MIMO Environmental Capacity Sensitivity

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Abstract

Wireless communications using multiple input multiple output (MIMO) systems enable increased spectral efficiency for a given total transmit power. The increased capacity is achieved through the introduction of additional spatial channels (space-time coding). In this paper, MIMO capacity is calculated as a function of environmental factors, including channel complexity, external interference, and channel estimation error. Capacity of MIMO systems, where both transmitter and receiver know the channel (channel estimate feedback), is compared with single input multiple output (SIMO) and MIMO systems, where only the receiver knows the channel. Channel complexity is studied using a simple statistical physical scattering model. Finally, an expression for capacity loss particular to channel estimation error at the transmitter is introduced.

1 Introduction

Multiple input multiple output (MIMO) systems are a natural extension of developments in antenna array communications. While the advantages of multiple receive antennas, such as gain and spatial diversity, have been known and exploited for some time [5, 1], the advantages of MIMO communications, exploiting the physical channel between many transmit and receive antennas, have recently received significant attention [2]. While it is possible for the channel to be so nonstationary that it cannot be estimated in any useful sense [4], in this paper a quasistationary channel assumption will be employed. In implementing MIMO systems one must decide whether channel estimation information will be fed back to the transmitter so that it can adapt. Most MIMO communications research has focused on systems without feedback. A MIMO system with an *uninformed transmitter* (without feedback) is logistically simpler to implement, and at high signal-to-noise-ratio (SNR) its capacity approaches that of an *informed transmitter*. If the system must operate over a range of SNR, incorporating feedback may be a useful option. The *informed transmitter* approach suffers from increased sensitivity to channel stationarity as the channel must be stationary long enough for it to be estimated and for the estimate to be fed back. In this paper the narrowband capacity of 1-to-M single input multiple output (SIMO), *uninformed transmitter* M-to-M, and *informed transmitter* M-to-M MIMO systems are compared as a function of environment and channel estimation error, where M is the number of antennas.

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1.1 MIMO

MIMO systems provide a number of advantages over single antenna communications. Sensitivity to fading is reduced by the spatial diversity provided by multiple spatial paths. Under certain environmental conditions, the power requirements associated with high spectral efficiency communications can be significantly reduced by avoiding the compressive region of the information theoretic capacity. Capacity increases linearly with SNR at low SNR, but increases logarithmically with SNR at high SNR. A given total transmit power can be divided among multiple spatial paths (or modes), driving the capacity closer to the linear regime for each mode, thus increasing the aggregate spectral efficiency. As seen in Figure 1, which assumes an optimal MIMO channel (full rank channel matrix), MIMO systems enable high spectral efficiency at much lower required energy per bit. Finally, because MIMO systems use antenna arrays, interference can be naturally mitigated.

1.2 Environment

The environmental factors that affect MIMO system capacity, channel complexity, external interference, and channel estimation error, are addressed in this paper.

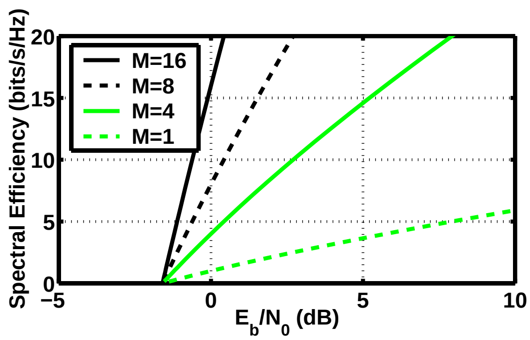

Figure 1. Spectral efficiency as a function of energy per bit comparison of M×M MIMO systems in an ideal environment.

The first category, channel complexity, is a function of the richness of environmental scatterers. In general, capacity increases as the singular values of the channel matrix increase. The distribution of singular values is a measure of the usefulness of various spatial paths through the channel.

The second category, external interference, adversely affects the usefulness of various paths through the channel. Given that the useful portion of the channel lives in a subspace of the channel matrix, capacity loss is a function of the overlap of the interference with this subspace.

The third category is channel estimation error. If the environment is stationary, then asymptotically channel estimation error vanishes. However, in practical systems channel stationarity limits the useful period over which a channel can be estimated. The relative capacity loss of an *informed* versus *uninformed transmitter* due to channel estimation error is considered.

1.3 Information Theoretic Capacity

The information theoretic capacity of MIMO systems has been widely discussed, for example in [6]. The development of the *informed transmitter* "water filling" approach is repeated here as an introduction.

1.3.1 Informed Transmitter (IT)

For narrowband MIMO systems, the coupling between the transmitter and receiver can be modeled using

$$\vec{z} = \mathbf{H}\vec{x} + \vec{n} \,, \qquad (1)$$

where $\vec{z}$ is the complex receive array output, $\mathbf{H}$ is the $n_{Rx} \times n_{Tx}$, number of receive by transmit antenna, channel correlation matrix, $\vec{x}$ is the transmit array vector, and $\vec{n}$ is additive Gaussian noise.

Capacity is determined by maximizing the mutual information given by $h(\vec{z}) - h(\vec{z}|\vec{x})$, where the entropy for $\vec{z}$, assuming Gaussian distributions, is given by

$$h(\vec{z}) = \log_2 \left| < \vec{z}\vec{z}^\dagger > \right| + const, \qquad (2)$$

$$h(\vec{z}|\vec{x}) = \log_2 \left| < \vec{n}\vec{n}^\dagger > \right| + const. \qquad (3)$$

The expectation value and determinant are indicated using the notations $< \cdots >$ and $|\cdots|$, respectively. Assuming spatially white additive Gaussian noise with power σ_n^2 per array element, the capacity (bit/s/Hz) is given by optimizing over available parameters:

$$C = \sup \log_2 \frac{|\sigma_n^2 \mathbf{I} + \mathbf{H} < \vec{x}\vec{x}^\dagger > \mathbf{H}^\dagger|}{|\sigma_n^2 \mathbf{I}|} . \qquad (4)$$

If the transmitter and receiver have accurate estimates of the channel matrix, the theoretical capacity of a MIMO system, assuming a total transmit power, P_o, is

$$C_{IT} = \sup_{\mathbf{P};\, \mathrm{tr}(\mathbf{P}) = P_o} \log_2 |\mathbf{I} + \mathbf{H}\mathbf{P}\mathbf{H}^\dagger| \,. \qquad (5)$$

The $n_{Tx} \times n_{Tx}$ matrix $\mathbf{P}$ contains the transmitter antenna element-to-element noise-normalized covariance coefficients. The total transmitted noise-normalized power is given by $\mathrm{tr}(\mathbf{P})$. To avoid radiating negative power, the additional constraint that $\mathbf{P} > \mathbf{0}$ is imposed by choosing to use only a subset of modes.

Substituting the magnitude-ordered singular value decomposition of $\mathbf{H}$ as $\mathbf{U}\mathbf{S}\mathbf{W}^\dagger$, Equation (5) can be written:

$$C_{IT} = \sup_{\mathbf{Q};\, \mathrm{tr}\{\mathbf{Q}(\mathbf{S}^\dagger\mathbf{S})^{-1}\} = P_o} \log_2 |\mathbf{I} + \mathbf{Q}| \qquad (6)$$

$$\mathbf{Q} \equiv \mathbf{S}\mathbf{W}^\dagger\mathbf{P}\mathbf{W}\mathbf{S}^\dagger \,. \qquad (7)$$

Maximizing Equation (6) under the total and positive power constraints gives the optimum $\mathbf{Q}_{IT}$,

$$\mathbf{Q}_{IT} = \begin{pmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \qquad (8)$$

$$\mathbf{\Delta} = \left(\frac{P_o + \mathrm{tr}(\mathbf{D}^{-1})}{n_{modes}} \right) \mathbf{D} - \mathbf{I} \,, \qquad (9)$$

where the entries, d_m, in the diagonal matrix, $\mathbf{D}$, contain the n_{modes} top eigenvalues of $\mathbf{S}\mathbf{S}^\dagger$, or equivalently of $\mathbf{H}\mathbf{H}^\dagger$, satisfy

$$\mathbf{\Delta}\mathbf{D}^{-1} > 0 \,, \qquad (10)$$

$$d_m > \frac{n_{modes}}{P_o + \mathrm{tr}\{\mathbf{D}^{-1}\}} . \qquad (11)$$

This results in a capacity of

$$C_{IT} = \log_2 \left| \frac{P_o + \mathrm{tr}\{\mathbf{D}^{-1}\}}{n_{modes}} \mathbf{D} \right| . \qquad (12)$$

The receive and transmit beamforming pairs are given by the columns of $\mathbf{U}_\Delta$ and $\mathbf{W}_\Delta$ associated with the selected eigenvalues contained in $\mathbf{D}$.

1.3.2 Uninformed Transmitter (UT)

If the channel is not known at the transmitter, then the optimal transmission strategy is to send equal power to all antennas. Assuming that the receiver can accurately estimate the channel, the capacity is given by

$$C_{UT} = \log_2 \left| \mathbf{I} + \frac{P_o}{n_{Tx}} \mathbf{H}\mathbf{H}^\dagger \right| . \qquad (13)$$

1.3.3 External Interference

Assuming a temporally white Gaussian model for external interference, its effect on capacity is equivalent to spatially colored noise. Adding an interference term with covariance $\sigma_n^2 \mathbf{R}$ to Equation (1) results in the simple spatial whitening of $\mathbf{H} \rightarrow \tilde{\mathbf{H}} = (\mathbf{I} + \mathbf{R})^{-1/2}\,\mathbf{H}$ in Equation (5):

$$C_{IT,int} = \sup_{\mathbf{P};\,\mathrm{tr}(\mathbf{P})=P_o} \log_2 |\mathbf{I} + \tilde{\mathbf{H}}\mathbf{P}\tilde{\mathbf{H}}^\dagger| . \qquad (14)$$

The evaluated optimal capacity in the presence of interference has the a form identical to Equation (12) with $\mathbf{D} \rightarrow \tilde{\mathbf{D}}$ now containing the eigenvalues of $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger$. Similarly, the *uninformed transmitter* capacity in the presence of noise is given by the same transformation of $\mathbf{H} \rightarrow \tilde{\mathbf{H}}$. In the limit of strong interferers, the spatial whitening approaches subspace projection that excises the spatial subspace associated with the interference.

2 Channel Complexity

The eigenvalue distribution of a 2×2 narrowband MIMO system in the absence of environmental scatterers is discussed here as a toy example. In order to visualize the example, imagine two receive antennas and two transmitting antennas located at the corners of a rectangle. The ratio of channel matrix eigenvalues can be changed by varying the shape of the rectangle. In principle the eigenvalues are a function of the lengths of the sides of the rectangle and the wavelength; however, this can be reduced to a single parameter. The columns of the channel matrix, $\mathbf{H}$, can be viewed as the receiver array response vectors, one vector for each transmitting antenna, $\mathbf{H} = (\vec{v}_1\,\vec{v}_2)$. Using this definition the separation between receive array responses can be described in a convenient form in terms of generalized beamwidths,

$$b_{mn} = \frac{2}{\pi} \arccos \left\{ \frac{||\vec{v}_m^\dagger \vec{v}_n||}{||\vec{v}_m||\,||\vec{v}_n||} \right\} , \qquad (15)$$

where the norm is denoted by $||\cdots||$. For small angular separations this definition of beamwidths is equivalent to physical beamwidths. The ratio of the smaller eigenvalue, λ_{min}, to the sum of eigenvalues is displayed in Figure 2. When the transmit and receive arrays are small, indicated by a small separation in beamwidths, one eigenvalue is dominant. As the array apertures become larger, indicated by larger separation, one array's individual elements can be resolved by the other array. Consequently, the smaller eigenvalue increases, resulting in increased capacity.

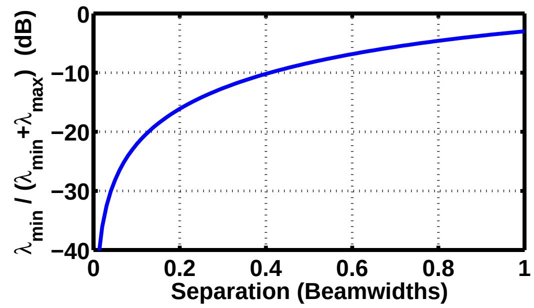

Figure 2. Ratio of smaller eigenvalue of $\mathbf{H}\mathbf{H}^\dagger$ to the sum of eigenvalues for a 2$\times$2 matrix example.

2.1 Channel Matrix Eigenvalue Distributions

In complicated multipath environments, small arrays can employ scatterers to create virtual arrays of a much larger effective aperture. The effect of the scatterers upon capacity depends on their number and distribution in the environment. Using a narrowband version of a simple statistical scattering model that was relatively successful in matching the spatial decorrelation of antenna elements measured at cellular phone frequencies and bandwidths [3], distributions of channel matrix eigenvalues are estimated. In the statistical model used to produce the results reported here, an ensemble of realizations of three environments were simulated. The first assumes a random channel matrix (a common assumption in the literature), where the distribution of the entries in $\mathbf{H}$ are independent complex Gaussians. The second environment assumes a dense field of scatterers, $10/\mathrm{km}^2$, consistent with previous experimental results. The 8×8 MIMO arrays are separated by 1 km and have half-wavelength spacing with a 1 GHz carrier frequency. The scattering field has width and length of 2 km. The third environment assumes the same parameters with a sparse field of scatterers, $1/\mathrm{km}^2$.

In Figure 3 the channel matrix eigenvalue distributions for $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger$ in the presence of 0, 2, or 4 strong interferers are displayed. As one would expect, in the absence of interferers the eigenvalue distribution for the random channel is relatively flat, while the distribution for the sparse-scatterer channel falls off quickly. In the case of the sparse-scatterer channel, the shape of the distribution is determined by the relatively few resolvable scatterers in the environment, limiting the number of large eigenvalues that the channel can produce. As interferers are introduced and their associated subspaces are removed from the channel, the eigenvalue distribution becomes truncated. The interference in effect reduces differences between the various channel types.

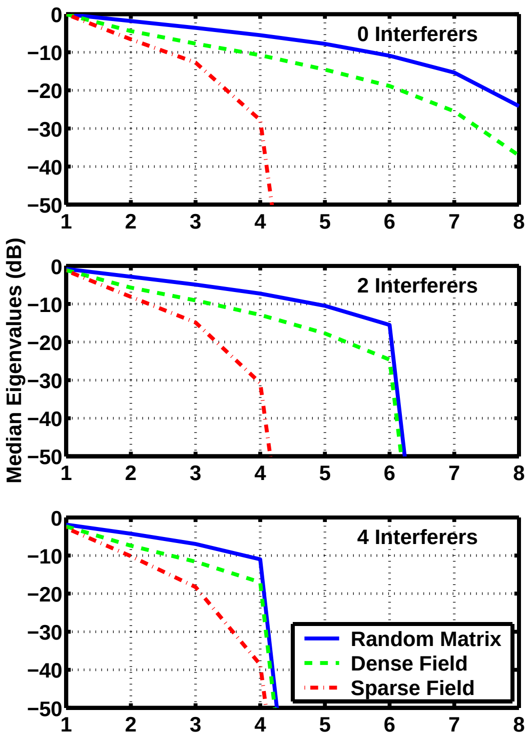

Figure 3. Eigenvalue distributions of $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\dagger}$ for an 8×8 channel for random, dense and sparse scattering fields, assuming 0, 2, and 4 strong interferers.

2.2 Capacity Implications

It is interesting to compare the capacity of a 1×8 SIMO communication system with an 8×8 MIMO system, under the constraint that the total transmit power is equal. The capacity ratio,

$$\frac{C\left(8 \times 8; \mathrm{tr}\{\mathbf{P}\} = P_o\right)}{C\left(1 \times 8; P_1 = P_o\right)}, \tag{16}$$

is displayed in Figure 4 for both *informed* and *uninformed transmitter* capacities. In the figure, for each environment type, the total transmit power is held constant across capacity curves. In general the transmit powers between environments are not equal. The horizontal axis displays the optimal receive SNR, $\mathrm{tr}\{\mathbf{Q}\}$, when the total noise-normalized power, P_o, is transmitted by the *informed transmitter*. Given that the transmit power is held constant, the total received power for the *uninformed transmitter* and the single transmitter will be lower than that received by the *informed transmitter*. This choice of total receive power normalization is consistent with the traditional normalization used when expressing single channel capacities.

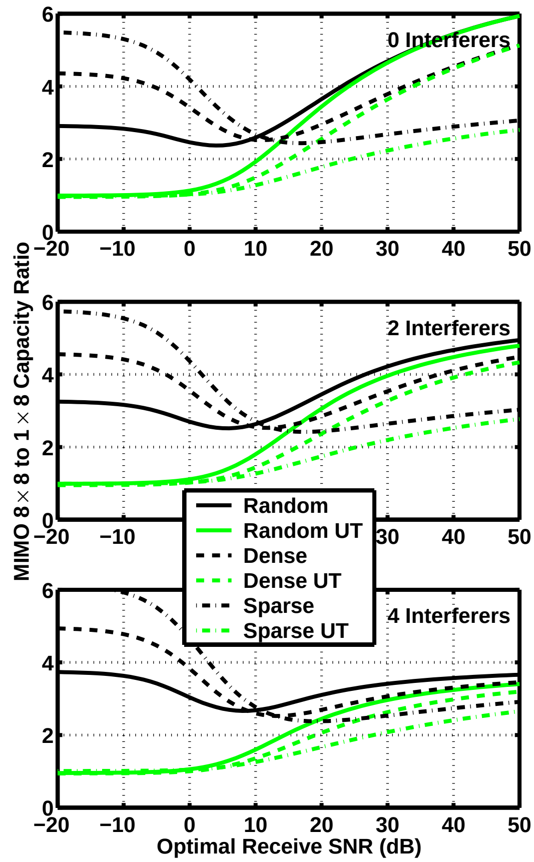

Figure 4. Capacity ratio of 8×8 MIMO to 1×8 SIMO for random, dense and sparse scattering fields, assuming 0, 2, and 4 strong interferers.

In Figure 4 the sensitivity of MIMO capacity to environment is demonstrated. At very high SNR the *uninformed transmitter* capacity and *informed transmitter* capacity converge, which is the result of P_o dominating $\mathrm{tr}\{\mathbf{D}^{-1}\}$ in Equation (12) at high SNR. At low SNR the *informed transmitter* avoids modes with small singular values, while the *uninformed transmitter* randomly spreads energy between modes. The loss is most significant for environments with a relatively few large channel matrix singular values.

3 Channel Estimation Error

Channel estimation accuracy is limited by channel stationarity. For the sake of this discussion, channel estimation error will be modeled as a perturbing matrix, $\mathbf{\Sigma}$, with independently distributed elements. The estimated channel is then given by $\hat{\mathbf{H}} \equiv \mathbf{H} + ||\mathbf{H}||\mathbf{\Sigma}$. Here $|| \cdots ||$ indicates the Frobenius norm. While both *informed* and *uninformed transmitter* MIMO systems suffer loss in capacity as a result of channel estimation error, the *informed transmitter* suffers a loss due to using incorrect transmit spatial coding.

The losses peculiar to *informed transmitter* MIMO systems can be investigated by assuming that the receiver has an accurate estimate of the channel, but the transmitter has an inaccurate estimate. This model is reasonable for nonstationary channels. Assuming data is transmitted in blocks, the receiver can perform channel estimation using the current block of data. However, the transmitter must wait for that information to be fed back. Ignoring the possibility of prediction, the transmitter will employ channel estimates from a previous block. Using this estimated channel with error, $\mathbf{\Sigma}$, the optimal noise-normalized transmit covariance is given by $\hat{\mathbf{P}} = \hat{\mathbf{W}}_\Delta \hat{\mathbf{\Lambda}} \hat{\mathbf{W}}_\Delta^\dagger$, where $\hat{\mathbf{\Lambda}}$ is a diagonal matrix with elements given by solving for $\mathbf{P}$, using Equations (7-8), assuming the estimated channel is the true channel,

$$\hat{\mathbf{\Lambda}} = \frac{P_o + \mathrm{tr}(\hat{\mathbf{D}}^{-1})}{\hat{n}_{modes}} \mathbf{I} - \hat{\mathbf{D}}^{-1} . \tag{17}$$

As a result the capacity with channel estimation error at the transmitter is given by

$$C_{TxErr} = \log_2 |\mathbf{I} + \mathbf{H}\hat{\mathbf{P}}\mathbf{H}^\dagger| . \tag{18}$$

In Figure 5 the fraction of the optimal capacity assuming transmit channel estimation error for $||\mathbf{\Sigma}||^2$ = 0.01, 0.1, and 1 is displayed as a function of optimal received SNR. For this analysis an ensemble of errors and realizations of the dense scatterer environment are used. For comparison, the capacity of the *uninformed transmitter* is presented. The transmit power is held constant between capacity results at a given optimal receive SNR. In general the total received SNR for the *uninformed transmitter* and the erroneous transmitters is lower than for the optimal transmitter. At high SNR MIMO capacity is very forgiving of transmit channel estimation error for the same reason that the *uninformed transmitter* capacity approaches the optimal capacity at high SNR. At very high SNR all modes are treated equally at transmit. At low SNR the capacity remains remarkably insensitive to channel estimation error. Here relatively few modes are used by the optimal transmitter. It is apparently difficult for random noise to significantly disturb the transmit beamformers even when the channel estimation error and the channel have the same Frobenius norm.

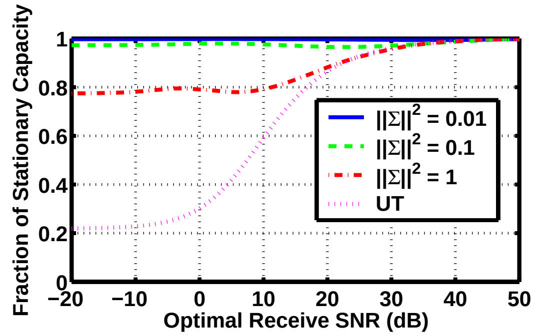

Figure 5. Fraction of stationary capacity for an 8×8 MIMO system with transmitter channel estimation error, assuming a dense scattering field and no interferers.

4 Summary

In this paper the sensitivity of channel capacity to environmental factors has been discussed. The effects of environmental complexity and interference have been investigated. The well-known advantages of *uninformed transmitter* capacity at high SNR were again demonstrated. However, for situations where the communication system must operate over a wide range of quasi-stationary channel environments and SNR, *informed transmitter* MIMO techniques may offer a more robust approach.

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