MULTIPLE SIGNAL DETECTION USING THE BENJAMINI-HOCHBERG PROCEDURE

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ABSTRACT

We treat the detection problem for multiple signals embedded in noisy observations from a sensor array as a multiple hypothesis test based on log-likelihood ratios. To control the global level of the multiple test, we apply the false discovery rate (FDR) criterion proposed by Benjamini and Hochberg. The power of this multiple test has been investigated through narrow band simulations in previous studies. Here we extend the proposed method to broadband signals. Unlike the narrow band case where the test statistics are characterized by F-distribution, in the broadband case the test statistics have no closed form distribution function. We apply the bootstrap technique to overcome this difficulty. Simulations show that the FDR-controlling procedure always provides more powerful results than the familywise error-rate (FWE) controlling procedure. Furthermore, the reliability of the proposed test is not affected by the gain in power.

1. INTRODUCTION

This work is concerned with broadband signal detection using a multiple hypothesis test. Estimating the number of signals embedded in noisy observations is a key issue in array processing, harmonic retrieval, wireless communication and geophysical application. In [2] [7], a multiple testing procedure was suggested to determine the number of signals. Therein, a Bonferroni-Holm procedure [6] was used to control the familywise error-rate (FWE), the probability of erroneously rejecting any of the true hypotheses. As the control of FWE requires each test to be conducted at a significantly lower level, the Bonferroni-Holm procedure often leads to conservative results. To overcome this drawback, we adopted the false discovery rate (FDR) criterion suggested by Benjamini and Hochberg [1] to keep balance between type one error control and power [3]. In addition to the successful numerical results reported in [3], we examined carefully the independence condition required by the Benjamini and Hochberg procedure in [4].

Motivated by the promising results of previous work, we extend the multiple testing procedure to broadband signals. Unlike the narrow band case in which the test statistics are F-distributed under null hypothesis, there is no closed form expression for the distribution of test statistics in the broadband case. Therefore, we incorporate the powerful bootstrap technique to approximate the distribution under null hypothesis and estimate the observed significance level.

This paper is organized as follows. We give a brief description of the signal model in the next section. Then we present the multiple test procedure for signal detection. The bootstrap principle and its application to our problem are illustrated in section 4. In section 5 we introduce the idea of false discovery rate (FDR) and the Benjamini Hochberg procedure. Simulation results are presented and discussed in section 6. Our concluding remarks are given in section 7.

2. DATA MODEL

Consider an array of n sensors receiving m broad band signals emitted by far-field sources located at $\underline{\theta} = [\ \theta_1,\ldots,\theta_m]^T$. The array output $\underline{x}(t)$, $(t = 0,\ldots,T-1)$ within the kth observation interval (or snapshot) is short time Fourier-transformed

$$\underline{X}^{(k)}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} w(t)\underline{x}(t)e^{-j\omega t} \tag{1}$$

where $\{w(t)\}_{t=0}^{T-1}$ is a window function. For large number of samples T, we can describe the frequency domain data approximately by the following relation

$$\underline{X}^{(k)}(\omega) = \boldsymbol{H}(\omega;\underline{\theta})\underline{S}^{(k)}(\omega) + \underline{U}^{(k)}(\omega) \tag{2}$$

where the matrix $\boldsymbol{H}(\omega,\underline{\theta}) = [\underline{d}_1(\omega)\cdots\underline{d}_i(\omega)\cdots\underline{d}_m(\omega)] \in \mathbb{C}^{n\times m}$ consists of m steering vectors with the ith column $\underline{d}_m(\omega)$ corresponding to the ith incoming wave. In the following analysis, the signal waveform $\underline{S}^{(k)}(\omega)$ is considered to be unknown and deterministic. The noise $\underline{U}^{(k)}(\omega)$ results only from sensors. According to the asymptotic theory of Fourier transform, $\underline{X}^{(k)}(\omega_j)$, $(k = 1,\ldots,K, j = 1,\ldots,J)$ are independent, identically complex normally distributed with mean $\boldsymbol{H}(\omega_j;\underline{\theta})\underline{S}^{(k)}(\omega_j)$ and covariance matrix $\nu(\omega_j)\boldsymbol{I}$ where $\nu(\omega_j)$ is the unknown noise spectral parameter and $\boldsymbol{I}$ is an identity matrix of corresponding dimension. The problem of central interest is to determine the number of signals m embedded in the observations.

3. SIGNAL DETECTION USING A MULTIPLE HYPOTHESIS TEST

We formulate the problem of detecting the number of signals as a multiple hypothesis test. Let M denote the maximal number of sources. The following procedure detects one signal after another. More precisely, for $m = 1$,

$$\begin{aligned} H_1 \quad &: \quad \text{Data contains only noise.} \\ &\quad\ \underline{X}(\omega) = \underline{U}(\omega) \\ A_1 \quad &: \quad \text{Data contains at least 1 signals.} \\ &\quad\ \underline{X}(\omega) = \boldsymbol{H}_1(\omega;\underline{\theta}_1)\underline{S}_1(\omega) + \underline{U}(\omega) \end{aligned} \quad (3)$$

For $m = 2, \ldots, M$

$$\begin{aligned} H_m \quad &: \quad \text{Data contains at most } (m-1) \text{ signals.} \\ &\quad\ \underline{X}(\omega) = \boldsymbol{H}_{m-1}(\omega;\underline{\theta}_{m-1})\underline{S}_{m-1}(\omega) + \underline{U}(\omega) \\ A_m \quad &: \quad \text{Data contains at least } m \text{ signals.} \\ &\quad\ \underline{X}(\omega) = \boldsymbol{H}_m(\omega;\underline{\theta}_m)\underline{S}_m(\omega) + \underline{U}(\omega) \end{aligned} \quad (4)$$

We use the subscript $(m-1)$ or m for the steering matrix $\boldsymbol{H}(\omega, \underline{\theta})$ and the signal vector $\underline{S}(\omega)$ to emphasize their dimensions under the hypothesis H_m or alternative A_m.

Based on the likelihood ratio (LR) principle, we obtain the test statistic $T_m(\hat{\underline{\theta}}_m)$, $(m = 1, \ldots, M)$ as follows.

$$\begin{aligned} T_m(\hat{\underline{\theta}}_m) &= \frac{1}{J}\sum_{j=1}^{J} \log\left(\frac{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_{m-1}(\omega_j;\hat{\underline{\theta}}_{m-1}))\hat{\boldsymbol{R}}]}{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j;\hat{\underline{\theta}}_m))\hat{\boldsymbol{R}}]}\right) \\ &= \frac{1}{J}\sum_{j=1}^{J} \log\left(1 + \frac{n_1}{n_2}F_m(\omega_j;\hat{\underline{\theta}}_m)\right) \end{aligned} \quad (5)$$

where $\hat{\boldsymbol{R}} = \frac{1}{K}\sum_{k=1}^{K} \underline{X}^{(k)}(\omega_j)\underline{X}^{(k)}(\omega_j)^H$ represents a nonparametric power spectral estimate of sensor outputs over K snapshots and $\boldsymbol{P}(\omega_j;\hat{\underline{\theta}}_m)$ is the projection matrix onto the subspace spanned by the columns of $\boldsymbol{H}_m(\omega_j;\hat{\underline{\theta}}_m)$. When $m = 1$, we define $\boldsymbol{P}_0(\cdot) = \mathbf{0}$. $\hat{\underline{\theta}}_m$ represents the ML estimate assuming that m signals are present in the observation.

Under hypothesis H_m, the statistic $F_m(\omega_j;\hat{\underline{\theta}}_m)$ is F_{n_1,n_2}–distributed where the degrees of freedom n_1, n_2 are given by [7]

$$n_1 = K(2 + r_m), \quad n_2 = K(2r_{\underline{x}} - 2m - r_{m-1}) \quad (6)$$

with $r_{\underline{x}} = \dim(\underline{x}(t)) = n$ and $r_m = \dim(\underline{\theta}_m) = m$.

From eq. (5) it is easy to see that in the narrow band case, $J = 1$, the LR test is equivalent to the F-test proposed by Shumway [9]. However, in the broadband case, the distribution of the test statistic $T_m(\hat{\underline{\theta}}_m)$ under H_m can not be expressed in closed form. To overcome this difficulty, we shall apply the bootstrap method to approximate the distribution of $T_m(\hat{\underline{\theta}}_m)$.

4. THE BOOTSTRAP PRINCIPLE

The bootstrap [5], [10] requires little prior knowledge on the data model. The key idea of bootstrap is that, rather than repeating the experiment, one obtains the "samples" by reassignment of the original data samples. We give a brief description of the basic concept and then introduce our test procedure. For more details, the reader is referred to [10] and references therein.

Basic concept

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_M\}$ be an i.i.d. sample set from a completely unspecified distribution F. Let ϑ denote an unknown parameter, such as the mean or variance, of F. The goal of the following procedure is to construct the distribution of an estimator $\hat{\vartheta}$ derived from $\mathcal{Z}$.

The bootstrap principle

1. Given a sample set $\mathcal{Z} = \{z_1, z_2, \ldots, z_M\}$
2. Draw a bootstrap sample $\mathcal{Z}^* = \{z_1^*, z_2^*, \ldots, z_M^*\}$ from $\mathcal{Z}$ by resampling with replacement.
3. Compute the bootstrap estimate $\hat{\vartheta}^*$ from $\mathcal{Z}^*$.
4. Repeat 2. and 3. to obtain B bootstrap estimates
$$\hat{\vartheta}_1^*,\ \hat{\vartheta}_2^*, \ldots,\ \hat{\vartheta}_B^*.$$
5. Approximate the distribution of $\hat{\vartheta}$ by that of $\hat{\vartheta}^*$.

In step 2., a pseudo random number generator is used to draw a random sample of M values, with replacement, from $\mathcal{Z}$. A possible bootstrap sample might look like $\mathcal{Z}^* = \{z_{10}, z_8, z_8, \ldots, z_2\}$. Given the sample set $\mathcal{Z}$, the bootstrap procedure can be easily adapted to calculate a confidence interval of $\hat{\vartheta}$ or construct a hypothesis test.

For the problem testing the hypothesis $H_0 : \vartheta = \vartheta_0$ against $H_0 : \vartheta \neq \vartheta_0$, we define the test statistic as

$$\hat{\mathcal{T}} = \frac{|\hat{\vartheta} - \vartheta_0|}{\hat{\sigma}} \quad (7)$$

where $\hat{\sigma}^2$ is an estimator of the variance of $\hat{\vartheta}$. $\hat{\sigma}^2$ can be obtained through direct computation or nested bootstrap [10]. The inclusion of $\hat{\sigma}$ guarantees $\hat{\mathcal{T}}$ is asymptotically pivotal. Given a significance level α, the bootstrap test computes the threshold t_α based on the bootstrap approximation for the distribution of $\hat{\mathcal{T}}$ under H_0 [10].

When the observed significance level, denoted by p , is desired, for example, in the Benjamini-Hochberg procedure, one uses bootstrap samples to estimate the p-value through the relation

$$\hat{p} = P\Big\{ \frac{|\hat{\vartheta}^* - \hat{\vartheta}|}{\hat{\sigma}^*} \geq \frac{|\hat{\vartheta} - \vartheta_0|}{\hat{\sigma}} \Big\}. \quad (8)$$

where $P\{\cdot\}$ denotes the probability that the bootstrap estimates larger than the observed test statistic $\hat{\mathcal{T}}$. This method proceeds as follows.

The bootstrap procedure for estimation of p-values

1. *Resampling*: Draw a bootstrap sample $\mathcal{Z}^*$.
2. Compute the bootstrap statistic
$$\hat{\mathcal{T}}^* = \frac{|\hat{\vartheta}^* - \vartheta_0|}{\hat{\sigma}^*}.$$
3. Repeat 1. and 2. to obtain B bootstrap statistics.
4. *Ranking*: $\hat{\mathcal{T}}^*_{(1)} \leq \hat{\mathcal{T}}^*_{(2)} \leq \ldots \leq \hat{\mathcal{T}}^*_{(B)}$
5. Choose L so that
$$\hat{\mathcal{T}}^*_{(L-1)} \leq \hat{\mathcal{T}} \leq \hat{\mathcal{T}}^*_{(L)} \ldots \leq \hat{\mathcal{T}}^*_{(B)}.$$
Estimate the observed p-value by $\hat{p} = L/B$.

Application to multiple signal detection

The test statistic $T_m(\hat{\theta}_m)$ in eq. (5) is the sample mean of J samples

$$Z_j = \log\left(1 + \frac{n_1}{n_2} F_m(\omega_j; \underline{\hat{\theta}}_m)\right), (j = 1, \ldots, J). \qquad (9)$$

We consider Z_j, $(j = 1, \ldots, J)$ as i.i.d. samples of the random variable $Z_m = \log\left(1 + \frac{n_1}{n_2} F_m\right)$ because (1) $\underline{X}(\omega_j)$, $(j = 1, \ldots, J)$ are asymptotically independent. (2) $F_m(\cdot)$, $(j = 1, \ldots, J)$ are F_{n_1,n_2}-distributed. Furthermore, under H_m, the mean of Z_m is given by [8]

$$\mu_m = EZ_m = \Psi(\frac{n_1}{2} + \frac{n_2}{2}) - \Psi(\frac{n_2}{2}) \qquad (10)$$

where $\Psi(s) = (\log \Gamma(s))'$ represents the first derivative of logarithm of the gamma function.

Under the bootstrap framework, the hypothesis test (4) can be reformulated as

$$\begin{aligned} H_m : \quad & E[\, T_m(\hat{\theta}_m)] = \mu_m \\ A_m : \quad & E[\, T_m(\hat{\theta}_m)] \neq \mu_m \,. \end{aligned}$$

The test statistic $T_m(\hat{\theta}_m)$ is an estimator for the mean. Then we apply the bootstrap technique discussed previously to find the observed significance value of the statistic $T_m(\hat{\theta}_m)$ in eq. (5).

5. CONTROL OF THE FALSE DISCOVERY RATE

The control of type one error is an important issue in multiple inferences. A type one error occurs when the null hypothesis H_m is wrongly rejected. The traditional concern in multiple hypothesis problems has been about controlling the familywise error-rate (FWE). Given a certain significance level α, the control of FWE requires each of the M tests to be conducted at a lower level. When the number of tests increases, the power of the FWE-controlling procedures such as Bonferroni-type procedures [6] is substantially reduced. The false discovery rate (FDR), suggested by Benjamini and Hochberg [1], is a completely different point of view for considering the errors in multiple testing. The FDR is defined as the expected proportion of errors among the rejected hypotheses. If all null hypotheses $\{H_1, H_2, \ldots, H_M\}$ are true, the FDR-controlling procedure controls the traditional FWE. But when many hypotheses are rejected, an erroneous rejection is not as crucial for drawing conclusion from the whole family of tests, the FDR is a desirable error rate to control.

Assume that among the M tested hypotheses $\{H_1, H_2, \ldots, H_M\}$, m_0 are true null hypotheses. Let $\{p_1, p_2, \ldots, p_M\}$ be the p-values (observed significance values) corresponding to the test statistics $\{T_1, T_2, \ldots, T_M\}$. By definition, $p_m = 1 - P_{H_m}(T_m)$ where P_{H_m} is the distribution function under H_m. Benjamini and Hochberg showed that when the test statistics *corresponding to the true null hypotheses* are independent, the following procedure controls the FDR at level $q \cdot m_0/M \leq q$[1].

The Benjamini Hochberg Procedure

Define
$$k = \max\left\{m : p_{(m)} \leq \frac{m}{M} q\right\} \qquad (11)$$
and reject $H_{(1)} \ldots H_{(k)}$. If no such k exists, reject no hypothesis.

6. SIMULATION

We test the proposed algorithms by numerical experiments. A uniformly linear array of 15 sensors with inter-element spacings of half a wavelength is considered. The number of selected frequency bins $J = 10$ and the number of snapshots $K = 5$. Each experiment performs 100 trials.

In the first experiment, the broadband signals are generated by $m = 12$ sources of equal amplitudes. The SNR varies from -14 dB to 6 dB in a 1 dB step. For comparison, the simulated data is applied to the Bonferroni-Holm procedure [6] as well. The sequentially rejective Bonferroni-Holm procedure keeps the FWE at the same level α as the classical Bonferroni test but is more powerful than it. The significance level of each test is given by $\alpha/(M+1-m)$. We use $q = 0.05$ and $\alpha = 0.05$ in the simulation.

Fig. 1 presents results for 12 sources that are well separated in arrival angle. The probability of correct detection $P(\hat{m} = 12)$ increases with increasing SNRs. The FDR-controlling procedure has a lower SNR threshold and a higher probability of detection in the threshold region.

In the second experiment, we consider a more difficult scenario: two of the signal sources are closely located at $\theta_1 = 9°$ and $\theta_2 = 12°$ relative to broadside. From the results shown in fig. 2 we can easily observe that both procedures perform worse than in fig. 1. The gap between FDR- and FWE-controlling procedures has widened in the threshold region. This implies that the FDR-controlling procedure is more useful in critical situations.

When we apply both procedures to simulated data containing only noise, the probability of correct decision (implying no signal is detected), $P(\hat{m} = 0)$, is always 1.

In summary, the FDR-controlling procedure leads to a higher probability of correct detection than the FWE-controlling procedure. In particular, for situations involving closely located signals, the difference between these two procedures become larger. In the noise only case, the proposed detection scheme has a false alarm rate of 0 for a choice of FDR $q = 0.05$.

7. CONCLUSION

We discuss broadband signal detection using a multiple hypothesis test under an FDR consideration of Benjamini and Hochberg. Compared to the classical FWE criterion, the FDR criterion leads to more powerful tests and controls the errors at a reasonable level. Unlike the narrow band case where the test statistics are characterized by F-distribution, the test statistics have no closed form distribution in the broadband case. We apply the bootstrap technique to determine the distribution numerically. Simulations show that the FDR-controlling procedure has always a higher probability of detection than the FWE controlling procedure. More importantly, the reliability of the proposed test is not affected by the gain in power.

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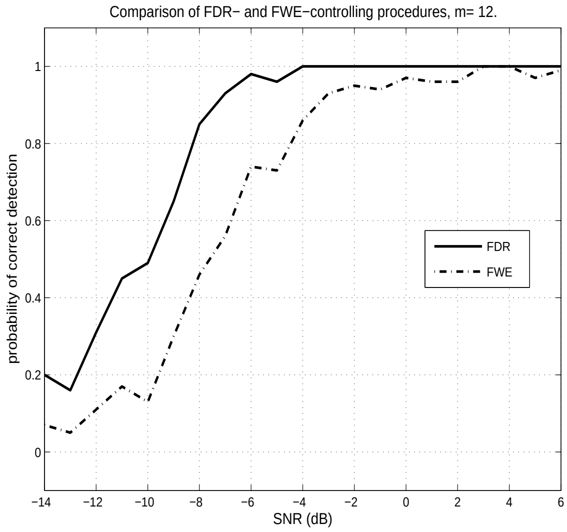

Fig. 1. Probability of correct detection. $M = 12$, SNR $= [-14 : 1 : 6]$ dB, number of frequency bins $J = 10$, number of snapshots $K = 5$. All sources are apart more than 7° .

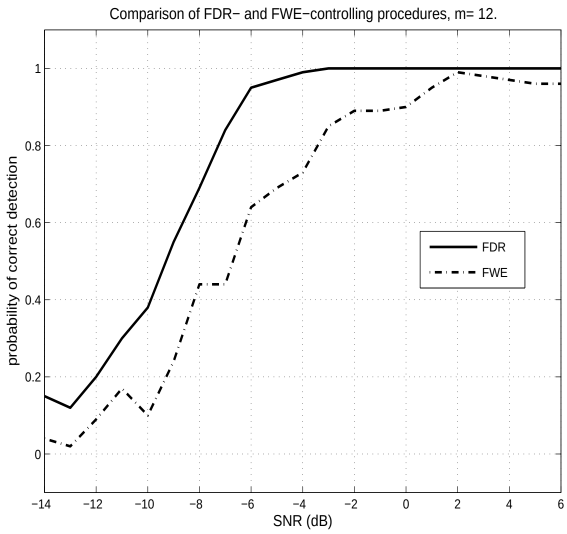

Fig. 2. Probability of correct detection. $M = 12$, SNR $= [-14 : 1 : 6]$ dB, number of frequency bins $J = 10$, number of snapshots $K = 5$. Two sources are closely located.