Correction to 'On decentralized estimation with active queries'

David A. Castañón and Theodoris Tsiligkaridis and Alfred O. Hero III

***Abstract*—In this paper, we provide a counterexample to a key lemma used in the proofs of the convergence of decentralized estimation algorithms in [2]. We also provide an alternative lemma that establishes a new proof of the convergence results in the paper [2].**

I. Introduction

The problem of random binary search by a collection of agents has been studied recently in a number of papers [1], [2], [3]. An interesting approach, proposed in Tsiligkaridis et al [2], described a protocol whereby agents do not share their measurements, but instead perform local processing and exchange probability measures, referred to as belief densities, of the location of the object of interest. These belief densities evolve as an average of the Bayesian update using the agent's individual measurements and the belief densities received from the neighbors. The main results of [2] establish that, as the number of measurements increase, each agent's belief density converges to a common density. Furthermore, these belief densities converge to a distribution that is concentrated at the true object location.

The purpose of this note is to provide a counterexample to a key lemma that is used in the proof of the main convergence result in [2]. After discussing the counterexample, we prove a new lemma that leads to an alternate proof of the main results in [2].

II. Formulation

We follow closely the notation of [2]. Define X be a random variable denoting the true target location in $\mathcal{X} = [0,1]$, with $\mathcal{B}(\mathcal{X})$ denoting the Borel-measurable subsets of $\mathcal{X}$. There are M agents, which seek to locate the target using a random binary search model introduced in [4] as follows: at discrete times t, each agent i selects an interval $A_{i,t} = [0, X_{i,t}]$ and constructs the binary query "is $X \in A_{i,t}$"? Agent i receives a noisy binary response $Y_{i,t+1}$ to the query, which is correct with probability $1-\epsilon_i$, where $\epsilon_i < 0.5$. We assume the existence of an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that generates the target location X and the observations $Y_{i,t}$, $i = 1, \ldots, M, t = 1, 2, \ldots$.

Let $\mathbf{Y}_t$ denote the vector of observations $[Y_{1,t}, \ldots, Y_{M,t}]^T$, and $\mathcal{A}_t$ denote the collection of queries at time t. We assume that the components of $\mathbf{Y}_{t+1}$ are conditionally independent given $X = x, \mathcal{A}_t$, so that

$$\mathbb{P}(\mathbf{Y}_{t+1} = \mathbf{y}_{t+1} | x, \mathcal{A}_t) = \prod_{i=1}^{M} \mathbb{P}(y_{i,t+1} | x, A_{i,t}).$$

Furthermore, we assume conditional independence of $\mathbf{Y}_t$ across time, so that

$$\mathbb{P}(\mathbf{y}_{t+1}, \mathbf{y}_{s+1}, | x, \mathcal{A}_t, \mathcal{A}_s) = \mathbb{P}(\mathbf{y}_{t+1} | x, \mathcal{A}_t) \mathbb{P}(\mathbf{y}_{s+1} | x, \mathcal{A}_s),$$

for all $s \neq t$.

We define the sequence of event spaces $\mathcal{F}_t, t \geq 1$ to be the σ-field generated by the random variables $\mathbf{Y}_1, ..., \mathbf{Y}_t$, along with the corresponding queries $\mathcal{A}_1, \ldots, \mathcal{A}_{t-1}$. Note that this is an increasing sequence ($\mathcal{F}_t \subset \mathcal{F}_{t+1}$). Denote by $l_i(y|x, A_{i,t}) = \mathbb{P}(y_{i,t+1}|x, A_{i,t})$ the probability that $Y_{i,t+1} = y$ given the true location $X = x$ and the query $A_{i,t}$. Then,

$$l_i(y|x, A_{i,t}) = \begin{cases} (1-\epsilon_i) I(x \in A_{i,t}) + \epsilon_i I(x \notin A_{i,t}), & y = 1 \\ \epsilon_i I(x \in A_{i,t}) + (1-\epsilon_i) I(x \notin A_{i,t}), & y = 0. \end{cases} \quad (1)$$

where $I(\cdot)$ is the indicator function. Define $\mathcal{Z}_{i,t}(y)$ as

$$\mathcal{Z}_{i,t}(y) = \int_{x \in \mathcal{X}} l_i(y|x, A_{i,t}) p_{i,t}(x) dx. \quad (2)$$

Each agent i keeps a probability density $p_{i,t}(x)$ on $\mathcal{X}$, which is its belief density on the location X of the target. Initially, each agent knows $p_{i,0}(x)$, which is assumed to be strictly positive on $[0,1]$. At each time $t = 0, 1, \ldots$, agent i selects its query point $X_{i,t}$ to be the median of its belief density, and generates query $A_{i,t}$ to collect observation $Y_{i,t+1}$ with value $y_{i,t+1}$.With such choice of query points, it is shown in [2] that $\mathcal{Z}_{i,t}(0) = \mathcal{Z}_{i,t}(1) = 0.5$ for all $i = 1, \ldots, m, t \geq 0$. The belief densities evolve over time as observations are collected by the agents according to the following social learning update rule

$$p_{i,t+1}(x) = a_{i,i} p_{i,t}(x) \frac{l_i(y_{i,t+1}|x, A_{i,t})}{\mathcal{Z}_{i,t}(y_{i,t+1})} + \sum_{j=1, j\neq i}^{M} a_{i,j} p_{j,t}(x), \quad (3)$$

where a_{ij} are non-negative coefficients with $a_{i,i} > 0$ and $\sum_{j=1}^{M} a_{i,j} = 1$. The matrix $\mathbf{A}$ is a stochastic matrix, called the social interaction matrix, which we assume is irreducible, corresponding to a single strongly connected class of agents.

The evolution (3) can be summarized in vector form as in [2]:

$$\mathbf{p}_{t+1}(x) = (\mathbf{A} + \mathbf{D}_t(x)) \mathbf{p}_t(x), \quad (4)$$

where $\mathbf{D}_t(x)$ is a diagonal matrix with elements

$$[\mathbf{D}_t(x)]_{i,i} = a_{i,i} \left(\frac{l_i(y_{i,t+1}|x_t, A_{i,t})}{\mathcal{Z}_{i,t}(y_{i,t+1})} - 1 \right).$$

Note $\mathbf{p}_t(x)$ is measurable with respect to the event space $\mathcal{F}_t$.

With this notation, Lemma 1 in [2] claims the following:

Lemma 1: [2] For any $B \in \mathcal{B}(\mathcal{X})$, we have

$$\mathbb{E}\left\{ \int_B \mathbf{D}_t(x) \mathbf{p}_t(x) dx \middle| \mathcal{F}_t \right\} = 0. \quad (5)$$

We provide a counterexample to the above lemma below. Assume two agents ($M = 2$) with uniform initial belief densities $p_{1,0}(x) = p_{2,0}(x) = 1$. Assume the true density of X is also uniform in $[0,1]$. Let $\epsilon_1 = \epsilon_2 = 0.25$. Then,

$$\mathbb{P}(y_{i,t+1}|x, A_{i,t}) = \begin{cases} \frac{1}{2} I(y_{i,t+1} = 1) + \frac{1}{4} & x \in A_{i,t} \\ \frac{1}{2} I(y_{i,t+1} = 0) + \frac{1}{4} & x \notin A_{i,t} \end{cases}$$

Let $\mathbf{A} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$.

According to the algorithm above, the medians are $X_{1,0} = X_{2,0} = 0.5$, leading to queries $A_{1,0} = A_{2,0} = [0, 0.5]$. Assume that the resulting measurements are $y_{1,1} = y_{2,1} = 1$. Then, using (3), we obtain

$$p_{1,1}(x) = \frac{5}{4} I(x \in [0, 0.5]) + \frac{3}{4} I(x \in [0.5, 1])$$
$$p_{2,1}(x) = \frac{5}{4} I(x \in [0, 0.5]) + \frac{3}{4} I(x \in [0.5, 1]).$$

D. Castañón is with Boston University, Boston, MA, email dac@bu.edu.
T. Tsiligkaridis is with MIT Lincoln Laboratory, Lexington, MA.
A. Hero is with the University of Michigan, Ann Arbor, MI.

The event space $\mathcal{F}_1$ is generated by actions and measurements $\{A_{1,0} = A_{2,0} = [0, 0.5], y_{1,1}, y_{2,1}\}$. Using Bayes' rule, the conditional probability density $p(x|\mathcal{A}_0, y_{1,1} = y_{2,1} = 1)$ is

$$p(x|\mathbf{A}_0, y_{1,1} = y_{2,1} = 1) = \frac{9}{5} I(x \in [0, 0.5]) + \frac{1}{5} I(x \in (0.5, 1]).$$

At time $t = 1$, the new queries are $A_{1,1} = A_{2,1} = [0, 0.4]$, based on the medians for $p_{1,1}(x), p_{2,1}(x)$, so that $\mathcal{Z}_{i,1}(y_{i,2}) = 0.5$. Note that these do not correspond to the median of the true probability density $p(x|\mathbf{A}_0, y_{1,1} = y_{2,1} = 1)$. To show that the conclusion of the lemma is incorrect, let $B = [0, 0.4]$ also. Then,

$$\begin{aligned}\int_B [\mathbf{D}_1(x)]_{1,1} p_{1,1}(x)dx &= 0.5 \int_B [2l(y_{1,2}|x, A_{1,1}) - 1] p_{1,1}(x)dx \\ &= 0.5 \int_0^{0.4} [2l(y_{1,2}|x, A_{1,1}) - 1] \frac{5}{4} dx \\ &= \frac{5}{8} \int_0^{0.4} [I(y_{1,2} = 1) - \frac{1}{2}] dx \\ &= \frac{1}{4} I(y_{1,2} = 1) - \frac{1}{8},\end{aligned}$$

because $B = A_{1,1} = A_{2,1}$. For the event $F \subset \mathcal{F}_1$ generated by $y_{1,1} = y_{2,1} = 1$, we have

$$\begin{aligned}&\mathbb{P}(y_{1,2} = 1|F, A_{1,1}) = \\ &\int_0^1 I(y_{1,2} = 1)\mathbb{P}(y_{1,2} = 1|x, F, A_{1,1}) p(x|F)dx \\ &= \frac{9}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{9}{5} \cdot \frac{1}{10} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{108 + 9 + 5}{200} = \frac{122}{200}.\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left\{ \int_B [\mathbf{D}_1(x)]_{1,1} p_{1,1}(x)dx \middle| F \right\} &= \frac{1}{4} [\mathbb{P}(y_{1,2} = 1|F, A_{1,1}) - \frac{1}{2}] \\ &= \frac{1}{4} \cdot \frac{22}{200} = \frac{11}{400} \neq 0.\end{aligned}$$

This contradicts the Lemma. The proof in [2] used the belief densities $p_{1,1}(x)$ and $p_{2,1}(x)$ instead of $p(x|\mathcal{F}_1)$ in computing the conditional expectation in (5), leading to the erroneous conclusion. The essence of the counterexample is to construct a situation where the local beliefs $p_{k,1}(x)$ differ from the centralized conditional probability $p(x|\mathcal{F}_1)$, so that the agents do not pick an accurate estimate of the median region. This situation is generic in these cases, so almost any choice of numbers in the above example would result in a counterexample. Note that Lemma 1 was needed for the proof of Lemma 2 in [2], so that lemma is also incorrect.

We establish a different lemma that can be used to prove the main results of [2] following the arguments in [5]. First, we define a different filtration $\mathcal{F}'_t$, consisting of the event space generated by the random variables $X, \mathbf{Y}_1, \ldots, \mathbf{Y}_t$ and the associated queries $\mathcal{A}_0, \ldots, \mathcal{A}_{t-1}$. Note that this filtration includes knowledge of X. Let $\mathbf{P}_t(B) = \int_B \mathbf{p}_t(x)dx$. Since $\mathcal{F}_t \subset \mathcal{F}'_t$, $\mathbf{P}_t(B)$ is measurable with respect to $\mathcal{F}'_t$. The new lemma is:

Lemma 2: Let B be a Borel set in $\mathcal{B}(\mathcal{X})$. Then, there exists a positive vector $\mathbf{v}$ such that

$$\mathbb{E}\{\mathbf{v}^T \mathbf{P}_{t+1}(B)|\mathcal{F}'_t\} \geq \mathbf{v}^T \mathbf{P}_t(B).$$

Furthermore, $\lim_{t\to\infty} \mathbf{v}^T \mathbf{P}_t(B)$ exists almost surely.

Proof: As in [2], select $\mathbf{v}$ as a strictly positive left eigenvector of the social interaction matrix $\mathbf{A}$ corresponding to the eigenvalue 1. Such eigenvector exists by the connectivity assumptions among the agents in [2] that guarantee that the interaction matrix $\mathbf{A}$ is an irreducible stochastic matrix. Then, by (3),

$$\begin{aligned}\mathbf{v}^T \mathbf{P}_{t+1}(B) &= \mathbf{v}^T \int_B \mathbf{p}_{t+1}(x)dx \\ &= \mathbf{v}^T \mathbf{A} \int_B \mathbf{p}_t(x)dx + \mathbf{v}^T \int_B \mathbf{D}_t(x)\mathbf{p}_t(x)dx \\ &= \mathbf{v}^T \int_B \mathbf{p}_t(x)dx + \mathbf{v}^T \int_B \mathbf{D}_t(x)\mathbf{p}_t(x)dx.\end{aligned}$$

Using the definition of $\mathbf{D}_t(x)$ and $\mathcal{Z}_{i,t}(y) = 0.5$, we get

$$\begin{aligned}&E\left\{ \int_B [\mathbf{D}_t(x)]_{i,i} p_{i,t}(x)dx \middle| \mathcal{F'}_t \right\} \\ &\quad = a_{i,i} \int_B p_{i,t}(x) E\{[2\mathbb{P}(y_{i,t+1}|x, A_{i,t}) - 1]|\mathcal{F}'_t\}dx.\end{aligned}$$

Since the function $1/x$ is convex for non-negative x, Jensen's inequality yields

$$E\{2\mathbb{P}(y_{i,t+1}|x, A_{i,t})|\mathcal{F}'_t\} \geq \left[E\left\{ \frac{1}{2\mathbb{P}(y_{i,t+1}|x, A_{i,t})} \middle| \mathcal{F}'_t \right\} \right]^{-1}.$$

Using the conditional independence assumptions, since $\mathcal{F}'_t$ includes X as a generator, we obtain $\mathbb{P}(y_{i,t+1}|\mathcal{F}'_t) = \mathbb{P}(y_{i,t+1}|x, A_{i,t})$. Thus,

$$\begin{aligned}&E\left\{ \frac{1}{2\mathbb{P}(y_{i,t+1}|x, A_{i,t})} \middle| \mathcal{F}'_t \right\} \\ &\quad = \sum_{y_{i,t+1}=0}^{1} \frac{1}{2\mathbb{P}(y_{i,t+1}|x, A_{i,t})} \mathbb{P}(y_{i,t+1}|x, A_{i,t}) = 1.\end{aligned}$$

This implies

$$\int_B p_{i,t}(x) E\{[2\mathbb{P}(y_{i,t+1}|x, A_{i,t}) - 1]|\mathcal{F}'_t\}dx \geq 0,$$

for every i, so

$$E\left\{ \mathbf{v}^T \int_B D_t(x)\mathbf{p}_t(x)dx \middle| \mathcal{F}'_t \right\} \geq 0.$$

The above inequality establishes that $\mathbf{v}^T \mathbf{P}_t(B)$ a submartingale with respect to the filtration $\mathcal{F}'_t$, bounded above by the L_1 norm of $\mathbf{v}^T$, and hence it converges almost surely, establishing the Lemma.

Lemma 2 is sufficient to establish the rest of the results in [2]. Let $\xi_t(B) = e^{\mathbf{v}^T \mathbf{P}_t(B)}$. Then,

$$\begin{aligned}E\{\xi_{t+1}(B)|\mathcal{F}'_t\} &= E[e^{\mathbf{v}^T \mathbf{P}_{t+1}(B)}|\mathcal{F}'_t] \\ &\geq e^{E\{\mathbf{v}^T \mathbf{P}_{t+1}(B)|\mathcal{F}'_t\}} \geq e^{\mathbf{v}^T \mathbf{P}_t(B)} = \xi_t(B),\end{aligned}$$

so $\xi_t(B)$ is a non-negative submartingale that is bounded by $e^{\|\mathbf{v}\|_1}$, so it converges almost surely, as required in the proof of Lemma 3 in [2], which provides the foundation for the remaining proofs in the paper.

References

[1] T. Tsiligkaridis, B. M. Sadler, and A. O. Hero III, "Collaborative 20 questions for target localization," *IEEE Trans. Information Theory*, vol. 60, no. 4, pp. 2233–2252, 2014.

[2] ——, "On decentralized estimation with active queries," *IEEE Trans. Signal Processing*, vol. 63, no. 10, pp. 2610–2622, 2015.

[3] H. Ding and D. A. Castañón, "Optimal solutions for classes of adaptive search problems," in *Proc. IEEE Conf. Decison and Control*, Osaka, Japan, December 2015.

[4] B. Jedynak, P. I. Frazier, and R. Sznitman, "Twenty questions with noise: Bayes optimal policies for entropy loss," *Journal of Applied Probability*, vol. 49, pp. 114–136, 2011.

[5] A. Jadbabaie, P. Molavi, A. Sandroni, and A. Tabaz-Salehi, "Non-Bayesian social learning," *Games Econ. Behav.*, vol. 76, pp. 210–225, 2012.