ term in Eq. (11) involves quantities which are not known. On the other hand Eq. (11) can be used to prove certain asymptotic results for a fairly general class of upcrossing processes, which we will now undertake to show. To motivate these results the following argument is useful. Referring to Eq. (11), assume that N^m converges to an independent increment point process $\widetilde{N}$ as $m \to \infty$. Then $q^m(t_i)$ converges to zero for all i and by Eq. (31), Lemma 1.2 and the finiteness of the mean number of upcrossings, $q^m(t_i)$ is summable over i as the t_i become dense in $[t_o, t)$. Dominated convergence then assures that $Q(t) = 0$ and Eq. (11) becomes equivalent to a linear first order, homogeneous differential equation with coefficient $\rho(\tau)$ and initial condition $p(t_o) = 0$. Eq. (11) then has the solution

$$p(t) = 1 - exp\left(-\int_{t_o}^{t} \rho(\tau)d\,\tau\right) \tag{32}$$

Eq. (32) is of course valid for any semiclosed subinterval of $[t_o, t)$. Hence, by the independence of N over disjoint intervals, the upcrossing process must actually be an inhomogeneous Poisson process with intensity ρ.

Unfortunately the above argument is fallacious since, roughly speaking, for non-zero N on bounded intervals, the independent increment property of N is incompatible with the sample function continuity of X so that Theorem 1.1 does not even apply. Clearly the pointwise convergence of N^m to an independent increment process $\widetilde{N}$ is an overly strong imposition on X. However in the following it will be shown that for a sequence of "thinned out" upcrossing count processes, $N^m(t_o, t)$ associated with X, a related (time normalized) random counting process can be defined which converges in distribution to a Poisson random process defined on the interval $[0,1)$ as $m \to \infty$. These results will depend on additional assumptions, such as mixing, on the distributions of X.

The basic idea is as follows. Let $X_0 \triangleq \{X(t): t \in [0,1]\}$ be a given random process with upcrossing intensity $\{\rho(t): t \in [0,1]\}$. Define a sequence of increasingly long time intervals, I_m, of length $T_m > 1$, $I_m \triangleq [0, T_m]$. On the interval I_m let X_m be a random process with an upcrossing intensity function $\{\rho^m(t): t \in [0, T_m]\}$ where ρ^m is related to ρ by: $\rho^m(t) \triangleq \frac{1}{T_m} \rho(\frac{t}{T_m})$, $t \in [0, T_m]$. Note, the average number of upcrossings by $\{X(t): t \in [0,1]\}$ is over $[0,1]$ is identical to the average number by $\{X_m(t): t \in [0, T_m]\}$ over $[0, T_m]$, while the intensity ρ^m is a stretched and downscaled (thinned) version of ρ. In this way the upcrossings by X_m differ from those of X only in that the average inter-event spacing has been uniformly increased, i.e. upcrossings by X_m become "rare events" over time. As we increase T_m out to infinity, the upcrossings will become approximately independent since, with probability close to one, the events are separated in time by an amount exceeding the "inter-

dependence time" (correlation time for Gaussian case) of X_m which can be specified by a mixing condition. Then, with the aid of some additional regularity conditions, Theorem 1.1 can be used to give the solution (32).

For simplicity, and without restricting the generality of the results, we set t_o in Eq. (32) to zero. In general, when there is multiple indexing, subscripts indicate indexing with respect to the partition, $\{t_i\}$, of the time interval under consideration and superscripts index the quantity with respect to the infinite sequences $\{X_m\}$ and $\{I_m\}$. Thus $N_n^m(t)$ denotes the number of upcrossings of zero by the polygonal approximation to X_m, ξ_n^m, over the interval $[0,t)$, $t \in I_m$. Likewise N^m is the number of upcrossings associated with X_m itself. Analogously to the development of Theorem 1.1 define $\Phi_{\sigma,\nu}^m$, the σ-field generated by X_m on $[\sigma,\nu)$; B_{t_i,t_j}^m , the event $N_n^m(t_i,t_j) > 0$, where t_i and t_j are points contained in the 2^n-th order partition of I_m ; and $p_m(t)$, the probability that X_m upcrosses zero on $[0,t) \subseteq [0,T_m)$.

Throughout the sequel of this section, we assume the intensity associated with N^m , ρ^m exists for all m and is defined in terms of the intensity associated with N^0, ρ, as follows

$$\rho^m(\tau) \triangleq \frac{1}{T_m} \ \rho(\frac{\tau}{T_m} \ ) , \quad m = 0, 1, \cdots \tag{33}$$

The next section is concerned with the various conditions which will be imposed on X_m for asymptotic independence of widely separated segments of the trajectory and Poisson-like behavior of the upcrossings. While not necessarily the most compact set of sufficient conditions, the following contribute to a clear and simple proof of the asymptotic theorem. Several comments will be made concerning simpler sufficient conditions during the discussion.

Asymptotic Conditions:

A mixing condition is a statement concerning the asymptotic independence of the trajectories of a random process on disjoint intervals $[\sigma,\nu)$ and $[s,\tau)$ as $|s - \nu| \to \infty$. For example X is "strong mixing" [16] if

$$\sup_\tau |P(A \cap B) - P(A)P(B)| < \beta_l \tag{34}$$

where A and B are arbitrary events

$$A \in \Phi_{\tau,\infty} , \quad B \in \Phi_{-\infty,\tau-l}$$

and

$$\lim_{l \to \infty} \beta_l = 0$$

The major weakness of "strong mixing" is that (34) becomes vacuous if either A or B are of vanishingly small probability. Indeed in the present context the event A will be contained in the event that an upcrossing of zero occurs in an exceedingly small interval, which of course has exceedingly small probability. The needed condition here is the summability to zero of the

differences below

$$\lim_{m,n \to \infty} \sum_{i=0}^{2^n-1} |P(A_i^m \cap B_i^m) - P(A_i^m)P(B_i^m)| = 0 \tag{35}$$

$$A_i^m \in \Phi_{-\infty,t_i-l_m}^m \ , \quad B_i^m \in \Phi_{t_i,t_{i+1}}^m$$

$$l_m \to \infty \ , \ l_m = o(T_m) \ as \ n,m \to \infty$$

$$\{t_i\}_{i=0}^{2^n} \ , \quad an \ increasingly \ dense \ partition \ of \ [0,T_m)$$

A sufficient condition for (35), if the quantities $P(B_i^m)$ are summable over i, is the following form of so called "uniform mixing" [16].

Mixing Condition

With $\Phi_{\sigma,\nu}^m$ the σ-field generated by the trajectories of X_m in $[\sigma,\nu)$, T_m a monotonic sequence increasing to infinity, X_m is said to be uniform-asymptotically mixing (u-a mixing) if

$$|P(A_i^m) - P(A_i^m|B_i^m)| < \alpha_{m,l_m} \tag{36}$$

where

$$A_i^m \in \Phi_{-T_m,t_i-l_m}^m \ , \quad B_i^m \in \Phi_{t_i,t_{i+1}}^m$$

and

$$\lim_{m \to \infty} \alpha_{m,l_m} = 0$$

with

$$l_m \to \infty, \ l_m = o(T_m) \ , \ as \ n,m \to \infty$$

$$\{t_i\}_{i=0}^{2^n} \ , \quad an \ increasingly \ dense \ partition \ of \ [0,T_m)$$

Note that for a dense partition $\{t_i\}$ the conditioning in (36) will be on the zero probability event $X(\tau) = 0$ at some specific point τ, viewed as a limit through a horizontal window [8]. Thus in the limit of dense partitions, although the conditional probability may not be well defined in the conventional sense, (36) is well defined. We state the following lemma [5] which generalizes the uniform mixing condition to multiple events.

Lemma 2.1

Assume that $X_m(t)$ is uniform mixing in the sense of (36). Fix $l > 0$ and for $r > 1$ let $E_1, E_2, \ldots, E_r$ be disjoint intervals indexed in increasing order, that is, $\sup\{\tau \in E_{i-1}\} < \inf\{\tau \in E_i\}$ for $i = 1, 2, \ldots, r$, and separated by at least l. Then

for $A_i^m \subset \Phi_{E_i}^m$

$$|P(\bigcap_{i=1}^{r} A_i^m) - \prod_{i=1}^{r} P(A_i^m)| < \alpha_{m,l_m} \sum_{i=2}^{r} P(A_i^m) \qquad (37)$$

For Gaussian processes X_m a sufficient condition for mixing is a rate of decay on its auto-correlation function: $R_{X_m}(t,t+l_m)\log l_m = \alpha_m(t) \to 0$ as $m,l_m \to \infty$ uniformly in t [14]. In order to make the upcrossings exceedingly rare events as $m \to \infty$ the following "rarefaction" condition is used

Rarefaction Condition

With $N_n^m(\sigma,\nu)$ the upcrossings of zero by the polygonal approximation ξ_n^m in $[\sigma,\nu) \subset [0,T_m)$ N_n^m satisfies a rarefaction condition if for $l_m \to \infty$, $l_m = o(T_m)$

$$\lim_{n,m \to \infty} \sum_{t_i=l_m}^{T_m} P(N_n^m(t_i,t_{i+1}) > 0, N_n^m(t_i-l_m,t_i) > 0) = 0 \qquad (38)$$

$\{t_i\}_{i=0}^{2^i}$, *an increasingly dense partition of* $[0,T_m)$

The above condition is a strong condition on the trajectories similar to, but possibly more restrictive than, the condition D_c' used in [12] for the stationary case. Eqn. (38) guarantees that the probability of more than a single level crossing over any $o(T_m)$ interval be exceedingly small as $m \to \infty$. The condition (38) is somewhat stronger than the property of a-*regularity* for $a = 2$ (see Lemma 2.3). (35) can be shown to hold if the hazard function,

$$h^m(u,\tau) \triangleq \lim_{h \to 0} \frac{1}{h} P(N^m(u,u+\tau) = 0 \mid N^m(u-h,u) > 0) , \quad \tau \geq 0 \qquad (39)$$

satisfies $1 - h^m(u,l_m) = o(\frac{1}{T_m})$ for all $u \in [0,T_m-l_m]$.

An additional condition needed is the following which is analogous to condition (4.6) in [12]

$$\frac{P(N_n^m(t,t+h) > 0)}{E[N_n^m(t,t+h)]} \to 1 \quad as \ n,m \to \infty \qquad (40)$$

$\{t_i\}_{i=0}^{2^i}$, $n = n(m)$, *an increasingly dense partition of* $[0,T_m)$

for some h_o , $0 < h < h_o$ and for all $t \in [0,T_m)$.

Condition (40) is stronger than a well known necessary condition for a process to be (asymptotically) Poisson: for infinitesimal intervals the probability of getting a point is

proportional to the expected number of points in the interval (linear in the length of the interval for stationary processes). The condition can be interpreted as an extension of this necessary condition to certain finite intervals.

We state here two easily verifiable conditions on the average number of crossings by X_0 over $I_0 = [0,1)$, $\mathbf{E}\{N(1)\}$, which are particular to the nonstationary situation. These properties guarantee that the behavior of the upcrossing process N^m be sufficiently uniform over time to exclude degeneration of the upcrossing probabilities to either probability 0 or probability 1 events over any $o(T_m)$ interval.

Uniform Denseness Condition

Let N be the number of upcrossings by X_0 on $I_0 = [0,1)$. Choose an interval A, a subset of $[0,1)$. The uniform denseness condition is satisfied if for any ϵ, K, $\epsilon > 0$, $1 < K < \infty$, there exist K subintervals of A, $\{J_i\}_{i=1}^{i=K}$, whose closures are disjoint, such that

$$|\mathbf{E}[N(J_i)] - \mathbf{E}[N(J_l)]| < \epsilon, \qquad i \neq l, \, i,l = 1, \ldots, K \tag{41}$$

Asymptotic Uniform Negligibility

Let N be as in the condition above and let $\{\tau_i\}_{i=0}^{K}$ be a uniformly spaced partition of $I_0 = [0,1)$ of size K. Then with $N_{\tau_k} = N(\tau_k, \tau_{k+1})$, the number of upcrossings within the k-th partition element, the uniform negligibility condition is satisfied if for all $l = 1, \ldots, K$

$$\lim_{K \to \infty} \frac{\mathbf{E}[N_{\tau_l}]}{\sum\limits_{k=1}^{K} \mathbf{E}[N_{\tau_k}]} = 0 \tag{42}$$

Loosely speaking (41) implies that the upcrossings are lean enough so that "similar" intervals, of similar order with respect to I_0, have associated with them a "similar" expected number of upcrossings. This will imply a continuity property on $P(N^m(\tau,\sigma) > 0)$ viewed as a function from the sets $[\tau,\sigma)$. Condition (42) guarantees that in no case will the total number of upcrossings over I_0 be dominated by upcrossings in small subintervals of I_0. The reader may be interested in the similarity between Asymptotic Uniform Negligibility and Feller's sufficient condition for a non-stationary Central Limit Theorem [22]. If the process X_0 were stationary, these two conditions, (41) and (42), would be trivially satisfied since the expected values of $N(J)$ and $N(I)$ are identical if J and I are intervals of equal length. For non-stationary X_0, a sufficient condition for (41) and (42) is that the (incomplete) intensity, $\rho = \rho_0$, satisfy $0 < \rho < M$ for some finite M.

Main Theorem

With the above conditions we are prepared to state the main result concerning the convergence of a certain normalized upcrossing process, associated with $X_m(t)$, to a Poisson process.

Theorem 2.1

Let the a.s. continuous processes $X_m(t)$ have absolutely continuous distributions for $m = 1, \cdots$. Assume each X_m satisfies Leadbetter's conditions (6) through (8), with, in addition, $l_m(z) = O(T_m^{-1})$ in (8), u-a mixing of the form (36) and conditions surrounding Eqs. (38) through (42). Also assume that the (incomplete) intensity, ρ^m, of the zero upcrossings by X_m over $[0,\tau)$, $\tau \in [0, T_m]$, $N^m(\tau)$, satisfy (33). Then if the time normalized count process $\widetilde{N}^m$ is defined: $\widetilde{N}^m(\tau) \triangleq N^m(\tau T_m)$, $\tau \in [0,1]$, we have

$$\widetilde{N}^m(\tau) \to N^*(\tau) \quad \text{in distribution} \tag{43}$$

where $N^*(\tau)$ is a non-stationary Poisson random process on $[0,1)$ with intensity ρ.

We will need the following proposition in order to use Theorem 1.1 for the proof of Theorem 2.1.

Proposition 2.1

Let $X_m(t)$ be a random process which satisfies the conditions in Theorem 2.1. Let $N^m(\sigma,\tau)$ and $N_n^m(\sigma,\tau)$ be the number of upcrossings of zero within $[\sigma,\tau)$ by X_m and the approximation to X_m, ξ_n^m, respectively. Further assume that $T_m \rho^m(\tau T_m) = \rho(\tau)$. Then for $p_m(t)$ the probability that $\widetilde{N}^m(0,t) \triangleq N^m(0,t T_m)$ exceeds zero and $p^*(t)$ the probability that a Poisson count process with intensity function ρ exceeds zero in the interval $[0,t) \in [0,1]$.

$$| p_m(t) - p^*(t) | \leq \lim_{n \to \infty} \sum_{i=0}^{2^n - 1} |q^m(t_i)| \, exp\left(-\int_0^t \rho(\tau) d\tau\right) \tag{44}$$

where

$$q^m(t_i) = P(B_{t_i,t_{i+1}}^m, \overline{B}_{0,t_i}^m) - P(B_{t_i,t_{i+1}}^m)P(\overline{B}_{0,t_i}^m) \tag{45}$$

$B_{t_i,t_j}^m \triangleq \{N_n^m(t_i,t_j) > 0\}$. *for an increasingly dense partition, $\{t_i\}_{i=0}^{2^n-1}$ of the interval $[0, T_m]$.*

Proof

Since X_m satisfies the assumptions of Thm. 1.1 and $P(N^m(t T_m) > 0) = p_m(t)$, $t \in [0, T_m]$

$$p_m(t) = \int\limits_0^{tT_m} \rho_m(\tau)(1 - p_m(\tau/T_m)) d\tau + \lim_{n \to \infty} \sum_{i=1}^{2^n - 1} q^m(t_i), \quad \tau \in [0,1] \tag{46}$$

With a change of variable $\tau/T_m \to \tau$, and the identity relating ρ to ρ_m (33)

$$p_m(t) = \int\limits_0^t \rho(\tau)(1 - p_m(\tau)) d\tau + \lim_{n \to \infty} \sum_{i=1}^{2^n - 1} q^m(t_i) \tag{47}$$

p^* is a Poisson probability measure having the form $p^*(t) = exp\,(-\int\limits_0^t \rho(\tau) d\tau)$ hence p^* satisfies the integral equation

$$p^*(t) = \int\limits_0^t \rho(\tau)(1 - p^*(\tau)) d\tau \tag{48}$$

An application of the triangle inequality to the difference: (47) minus (48), yields

$$|\, p_m(t) - p^*(t)| \le \int\limits_0^t \rho(\tau)|\, p(\tau) - p^*(\tau)| d\tau + \lim_{n \to \infty} \sum_{i=1}^{2^n - 1} |\, q^m(t_i)\,| \quad , \quad t' \in [0,t] \tag{49}$$

Let the last term in (49) be denoted $r(tT_m)$. Then r is monotone non-decreasing in $t \in [0,1]$ and replacement of $r(tT_m)$ by $r(T_m)$ can only weaken the inequality (49). Subsequent application of the Bellman-Gronwall inequality [18] to (49) finishes the proof.

∎

The following generalization of Lemma 2.2.3 in [12] is proven in [5].

Lemma 2.2

    Let X_m satisfy (6) through (8), be u-a mixing and satisfy (38) and (40). Then given $\epsilon > 0$, integers $r > 0$, $K > 0$, positive quantities l, $l = o(T_m/K)$ and t, $\dfrac{T_m}{K} < t < T_m$, we have for m sufficiently large

$$P(N_n^m(t-l,t) > 0, N_n^m(t-l) = 0) < \left(\frac{r}{r+1} \right) \frac{1}{r+1} + (2r-1)\alpha_{m,l} + \epsilon \tag{50}$$

With Lemma 2.2 and Proposition 2.1 we can easily prove the following weak form of the Poisson convergence result.

Proposition 2.2

    If the a.s. continuous processes $\{X_m(t)\}$ satisfy the conditions stated in the premise of Theorem 2.1 then for any interval I contained in [0,1]

$$P(\widetilde{N}^m(I) > 0) \to 1 - exp\,(-\int\limits_I \rho(\tau) d\tau) \tag{51}$$

Proof

First fix l greater than zero and K a positive integer. We reproduce Eq. (45) here for clarity. As in Eq. (12) of Theorem 1.1, for the sequence X_m, $m = 0, 1, \cdots$ we have the quantities $q^m(t_i)$ on the 2^n point grid $\{t_i\}_{i=0}^{2^n}$

$$q^m(t_i) = P(B_{t_i, t_{i+1}}^m, \bar{B}_{0,t_i}^m) - P(B_{t_i, t_{i+1}}^m) P(\bar{B}_{0,t_i}^m) \tag{52}$$

Partition the interval $[0, T_m)$ into K parts so that the sum on the right of Eq. (44) of Proposition 2.1 can be represented as

$$\sum_{i=0}^{2^n-1} |q^m(t_i)| = \sum_1 |q^m(t_i)| + \cdots + \sum_K |q^m(t_i)| \tag{53}$$

where $\sum_k$ denotes summation over the intersection of the grid $\{t_i\}_{i=0}^{2^n}$ and the k-th partition element, $k = 1, 2, \ldots, K$.

Fix $\epsilon > 0$ and let m be sufficiently large so that Lemma 2.2 holds. Consider the final $K-1$ terms in (53)

$$\sum_2 |q^m(t_i)| + \cdots + \sum_K |q^m(t_i)| \tag{54}$$

Now for each $q^m(t_i)$ in (54) we add and subtract terms so as to isolate the mixing dominated quantities of the form (36). That is we obtain via the triangle inequality

$$|q(t_i)| \leq |P(B_{t_i, t_{i+1}}, \bar{B}_{0, t_i-l}) - P(B_{t_i, t_{i+1}}) P(\bar{B}_{0, t_i-l})| \tag{55}$$

$$+ |P(B_{t_i, t_{i+1}}) P(\bar{B}_{0, t_i-l}) - P(B_{t_i, t_{i+1}}) P(\bar{B}_{0, t_i})|$$

$$+ |P(B_{t_i, t_{i+1}}, \bar{B}_{0, t_i}) - P(B_{t_i, t_{i+1}}, \bar{B}_{0, t_i-l})|$$

where we have suppressed dependencies on m for notational simplicity. Using the mixing condition (36) on the first term to the right of the inequality (55) and using simple set identities for the other two terms we have

$$|q(t_i)| \leq P(B_{t_i, t_{i+1}}) [\alpha_{m,l} + P(B_{t_i-l, t_i}, \bar{B}_{0, t_i-l})] + P(B_{t_i, t_{i+1}}, B_{t_i-l, t_i}, \bar{B}_{0, t_i-l}) \tag{56}$$

Finally applying Lemma 2.2 to the second term in brackets [ ] in (56) and noting that by monotonicity the third term in (56) is bounded

$$P(B_{t_i, t_{i+1}}^m, B_{t_i-l, t_i}^m, \bar{B}_{0, t_i-l}^m) \leq P(N_n^m(t_i, t_{i+1}) > 0, N_n^m(t_i-l, t_i) > 0) \tag{57}$$

we get by substituting the inequality (56) in (53)

$$\sum_{i=0}^{2^n-1} |q^m(t_i)| \tag{58}$$

$$< \sum_1 |q^m(t_i)| + [2r\,\alpha_l + (\frac{r}{r+1})^r \frac{1}{r+1} + \epsilon]\sum_{i=\varsigma}^{2^n-1} P(B^m_{t_i,t_{i+1}})$$

$$+ \sum_{i=\varsigma}^{2^n-1} P(N_n^m(t_i,t_{i+1}) > 0,\ N_n^m(t_i-l,t_i) > 0)$$

here $t_\varsigma \in \{t_i\}_{i=0}^{2^n}$ is the rightmost point contained in the first partition element, $[0, \frac{T_m}{K}]$, of the K-th order partition. Now applying the relation (31) and Lemma 1.2 to the first term to the right of the inequality (58) for n sufficiently large

$$\sum_1 |q^m(t_i)| \leq \sum_1 P(B^m_{t_i,t_{i+1}}) \leq \mathbf{E}[N^m_{r_1}] + \epsilon \tag{59}$$

where $N^m_{r_k}$ is as defined in Eq. (42). Likewise

$$\sum_{i=\varsigma}^{2^n-1} P(B^m_{t_i,t_{i+1}}) \leq \sum_{i=0}^{2^n-1} P(B_{t_i,t_{i+1}}) \leq \mathbf{E}[N^m(T_m)] + \epsilon \tag{60}$$

which gives via Eq. (58)

$$\lim_{n \to \infty} \sum_{i=0}^{2^n-1} |q^m(t_i)|$$

$$< \lim_{n \to \infty} \mathbf{E}[N^m_{r_1}] + [2r\,\alpha_{m,l} + (\frac{r}{r+1})^r \frac{1}{r+1} + \epsilon]\{\mathbf{E}[N^m(T_m)]\} \tag{61}$$

$$+ \lim_{n \to \infty} \sum_{i=\varsigma}^{2^n-1} P(N_n^m(t_i,\ t_{i+1}) > 0,\ N_n^m(t_i-l,\ t_i) > 0)$$

Therefore taking the limit as $m,l \to \infty$, $l = o(T_m)$, the first term to the right of (61) goes to zero by Uniform Negligibility, (42), and the finiteness of $\mathbf{E}[N^m(T_m)] = \mathbf{E}[\widetilde{N}(1)] = \int_0^1 \rho(\tau)\,d\tau$. The second term converges to a quantity not exceeding $[\frac{1}{r} + \epsilon]\,\mathbf{E}[\widetilde{N}(1)]$. However as m becomes unbounded r can be made arbitrarily large and ϵ can be made arbitrarily small, by Lemma 2.2, thus the second term is negligible. Finally the rarefaction condition, Eq. (38), asserts that the third term vanishes. Hence by Proposition 2.1, for $I = [0, t]$

$$p_m(t) \rightarrow p^*(t) = 1 - \exp(-\int_0^t \rho(\tau)d\tau) \quad , \quad t \in [0,1) \tag{62}$$

∎

Proposition 2.2 asserts that the probability that the normalized upcrossing process $\widetilde{N}^m$ is greater than zero in any interval contained in $[0,1)$ is the same as the corresponding probability for a Poisson counting process N^* in the limit as $m \rightarrow \infty$. To show the stronger result that $\widetilde{N}^m$ actually converges in distribution to a Poisson process we will follow Leadbetter [12] in making use of a theorem in [9]. Using the nomenclature in [9] a point process N is a –*regular* if for every collection of intervals I contained in $\Upsilon_{[0,1]}$, the Borel sets on $[0,1]$, there exists some array $\{I_{mk}\} \subset \Upsilon_{[0,1]}$ of finite covers of I (one for each $m = 1,2,\cdots$) such that

$$\lim_{n\rightarrow\infty} \limsup_{m\rightarrow\infty} \sum_k P(N^m(I_{nk}) \geq a) = 0 \tag{63}$$

We state the following special case of Theorem 4.7 in [9].

Lemma 2.3

Let $\widetilde{N}^m$ be a sequence of point processes and N^* a Poisson process both defined on $[0,1)$. Then $\widetilde{N}^m$ converges in distribution to N^* if and only if $\widetilde{N}^m$ is 2-regular and

$$\lim_{m\rightarrow\infty} P(\widetilde{N}^m(U) = 0) = P(N^*(U) = 0) \tag{64}$$

for all U of the form

$$U = \bigcup_{k=1}^{r} \Upsilon_i \quad , \quad \Upsilon_i \subset \Upsilon_{[0,1]}$$

for $r > 1$, and

$$\limsup_{m\rightarrow\infty} \mathbf{E}[\widetilde{N}^m(I)] \leq \mathbf{E}[N^*(I)] < \infty \tag{65}$$

for $I \subset \Upsilon_{[0,1]}$

We now proceed to the proof of Theorem 2.1 which at this point only involves showing that $\widetilde{N}^m$ of the theorem satisfies the conditions in Lemma 2.3.

Proof of Theorem 2.1

Without loss of generality we assume that the collection of intervals I in the a –*regularity* condition and in (65), and the Υ in (64) are sets of disjoint intervals. For each m, $m = 1, \cdots$, define the increasing set of disjoint covers of I: $\{J_{mk}\}$, $k = 1, 2, \cdots r_m$, with each J_{mk} of length l_m/T_m (recall $l_m = o(T_m)$). Assume for definiteness that J_{mk} are ordered such that the left endpoints are strictly increasing as k increases. With N^m as in

Theorem 2.1 and $\widetilde{N}^m(\tau) = N^m(\tau T_m)$ we have

$$\sum_{k=1}^{r_m} P(\widetilde{N}^m(J_{mk}) > 1) = \sum_{k=2}^{r_m} P(\widetilde{N}^m(J_{mk}) > 1) + P(\widetilde{N}^m(J_{m1}) > 1) \qquad (66)$$

$$\leq \lim_{n \to \infty} \sum_{k=1}^{r_m} \sum_{t_i \in J_{im} T_m} P(N_n^m(t_i^k, t_{i+1}^k) > 0, N_n^m(t_{i-l_m}^k, t_i^k) > 0) + \mathbf{E}\left[\widetilde{N}^m(J_{m1})\right]$$

where $\{t_i^k\}_{i=1}^{n_k}$ are increasingly dense partitions of J_{mk}, for $k = 1, \cdots r_m$ respectively. The first term on the right of the inequality (66) is bounded by the expression in the Rarefaction condition, Eq. (38) while the second term converges to zero by Uniform Negligibility, (42), and the finiteness of $\mathbf{E}[\widetilde{N}^m]$. Taking the limit of Eq. (66) as $m \to \infty$ we have that $\widetilde{N}^m$ is 2-*regular* in the sense of (63).

Fix $r > 0$. Because of the absolute continuity of the distributions of X_m the intervals Υ_i in (64) can be taken as having no common boundary points. Therefore by mixing, Lemma 2.1, for any collection of disjoint intervals $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_r$ contained in $[0,1]$

$$|P(\bigcap_{i=1}^r N^m(T_m \Upsilon_i) > 0) - \prod_{i=1}^r P(N^m(T_m \Upsilon_i) > 0)| \to 0, \quad as \ m \to \infty \qquad (67)$$

where we have adopted the operator notation for T_m: $T_m[\sigma,\nu) = [T_m \sigma, T_m \nu)$ for $0 \leq \sigma < \nu < 1$. Eq. (67) and Proposition 2.2 thus imply that

$$P(\bigcap_{i=1}^r N^m(T_m \Upsilon_i) = 0) \to \exp(-\sum_{i=1}^r \int_{\Upsilon_i} \rho(\tau) d\tau) \qquad (68)$$

Since, furthermore, $\widetilde{N}^m$ and N^* have identical intensity (recall (33)) the assumptions stated in Lemma 2 3 are valid for $\widetilde{N}^m$ and Proposition 2.1 establishes Theorem 2.1.

■

While the asymptotic theorem, Theorem 2.1, is an interesting result, Lemma 2.2 is more useful in applications. Let N be an upcrossing count process, on $[0,1]$, with (incomplete) intensity ρ. Lemma 2.2 states a bound on the "approximation error" of the Poisson model, $p^*(t) \triangleq P(N^*(t) > 0)$ and $p(t) \triangleq P(N(t) > 0)$.

$$|p^*(t) - p(t)| \leq \int_0^1 |q(\tau)| d\tau \exp(\int_0^1 \rho(\tau) d\tau) \qquad (69)$$

where the abstract integral has been defined

$$\int_0^1 q(\tau) d\tau \triangleq \lim_{n \to \infty} \sum_{i=1}^{2^n-1} |q(t_i)| \qquad (70)$$

Along with definition (12) for q, (70) asserts that as upcrossings become rare over time, $q \to 0$ and there is progressively smaller error involved in the Poisson approximation. Since

$q(\tau) \leq \rho(\tau)$ a uniform decrease in the intensity over time is sufficient for rarefaction. In the next section level crossings will correspond to large errors in an estimation problem involving signals and noises. In that context it can be shown that p is monotone decreasing in signal-to-noise-ratio. Thus a practical interpretation of Theorem 2.2 is that a Poisson model for large error is accurate for moderately large SNR and above. Furthermore, it can be shown that, for large ρ, $p^{*}(t) \geq p(t)$. This establishes p^{*} as an upper bound for p under large or small SNR conditions.

IV. APPLICATION TO PASSIVE ARRAYS

The performance of the correlator estimate of time delay in a two sensor passive array has received much attention in the past decade [2,20,10]. As is typical in non-linear estimation problems, exact expressions for the variance of any estimator are difficult to derive except under restrictive small error regimes [17]. In this section we will develop a global variance approximation which is directly motivated by a level crossing interpretation for large errors. In this context the asymptotic result presented in the last section has an interesting interpretation. For low probability of large error, the level crossings form a point process over the a priori region which have nearly Poisson statistics. While for large probability of error the Poisson model is conservative, i.e. $P(N > 0)$ is larger for a Poisson N than for the actual level crossing process N. This observation suggests building a conservative global variance estimate, via Poisson modeling, to complement the lower bounds such as the Ziv-Zakai and Cramer-Rao bounds.

Our observation model for the outputs of a two-sensor passive array is as follows. The outputs of two sensors, $x_1(t)$ and $x_2(t)$, are observed over a finite interval of time $t \in [0, T]$

$$
\begin{aligned}
x_1(t) &= s(t) + n_1(t) \\
x_2(t) &= s(t - D) + n_2(t)
\end{aligned}
\tag{71}
$$

The signal components, $s(t)$ and a delayed version $s(t - D)$, and the noises, $n_1(t)$ and $n_2(t)$, are zero mean, uncorrelated, stationary Gaussian random processes. The delay D is restricted to an *a priori* region of possible delay $[-D_M, D_M]$. The signal auto-correlation, $R_{ss}(\tau) = E[s(t)s(t+\tau)]$, is assumed to be essentially zero for $|\tau| > T_c$, where $T_c = k/B$ is the correlation time, k is an integer and B is the baseband bandwidth of the signal. As in most cases of interest, we assume that the uncertainty region $[-D_M, D_M]$ is large enough to make the time-bandwidth product $BD_M \gg 1$.

For flat signal and noise spectral densities, the correlation estimate of time delay, $\hat{D}$, is the location in time, within $[-D_M, D_M]$, at which the global maximum of the sample cross-correlation function occurs.

$$
\hat{D} \triangleq \underset{\tau \in [-D_M, D_M]}{argmax} \hat{R}_{12}(\tau)
\tag{72}
$$

$\hat{R}_{12}(\tau)$ is the sample cross-correlation function.

$$\hat{R}_{12}(\tau) \triangleq \frac{1}{T} \int_0^T x_1(t)x_2(t+\tau)dt \tag{73}$$

The correlator estimate is known to be asymptotically equivalent to the maximum likelihood estimate for D as $BT \rightarrow \infty$[10]. In [15] a simple small error approximation, the Cramer-Rao-Lower-Bound: σ^2_{CRLB}, was derived for the variance of $\hat{D}$. The CRLB is an accurate approximation to the true variance when $|\hat{D}-D| < \delta$ with high probability, where δ is a small constant dependent upon the signal and noise spectra. An exact expression for the global variance results from direct application of the "law of total probability" to $var\{\hat{D}\}$:

$$var\{\hat{D}\} = \sigma^2_{loc}(1 - P_e) + \sigma^2 P_e \tag{74}$$

In (74) $\sigma^2_{loc} = \min\{\sigma^2_{CRLB}, \frac{\delta^2}{3}\}$ and $\sigma^2 = \mathbf{E}\{(\hat{D}-D)^2 \,|\, (\hat{D}-D) \notin [-\delta,\delta]\}$ are the conditional expectations of the squared error given small (local) error and large error respectively, and $P_e = P(\hat{D}-D \notin [-\delta,\delta])$ is the probability of large error. The rest of this section deals with suitable approximations to the large error probability and the squared error $\sigma^2 Pe$.

The occurrence of a peak equal or greater in magnitude than the local maximum of the correlator within $[-\delta,\delta]$ is called a peak ambiguity, since it confounds the estimators search for the location of the local max occurring near the true delay D. A useful interpretation is that each peak ambiguity gives rise to a candidate for $\hat{D}$. In the exact model the candidates are the locations over the a priori interval where the peak ambiguities occur. From these candidates a single member is then selected for $\hat{D}$, the one which corresponds to the largest of the ambiguous peaks. We use the above interpretation to develop a different set of candidates, each of which corresponds to a level crossing location associated with each peak ambiguity. $\hat{D}$ is then modeled as equally likely to take on the identities of any of the candidates. With little loss of generality it will be assumed that $D = 0$ in the sequel [5].

Define the random level $m_\delta \triangleq \max_{u \in [-\delta,\delta]} \hat{R}_{12}(u)$ and the "ambiguity process" $\Delta R(\tau) \triangleq \hat{R}_{12}(\tau) - m_\delta$. m_δ is the magnitude of the desired local peak of the sample cross-correlation, while $\Delta R(\tau)$ must be negative over $[-D_M,D_M] - [-\delta,\delta]$ for no large error to occur. Next define the level crossing count process $N \triangleq \{N(\tau):\tau \in [-D_M,D_M]\}$ associated with the ambiguity process

$$N(\tau) \triangleq \begin{cases} N_u(-D_M,\tau) & \tau \in [-D_M,-\delta) \\ N_u(-D_M,-\delta) & \tau \in [-\delta,\delta] \\ N_u(-D_M,-\delta) + N_d(\delta,\tau) & , \tau \in (\delta,D_M] \end{cases} \tag{75}$$

where $N_u(t_1,t_2)$ is the number of up-crossings of zero by $\Delta R(\tau)$ over $\tau \in [t_1,t_2) \subset [-D_M,-\delta)$, and $N_d(t_1,t_2)$ is the number of down-crossings of zero by $\Delta R(\tau)$ over $\tau \in [t_1,t_2] \subset (\delta,D_M]$. The process N is merely the running sum, over time, of the total number of up-crossings to

the left of the true delay plus down-crossings to the right of the true delay.

With the above definitions, no large error occurs, i.e. $\hat{D} \in [-\delta,\delta]$, if and only if 1). $\Delta R(-D_M) < 0$ and $\Delta R(D_M) < 0$ and 2). $N(D_M) = 0$. Hence the probability of large error

$$P_e = 1 - P(N(D_M) = 0, \Delta R(-D_M) < 0, \Delta R(D_M) < 0) \tag{76}$$

Define the left continuous probability distribution function $F(x) \triangleq P(\Delta R(-D_M) < x)$. The probability of large error (76) can be shown to have the form [6]

$$P_e = 1 - \exp\{-\int_{-D_M}^{D_M} \lambda_c(\tau)d\tau\} \; F^2(0) \tag{77}$$

Where λ_c is the conditional (incomplete) intensity of N, $\lambda_c(\tau)d\tau \triangleq P(dN(\tau) > 0 \mid N(\tau)=0, \Delta R(-D_M) < 0)$ and $dN(\tau)$ is the infinitesimal increment in time of N at time τ.

Equation (77) is an exact representation of the probability of large error in terms of the level crossings N. However, while the presence of level crossings is (conditionally) equivalent to the presence of ambiguity, the level crossings alone do not uniquely specify the location of the global maximum. Hence one cannot expect the level crossings to be sufficient, by themselves, to give an exact expression for variance. As an approximation, we will use the following conditionally uniform model for the location of the global maximum given a particular sequence of level crossings. Let $\{\omega_1, \ldots, \omega_n\}$ be the ordered set of points in $[-D_M, D_M]$ where $N(\tau)$ increases. Conditioned on the occurrence of a large error, we will model $\hat{D}$ as follows: $\hat{D}$ takes values in the set $\{\omega_1, \ldots, \omega_n\}$ with equal probability if $N(D_M) = n$ and both $\Delta R(-D_M), \Delta R(D_M) < 0$; while $\hat{D}$ takes values in one of the sets: $\{\omega_1, \ldots, \omega_n, \pm D_M\}$, $\{\omega_1, \ldots, \omega_n, -D_M, D_M\}$, with equal probability if $N(D_M) = n$ and either: $\Delta R(-D_M) \geq 0$, $\Delta R(D_M) \geq 0$; or both, respectively.

Assume for simplicity that $N_d(\delta, D_M) = 0$ while $Nu(-D_m, -\delta) > 0$. The conclusions drawn for this case apply to the more general situation with no additional conceptual difficulty. Define $\{a_1, \ldots, a_N\} \triangleq \{\underset{[w_1, w_2)}{argmax \Delta R(\tau)}, \ldots, \underset{[w_N, -\delta)}{argmax \Delta R(\tau)}\}$ the ordered set of peak ambiguity locations. Since $|a_i| \leq |w_i|$, $i = 1, \ldots, N$, and the largest ambiguities tend to cluster in the vicinity of the high amplitude sidelobes of R_{ss}, the signal autocorrelation, occurring close to the true delay, the proposed model entails, at worst, a pessimistic estimate of the mean squared error of $\hat{D}$.

Under the conservative model described in the preceding paragraphs, the following inequality can be derived [6].

$$var\{\hat{D}\} \geq \sigma_{loc}^2 \exp\{- \int_{-D_M}^{D_M} \lambda_c(\tau)d\tau + \int_{-D_M}^{D_M} \tau^2 \rho(\tau) g(\tau)d\tau + D_M^2\alpha \tag{78}$$

In (78) ρ is the unconditional (incomplete) intensity of N, as defined in Section II, Eq. (9), and α is a small quantity given in [5] which is not significant in this discussion. The function $g(\tau)$ is defined as

$$g(\tau) \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} a_{k-1,n-k}(\tau) \tag{79}$$

and we have defined the bidirectional Palm measure

$$a_{k-1,n-k} \triangleq \lim_{h\to 0} \frac{1}{h} P(N(\tau) = k-1, N(D_M) = n \mid dN(\tau,h) > 0) \tag{80}$$

The bidirectional Palm measure corresponds to the probability that, given the occurrence of a (crossing) point at τ, this point is the k-th occurrence in a sequence of n points occurring over $[-D_M, D_M]$ (see [3] for a discussion of Palm measures).

The expression (78) consists of three factors. The first factor is the small error contribution to the global variance. The second term is the contribution of peak ambiguities which generate level crossings in $(-D_m, D_m)$, and the third term is the contribution of any peak ambiguity which does not generate a level crossing (i.e. corresponding to our conservative assignment $\hat{D} = \pm D_M$).

The asymptotic results cited in the previous sections suggest the feasibility of applying a Poisson approximation to the level crossing process N as these crossings become increasingly rare, i.e. for small ρ (Recall discussion at the end of Section III). Here we give more quantitative results concerning the actual error committed by the approximation. Let N^* be an inhomogeneous Poisson process with intensity $\rho(\tau)$ and define the signed difference $\Delta \triangleq P(N^*(\sigma,u) > 0) - P(N(\sigma,u) > 0)$. While sharp bounds on the deviation of Δ from zero can be derived for ΔR a nonstationary Gaussian process, using Theorem 1.1, we will concentrate on the following non-parametric bounds derived in [6].

$$\max\left\{-\frac{E^2\{N\}}{1+E\{N\}}, -e^{-E\{N\}}\right\} \leq \Delta \leq E\{\frac{1}{N+1}\} - e^{-E\{N\}} \tag{81}$$

Where, for compactness, N is shorthand for $N(\sigma,u)$. Note that all terms in the left and right hand inequalities of (81) depend on only the first moment of $N(\sigma,u)$, except for $E\{\frac{1}{N+1}\}$. This latter term can be upper bounded, however

$$E\left\{\frac{1}{N+1}\right\} \le \frac{G\{N\}}{1+G\{N\}} \tag{82}$$

where

$$G\{N\} \triangleq \frac{var\{N\}}{E^2\{N\}} + \frac{1}{E\{N\}} \tag{83}$$

The mean and variance of $N(\sigma,u)$ can be computed from the second and fourth order distributions of X if Leadbetters conditions are satisfied [11]. Thus if the variance of $N(\sigma,u)$ increases only as rapidly as $o(E^2\{N(\sigma,u)\})$ the bounds (81) and (82) guarantee that the error incurred in the Poisson approximation is near zero whenever $\exp\{-E\{N(\sigma,u)\}\}$ approaches either 1 or 0. In any case, for these extremal conditions the left hand inequality in (81) implies Δ is lower bounded by a small magnitude negative quantity, i.e. the Poisson model conserves the inequality (78) to a good approximation.

The application of the Poisson model to the level crossing process N gives the following simple relations for the probability of large error and the bound on the variance (78)

$$P_e = 1 - e^{-E_c\{N(D_M)\}}F^2(0) \tag{84}$$

$$var\{\hat{D}\} \le \sigma_{loc}(1-P_e) + \int_{-D_M}^{D_M} \tau^2 \hat{\rho}(\tau)d\tau (1 - e^{-E\{N(D_M)\}}) + D_M^2\alpha \tag{85}$$

In (85) we have defined the normalized intensity

$$\hat{\rho}(\tau) \triangleq \rho(\tau)/\int_{-D_M}^{D_M}\rho(u)du \tag{86}$$

and α is given by

$$\alpha \triangleq 2\left[\frac{1 - e^{-E\{N(D_M)\}}}{E\{N(D_M)\}} - \frac{1 - e^{-E_c\{N(D_M)\}}}{E_c\{N(D_M)\}}(1-F^2(0))\right]$$

Here

$$E_c\{N(D_M)\} = \int_{-D_M}^{D_M}\rho_c(\tau)d\tau \ , \quad E\{N(D_M)\} = \int_{-D_M}^{D_M}\rho(\tau)d\tau \tag{87}$$

are the conditional mean of $N(D_m)$ given $\Delta R(-D_m) > 0$ and the unconditional mean of $N(D_M)$ respectively.

The Poisson approximation (85) indicates that, as the intensity of peak ambiguities, ρ, increases, one must discount the small error variance, σ_{loc}^2 by P_e, adding an increasingly large quantity: the mean-square deviation of the locations of peak ambiguities overtime. In the following section we will explicitly calculate the intensities in (87) under a Gaussian assumption,

and analyze the resulting form of the Poisson variance approximation for simple bandpass signals.

V. NUMERICAL COMPARISONS

The intensity functions ρ_c and ρ in (87) can be derived under the following assumptions: a). $\hat{R}_{12}$ is a Gaussian random process with non-stationary mean and differentiable covariance function; and b). $\max_{u \in [-\delta,\delta]} \hat{R}_{12}(u) = \hat{R}_{12}(0)$. The Gaussian model is reasonable for large BT [5]. Since the exceedance of $\hat{R}_{12}(0)$ by $\hat{R}_{12}(\tau)$ for some $\tau \in [-D_M,D_M]-[-\delta,\delta]$ does not necessarily imply a peak ambiguity, assumption b). is pessimistic at worst.

Using the assumptions a). and b). the results are

$$\rho_c(\tau) = K_1 \int_0^\infty y \ \Phi(a_0 y + a_1) \ \phi(y+a_2) dy \qquad (88)$$

$$\rho(\tau) = K_2 \ \phi(a_3)[\phi(a_4) + a_4 \Phi(a_4)]$$

Here $K_1, K_2, a_0, \ldots, a_4$ are functions of τ given [6]. The functions Φ and ϕ are the standard Gaussian distribution and density functions respectively.

In [6] a simple explicit forms for (88), (84) and (85) was derived for flat lowpass signal and noise spectra. For these simple bandpass spectra the small error region over which the CRLB is accurate $[-\delta,\delta]$ is given by $\delta = 1/4f_o$. Here we only discuss numerical results for flat bandpass spectra. In Fig. 1 the intensity surface, is displayed for a bandpass signal at center frequency $f_o = 500Hz$, with bandwidth $B = 200Hz$, and $T = 8.0 secs$. Here the time window extends from the first zero crossing of the auto-correlation function of the signal at $\delta = 1/4f_o$, to approximately the fifth sidelobe away from the origin. In Fig. 1 the location of the global maximum of the autocorrelation is just beyond the rightmost point on the t axis. A distinctive feature of Fig. 1 is the SNR difference between the point, SNR_1 where a rapid rise in the intensity of ambiguity first begins, i.e. in the region of the first sidelobe, and the point, SNR_2 where a uniform increase of the ambiguity, over time, is in evidence. This implies the existence of at least three separate SNR thresholds which is consistent with studies of the Ziv-Zakai-Lower-Bound (ZZLB) for this problem [20].

We numerically evaluated the integrals in (88) and (85) for a flat bandpass signal with center frequency to bandwidth ratio $f_o/B = 10$, and $BD_M = 25$. The results are plotted in Figs. 2 and 3, along with plots of the CRLB and ZZLB, for $BT = 200$ and $BT = 80$ respectively. The Poisson approximation behaves similarly to the ZZLB in Fig. 2, both indicating the presence of three distinct SNR thresholds (e.g. SNR_{t1}, SNR_{t2} and SNR_{t3} in Fig. 2) of performance. For $SNR < SNR_{t1}$ the Poisson approximation becomes a much better predictor of variance than the CRLB. $[SNR_{t2}, SNR_{t1}]$ is a region where, with high probability, large errors are concentrated in the interval $\hat{D} \in [-T_c, T_c]$, the small error region for the envelope of the

bandpass signal. When $SNR < SNR_{t2}$ the error approaches that of a uniform random variable over $[-D_M, D_M]$: the estimate $\hat{D}$ is useless. For $BT = 80$, in Fig. 3 the Poisson approximation has moved away from the ZZLB relative to the case of $BT = 200$. Indeed it appears to hit an asymptote with increasing SNR, i.e. the correlator commits large errors even as the SNR approaches infinity. This behavior of the Poisson approximation corroborates the reported sub-optimality of the correlator estimate for small BT [7].

Finally the results of a simulation of correlator performance for a bandpass signal spectrum appears in Fig. 4. The relevant parameters are: $f_0/B = 2.5$, $BT = 50$ and $BD_M = 8$ and the vertical dimension of the "Φ" characters indicate approximate 95% confidence interval for the actual variance (obtained by simulation). Plotted for comparison are the CRLB, ZZLB and Poisson Approximation. The combination of the overly optimistic ZZLB and the overly pessimistic Poisson approximation jointly specify an admissible region of estimator variance. However, on the average, below a SNR of 5dB the Poisson approximation is significantly closer to the true variance than the ZZLB. Note in particular that at a SNR of -8dB the Poisson approximation is within the 95% confidence interval while the ZZLB is more than 5dB below this interval.

VI. CONCLUSION

Two results were derived in the context of level crossing probabilities. First, a representation of the probability of getting one or more upcrossings in an interval by a general random process was presented. This representation in effect isolates the portion of the upcrossing probability due to the intensity function of the upcrossings, from a correction term, which characterizes the deviation of the upcrossing probability from an associated inhomogeneous Poisson probability. The correction term depends on the degree to which the upcrossings can be modeled as an independent increment process. By identifying conditions which asymptotically force the correction term to zero a second result was made possible: that a certain time normalized version of the upcrossing process can be made to converge in distribution to the inhomogeneous Poisson law.

Future investigations of the of the correction term, $Q(t)$, associated with the probability representation of Theorem 1.1, should lead to useful expressions for the approximation error incurred by using such simple first moment approximations. For the asymptotic result, Thm. 2.1, the asymptotic conditions rarefaction and mixing play an important role. In particular, rarefaction could be replaced by conditions involving probability statements about the maximum process over the interval I, $\max_{\tau \in I} X(\tau)$, analogous to [12]. For specific probability models of the random process $X(t)$ of interest, e.g. Gauss-Markov or Rayleigh as in [12], one would expect the replacement condition to be more easily verified, than the conditions used here.

An application of the Poisson model to a problem in underwater acoustics, time delay estimation, yielded an approximation to the global variance of the estimate. Numerical results indicate the fidelity and conservativeness of this performance approximation relative to the Ziv-Zakai lower bound for bandpass spectra. Yet to be investigated is the feasibility of Poisson

approximations to large error in multiparameter estimation problems. In these situations the maximum likelihood procedure involves a search for a global maximum over an ambiguity surface. Thus the concept of level crossing becomes more difficult due to the lack of inherent directionality over the parameter space (points in the space are not well ordered). While this would not preclude the application of a Poisson spatial model for the locations of peak ambiguity, bounds on the approximation error, analogous to the one dimensional case, are not as simple to derive.

ACKNOWLEDGEMENT

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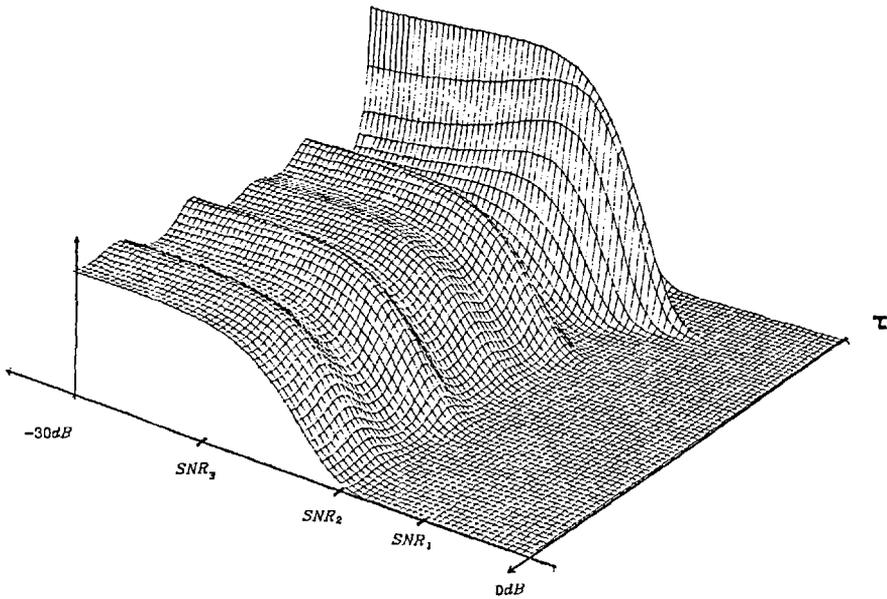22. P. Billingsly, *Probability and Measure,* Wiley, New York, 1979.

FIG. 1
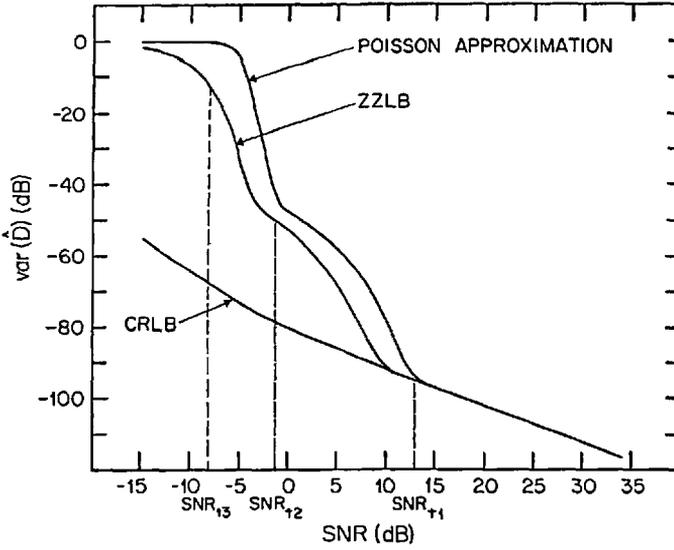Intensity surface, λ, for bandpass signal over time and SNR.

FIG. 2
Comparison of Poisson approximation with ZZLB and CRLB for bandpass signal spectrum. $f_o/B = 10$, $BD_m = 25$ and $BT = 200$. Variance, $var(D)$, normalized with respect to standard uniform distribution.
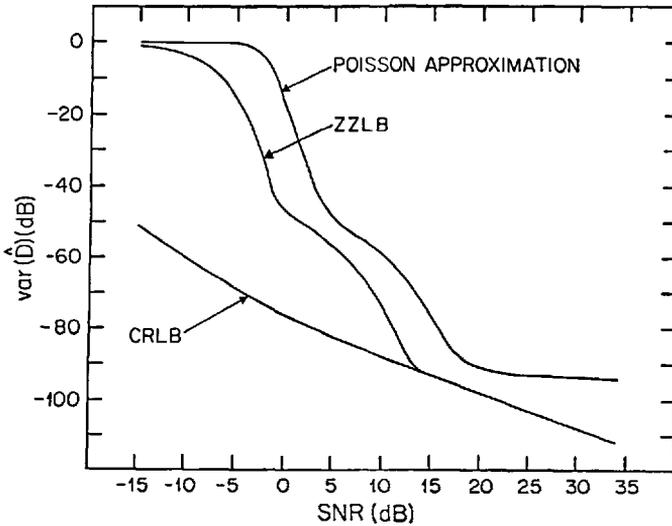

FIG. 3
Comparison of Poisson approximation with ZZLB and CRLB for bandpass signal spectrum. $f_o/B = 10$, $BD_m = 25$ and $BT = 80$.
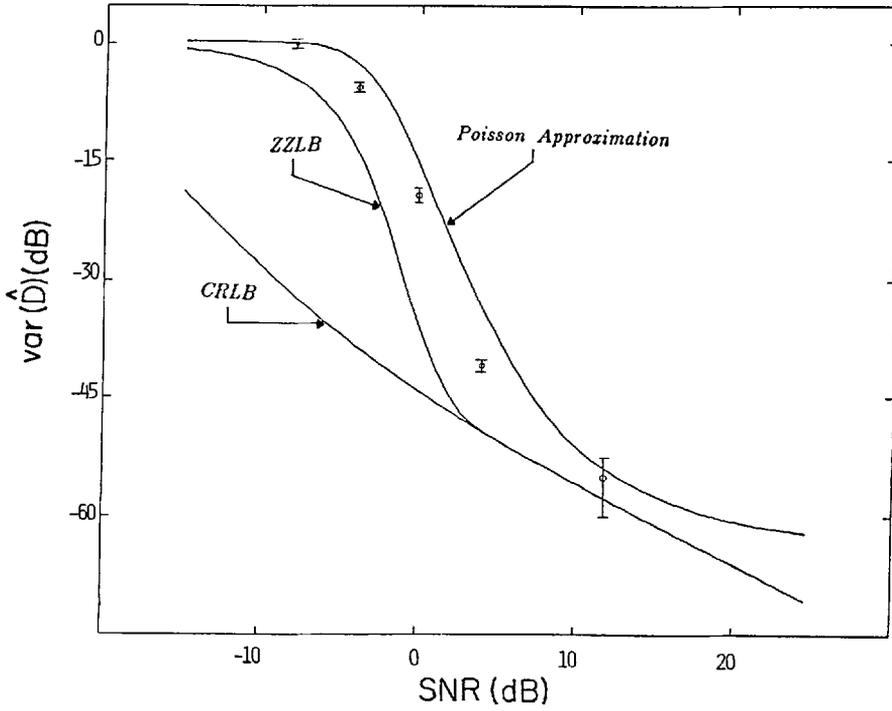
FIG. 4
Comparison of Poisson approximation with ZZLB and results of simulation, Φ, for $f_o/B = 2.5$, $BD_m = 8$ and $BT = 50$.