Entropic Graphs

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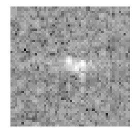
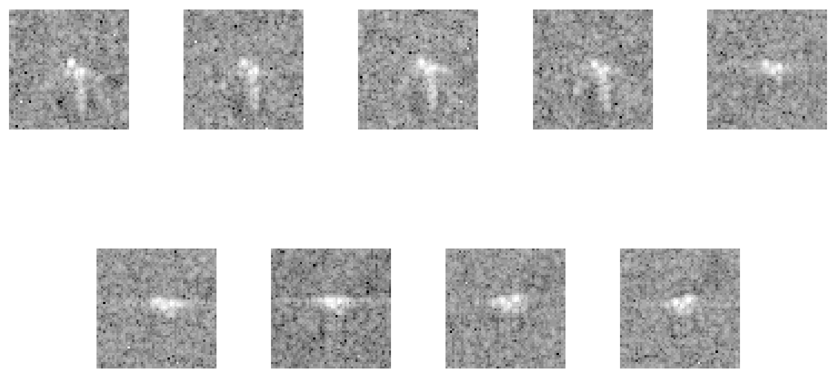
- Motivating Applications
- Entropic Euclidean Graphs
- A variant: K-point entropic graphs
- Application to US image registration

Pattern Matching and Image Retrieval



QUERY



DATABASE

Adaptive radar sensor management

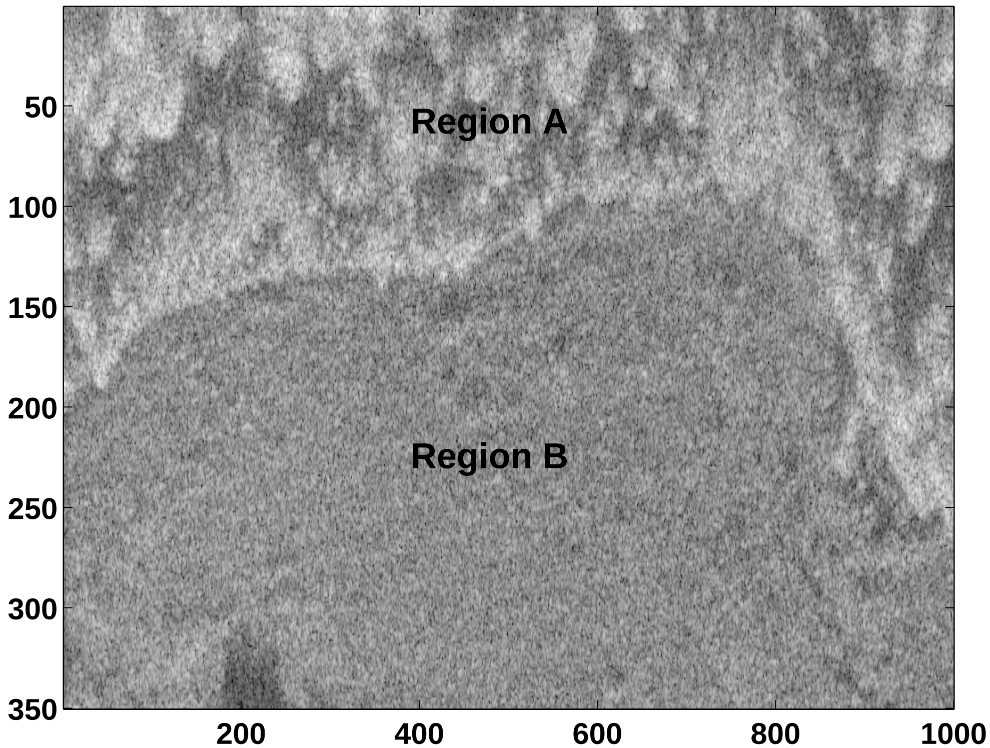
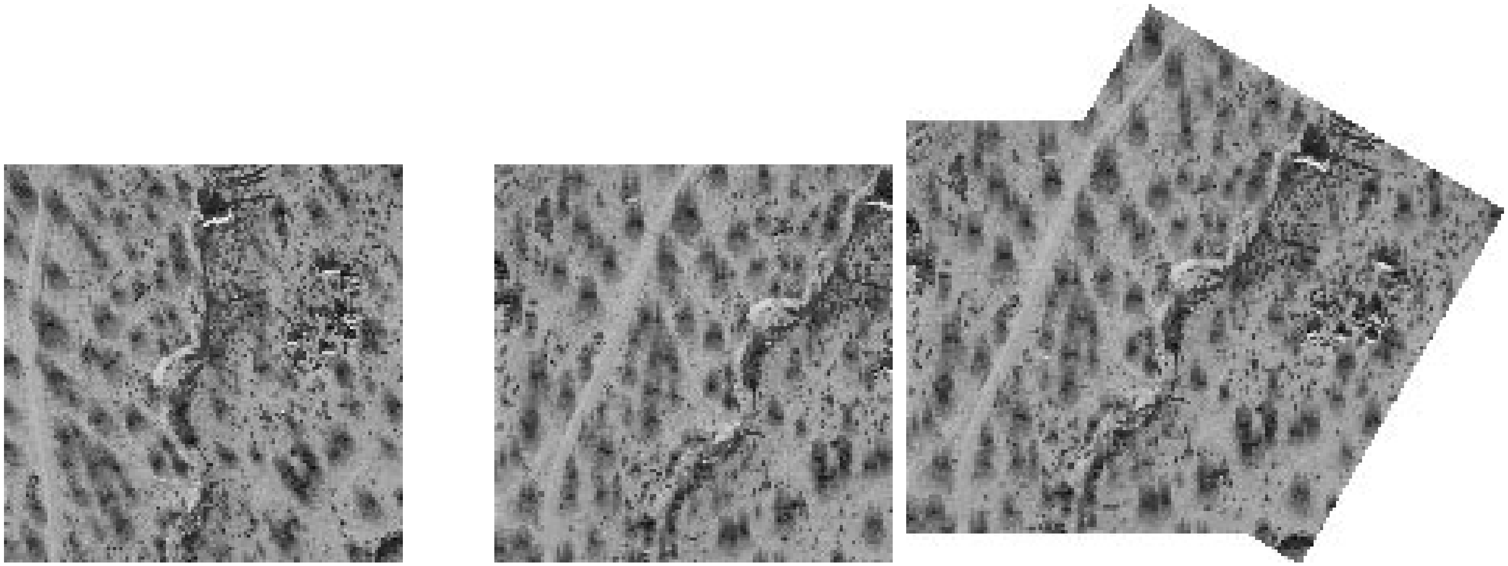


Figure 1: SAR clutter image, target on boundary at column 305.

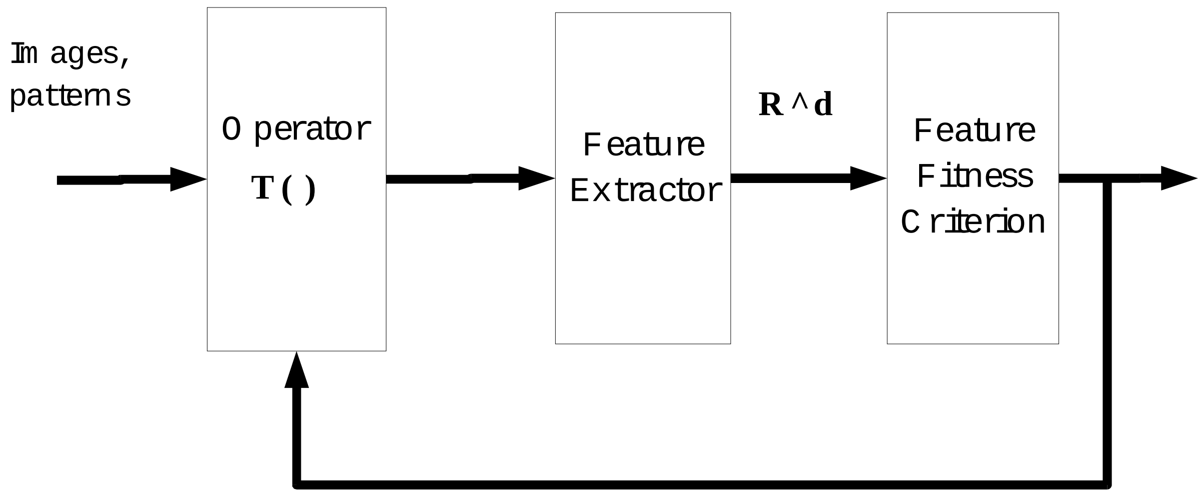
Image Registration



(a) Image I_1 (b) Image I_0 (c) Registration result

Figure 2: A multidate image registration example

Common Processing System



Objective: For given fitness criterion Q, find operator T which minimizes/maximizes Q

Our focus: entropic fitness criterion $Q(f)$

f: feature density over $x \in \mathbf{R}^d$

Some Popular Entropic Q's

1. Shannon Entropy of feature density f

$$Q(f) = H(f) = -\int f(x) \ln f(x)\, dx$$

2. KL Divergence between feature densities f, g

$$Q(f,g) = D(f \| g) = \int f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx$$

3. Jensen difference between feature densities f, g:

$$Q(f,g) = H(\varepsilon f + (1-\varepsilon) g) - \varepsilon H(f) - (1-\varepsilon) H(g)$$

4. Mutual information within joint feature density $f_{X,Y}$

$$Q(f_{X,Y}) = \mathrm{MI}(X,Y) = \int\int f_{X,Y}(x,y) \ln\left(\frac{f_{X,Y}(x,y)}{f_X(x)\, f_Y(y)}\right) dx$$

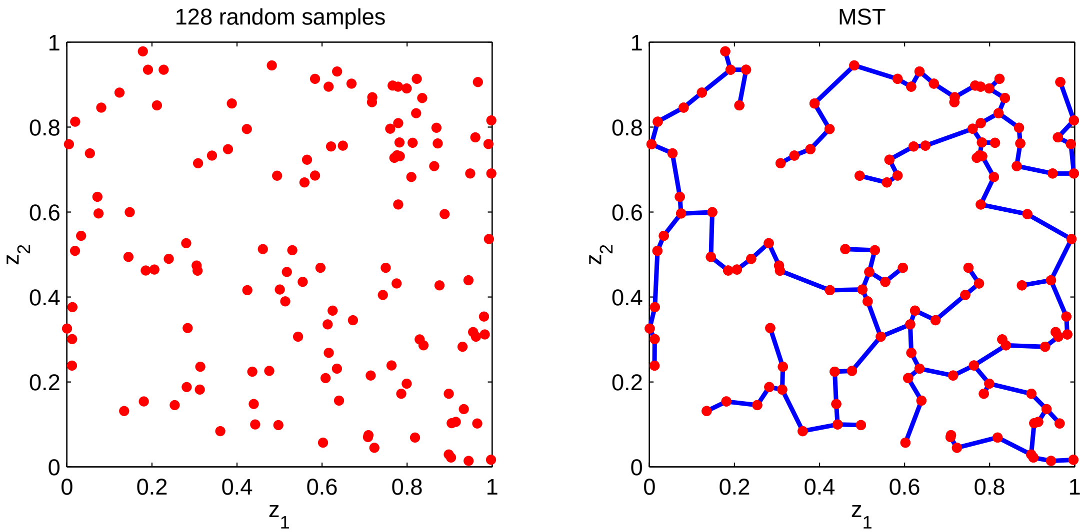
Issue: How to estimate entropic Q from measured data?

Some possibilities:

1. Assume parameteric models for f, g, $f_{X,Y}$
2. Quantize feature space and use histograms
3. Non-parameteric density estimation of f, g, $f_{X,Y}$

Our Strategy: construct "entropic graphs" on features

A Set of Feature Samples and a Euclidean Spanning Graph

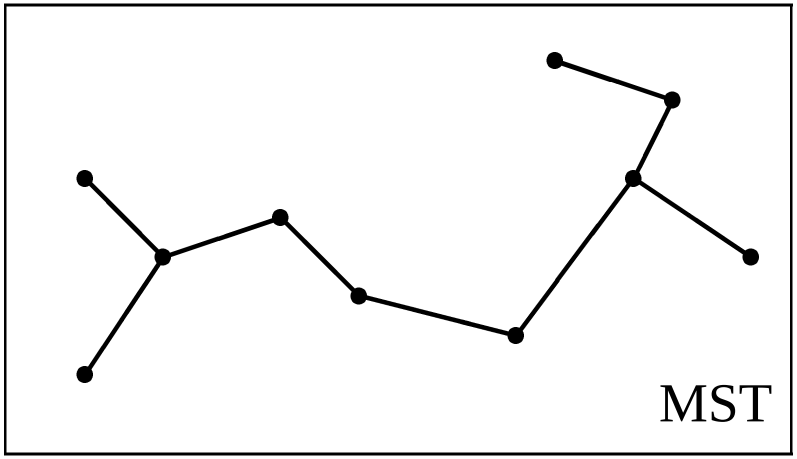


Minimal Euclidean Graphs: MST

Let $\mathrm{T}_n = \mathrm{T}(X_n)$ denote the possible sets of edges in the class of acyclic graphs spanning X_n (spanning trees).

The Euclidean Power Weighted MST achieves

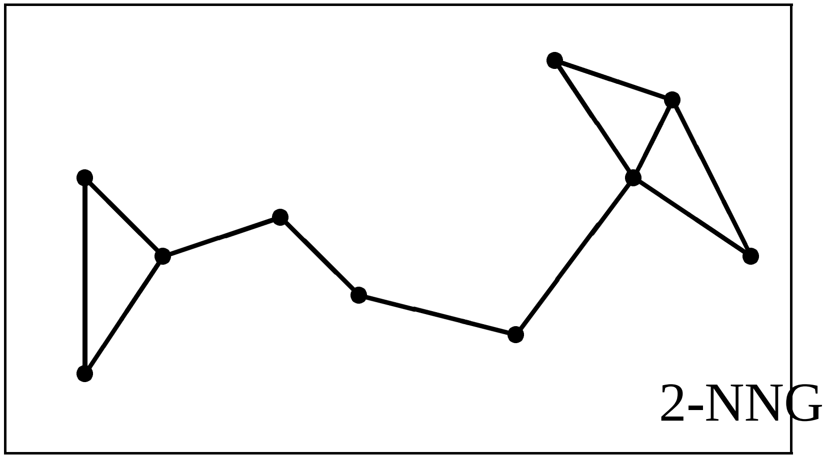
$$L_\gamma^{\mathrm{MST}}(X_n) = \min_{\mathrm{T}_n} \sum_{e \in \mathrm{T}_n} \|e\|^\gamma.$$



Minimal Euclidean graphs: k-NNG

Let $N_{k,i}(X_n)$ denote the possible sets of k edges connecting point x_i to all other points in X_n.
The Euclidean Power Weighted k-NNG is

$$L_\gamma^{k-NNG}(X_n) = \sum_{i=1}^{n} \min_{N_{k,i}(X_n)} \sum_{e \in N_{k,i}(X_n)} |e|^\gamma$$



MST for Two Different Samples

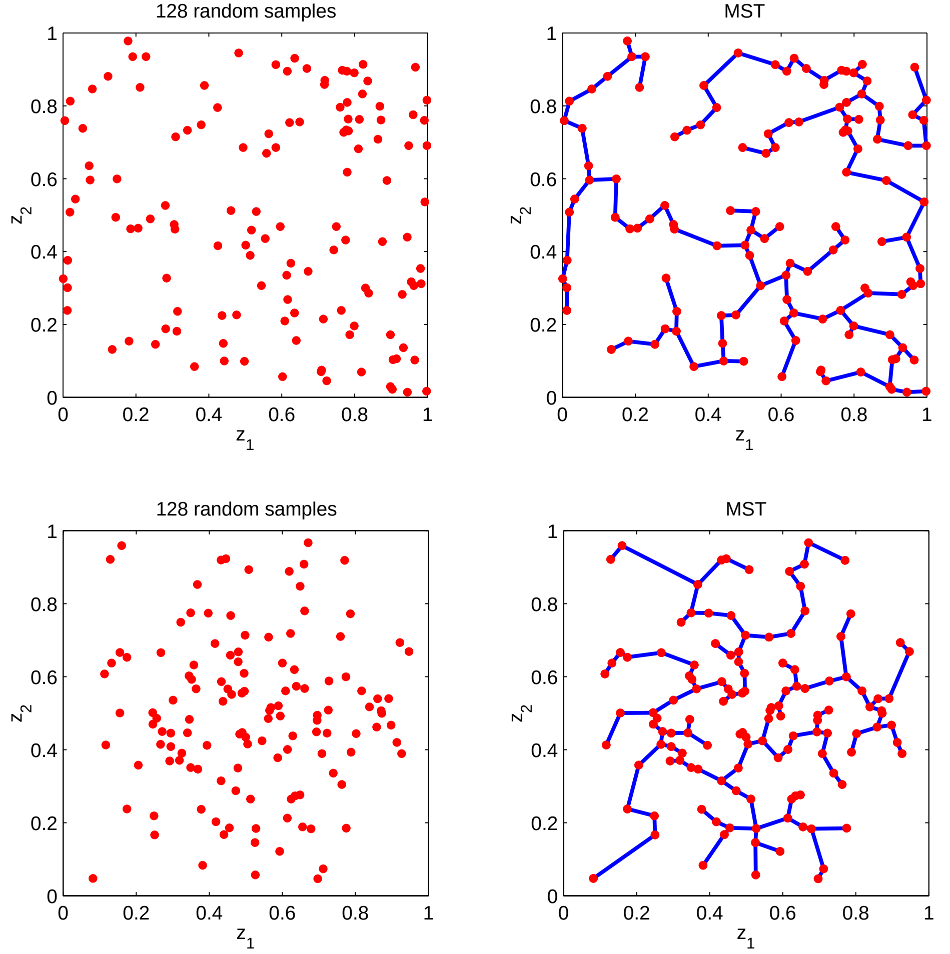


Figure 3:

Large *n* behavior of MST

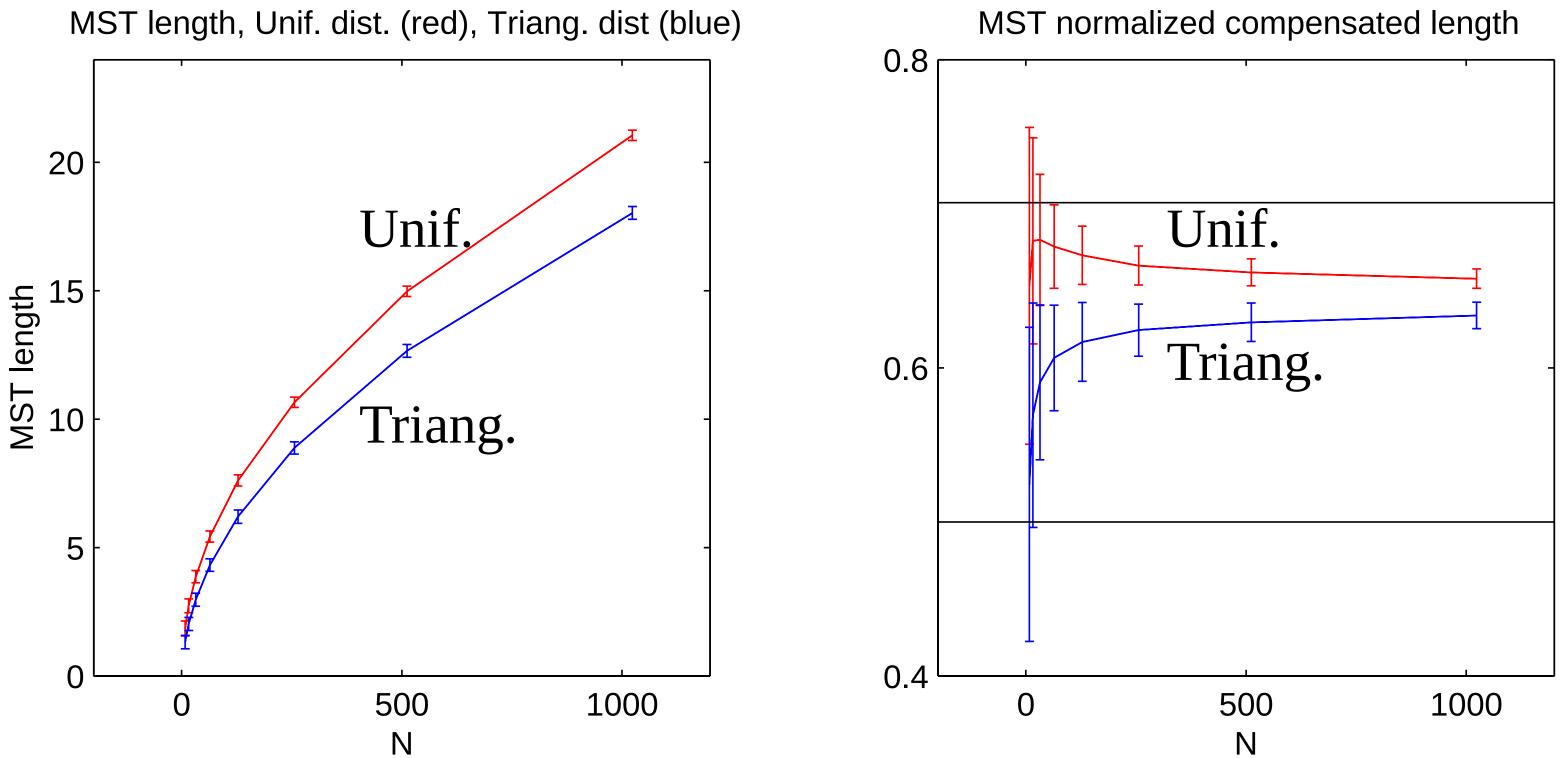


Figure: MST and log MST weights as function of the number of samples.

Asymptotics: the BHH Theorem

Define the MST length functional

$$L_\gamma(\mathcal{X}_n) = \min_{\mathrm{T}_n} \sum_{e \in \mathrm{T}_n} \|e\|^\gamma.$$

Theorem 1 *[Beardwood, Halton&Hammersley:1959] Let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ be an i.i.d. realization from a Lebesgue density f with support $S \subset [0,1]^d$.*

$$\lim_{n\to\infty} L_\gamma(\mathcal{X}_n)/n^{(d-\gamma)/d} \quad = \quad \beta_{L_\gamma,d} \int_S f(x)^{(d-\gamma)/d} dx, \qquad (a.s.)$$

Or, letting $\alpha = (d-\gamma)/d$

$$\frac{1}{1-\alpha} \ln L_\gamma(\mathcal{X}_n)/n^\alpha \quad \to \quad H_\alpha(f) + c \qquad (a.s.)$$

Rényi Entropy and Divergence

- Rényi Entropy of order α [Rényi:61,70]

$$H_\alpha(f) = \frac{1}{1-\alpha} \ln \int_S f^\alpha(x) dx$$

- Rényi α-divergence of fractional order $\alpha \in [0,1]$

$$\begin{aligned} D_\alpha(f_1 \parallel f_0) &= \frac{1}{\alpha-1} \ln \int_S f_0 \left(\frac{f_1}{f_0}\right)^\alpha dx \\ &= \frac{1}{\alpha-1} \ln \int_S f_1^\alpha f_0^{1-\alpha} dx \end{aligned}$$

  – α-Divergence vs. Kullback-Liebler divergence

$$\lim_{\alpha \to 1} D_\alpha(f_1 \| f_0) = \int f_1 \ln \frac{f_1}{f_0} dx.$$

Clustering via K-MST

Assume f is a mixture density of the form

$$f = (1-\varepsilon) f_1 + \varepsilon f_o,$$

where

- f_o is a known (uniform) outlier density
- f_1 is an unknown target density
- $\varepsilon \in [0,1]$ is unknown mixture parameter

K-point Minimal Spanning Tree (*K*-MST)

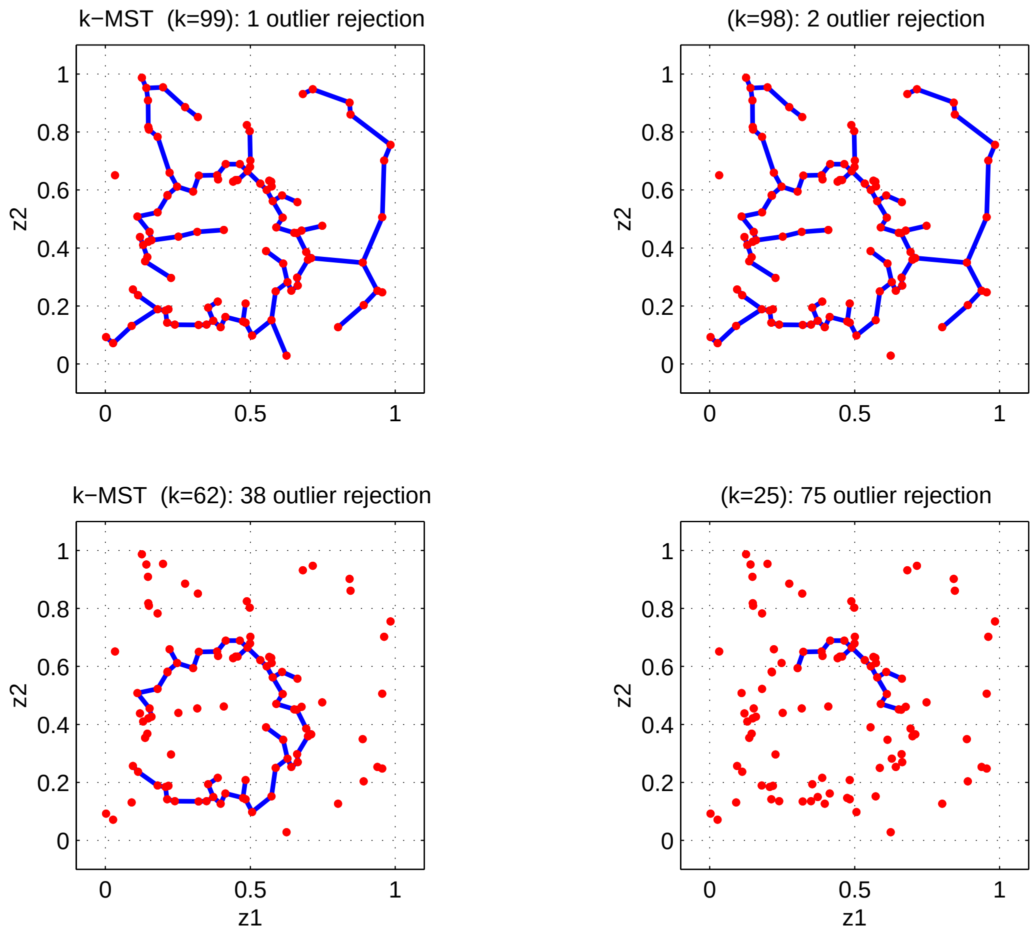


Figure 4: *Clustering an annulus density from uniform noise via k-MST.*

K-MST Stopping Rule

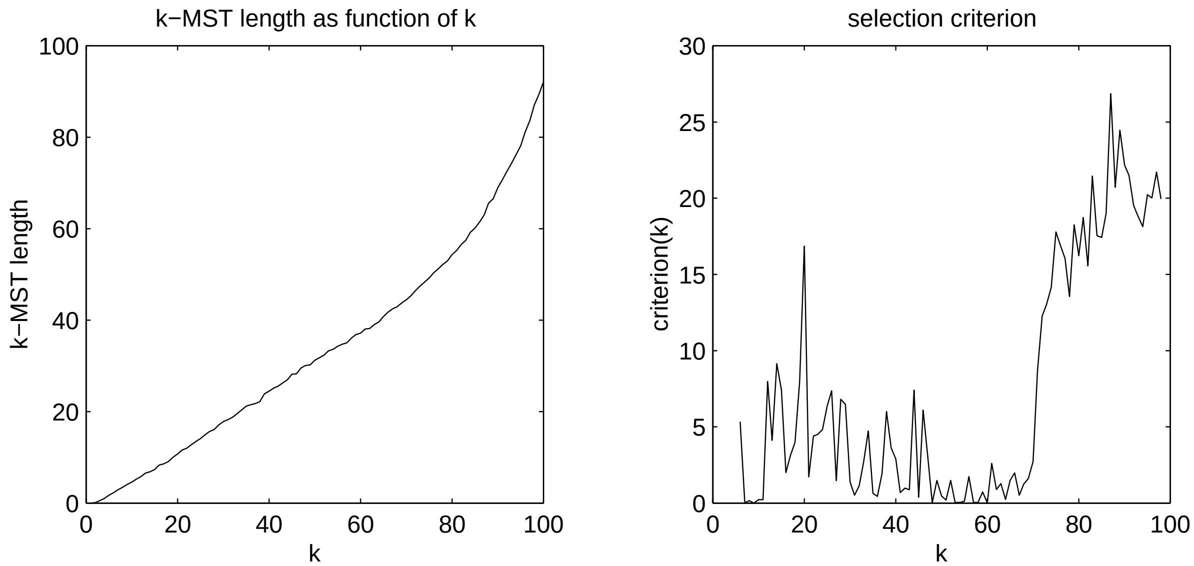


Figure 5: *Left: k-MST curve for 2D annulus density with addition of uniform "outliers" has a knee in the vicinity of* $n - k = 35$.

Greedy partioning approximation to K-MST

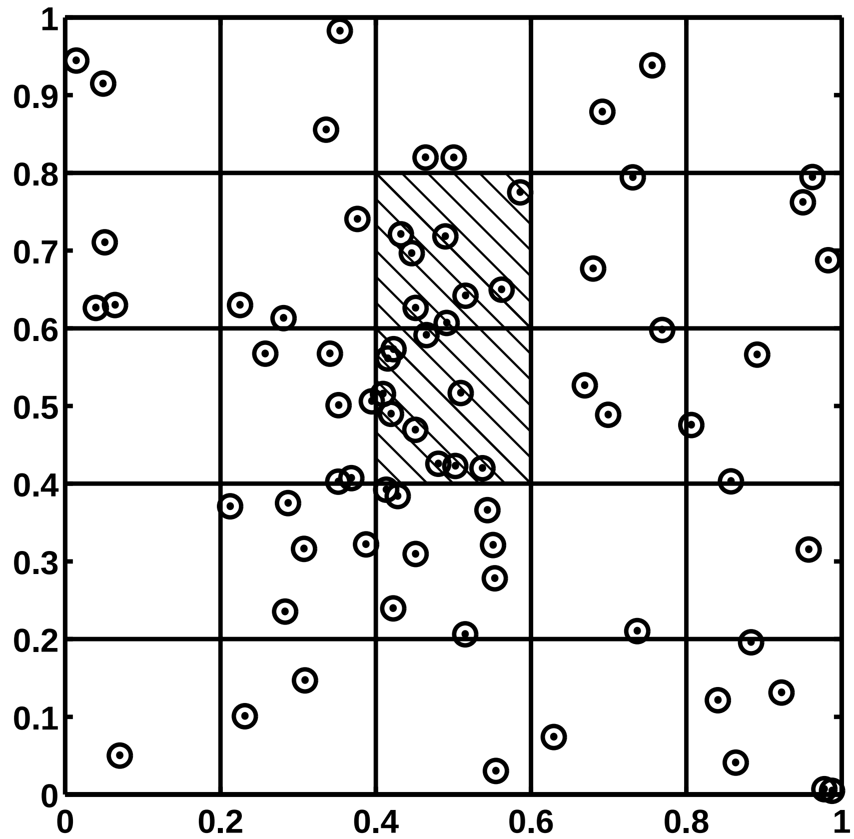


Figure 6: *A smallest subset* B_k^m *is the union of the two cross hatched cells shown for the case of* $m = 5$ *and* $k = 17$.

Extended BHH Theorem for Greedy K-MST

Fix $\rho \in [0,1]$. If $k/n \to \rho$ then the length of the greedy partitioning K-MST satisfies [Hero&Michel:IT99]

$$L_\gamma(X^*_{n,k})/(\lfloor \rho n \rfloor)^\alpha \to \beta_{L_\gamma,d} \min_{A:\int_A f \geq \rho} \int_S f^\alpha(x|x\in A)dx \qquad (a.s.)$$

or, alternatively, with

$$H_\alpha(f|x\in A) = \frac{1}{1-\alpha} \ln \int_S f^\alpha(x|x\in A)dx$$

$$\frac{1}{1-\alpha} \ln L_\gamma(X^*_{n,k})/(\lfloor \rho n \rfloor)^\alpha \to \beta_{L_\gamma,d} \min_{A:\int_A f \geq \rho} H_\alpha(f|x\in A) \qquad (a.s.)$$

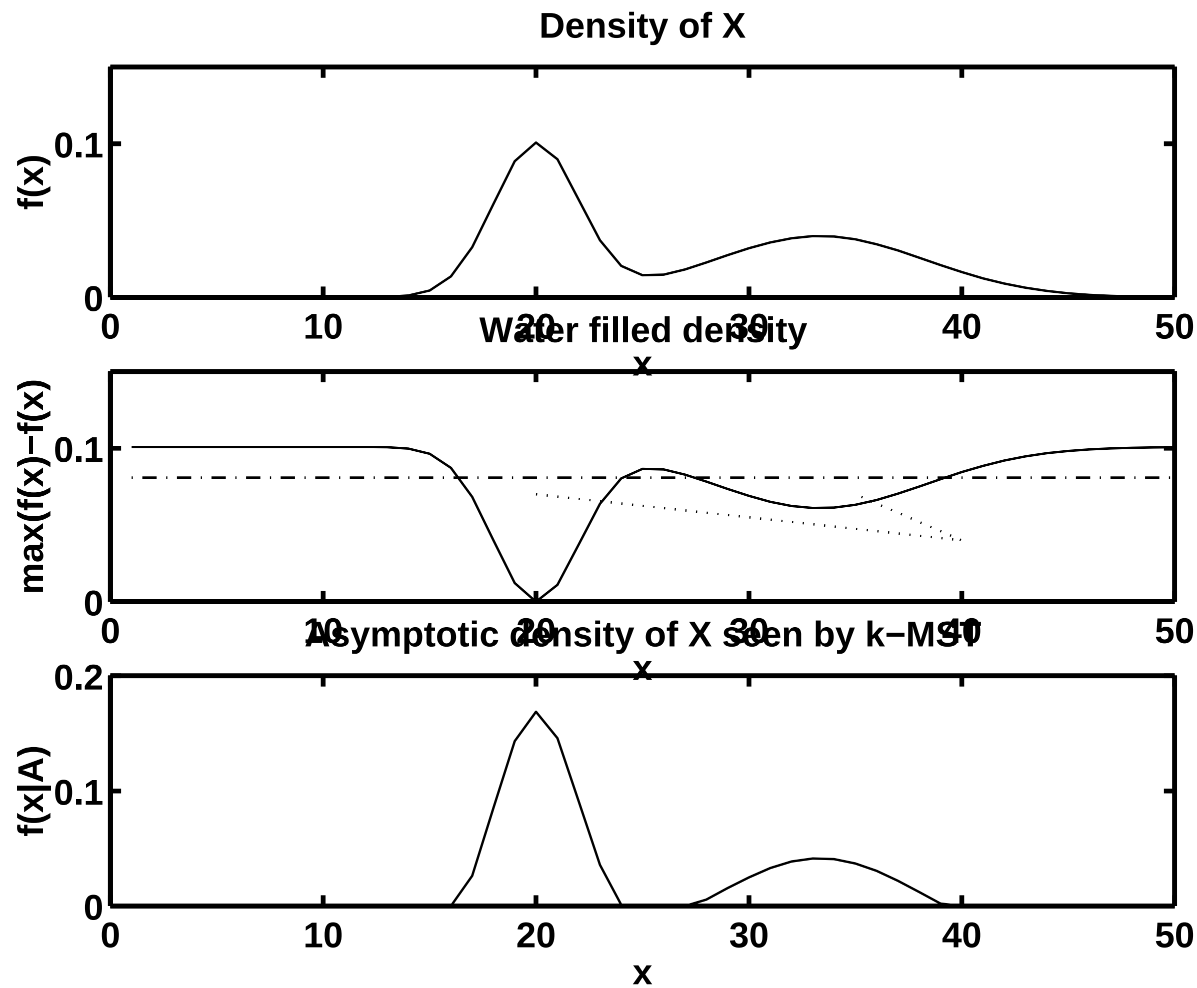


Figure 7: *Waterpouring contruction of minimum entropy density.*

k-MST Influence Function

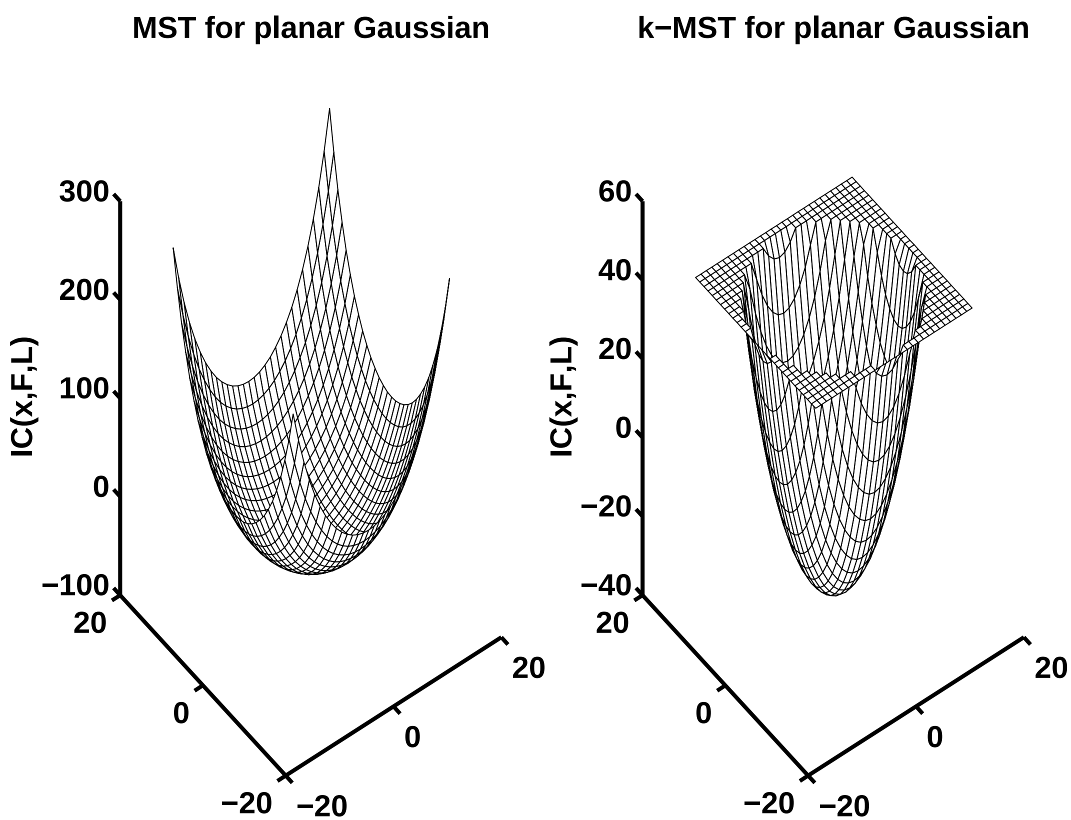


Figure 8: *MST and k-MST influence curves for Gaussian density on the plane.*

Extension of BHH to Divergence Estimation?

Question: How to go from

$$\frac{1}{1-\alpha}\ln\int f^{\alpha}(x)dx \quad \textbf{to} \quad \frac{1}{\alpha-1}\ln\int f^{\alpha}(x)g^{1-\alpha}(x)dx \ ?$$

- $g(x)$: a reference density on $\mathbf{R}^d$
- Assume $f \ll g$, i.e. for all x such that $g(x) = 0$ we have $f(x) = 0$.
- Make measure transformation $M(x)$ such that $dx \to g(x)dx$ on $[0,1]^d$. Then for $Y_n = M(X_n)$

$$L_\gamma(Y_n)/n^{\alpha} \quad \to \quad \beta_{L_\gamma,d}\int\left(\frac{f(x)}{g(x)}\right)^{\alpha} g(x)dx, \qquad (a.s.)$$

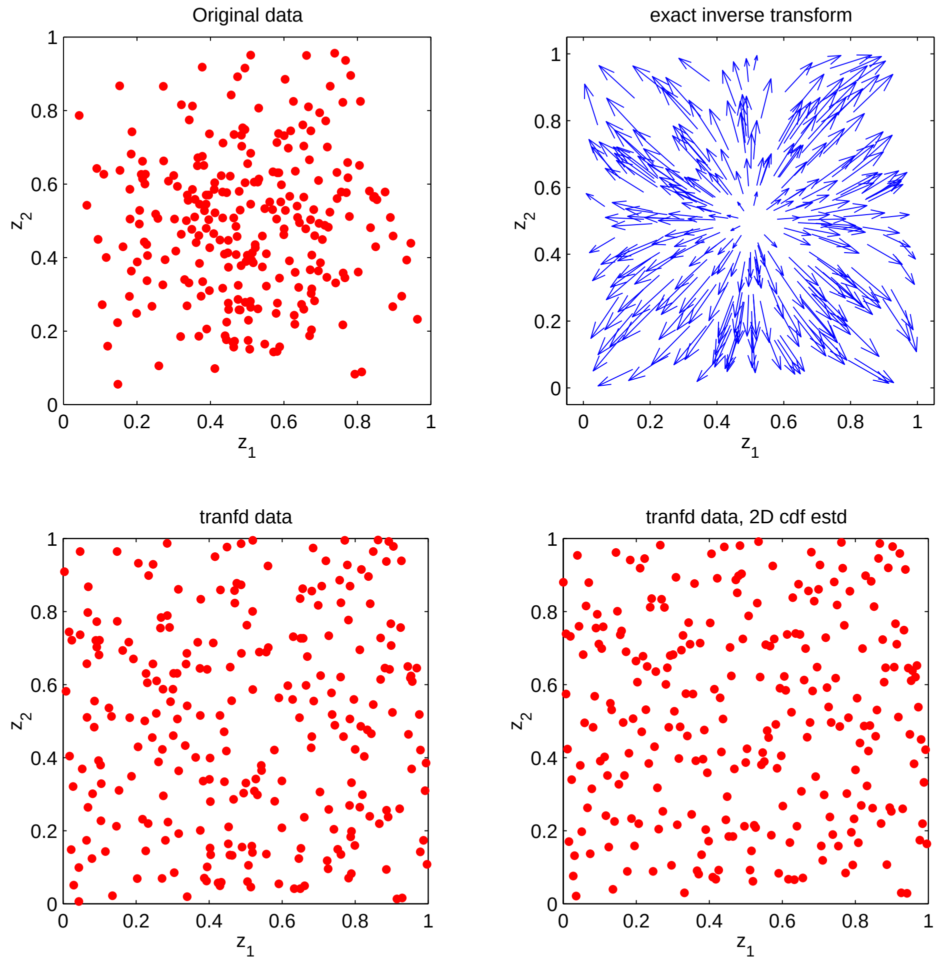


Figure 9: Top Left: i.i.d. sample from triangular distribution, Top Right: exact transformation, Bottom: after application of exact and empirical transformations.

Clustering Example

X_n is a sample from the mixture

$$f(x) = (1-\varepsilon)g(x) + \varepsilon h(x)$$

$h(x)$ is uniform density on $[0,1]^2$

$g(x)$ is triangular density on $[0,1]^2$

ε is unknown

Objective: Detect deviation of f from triangular and cluster the uniform variates in the sample

Illustration

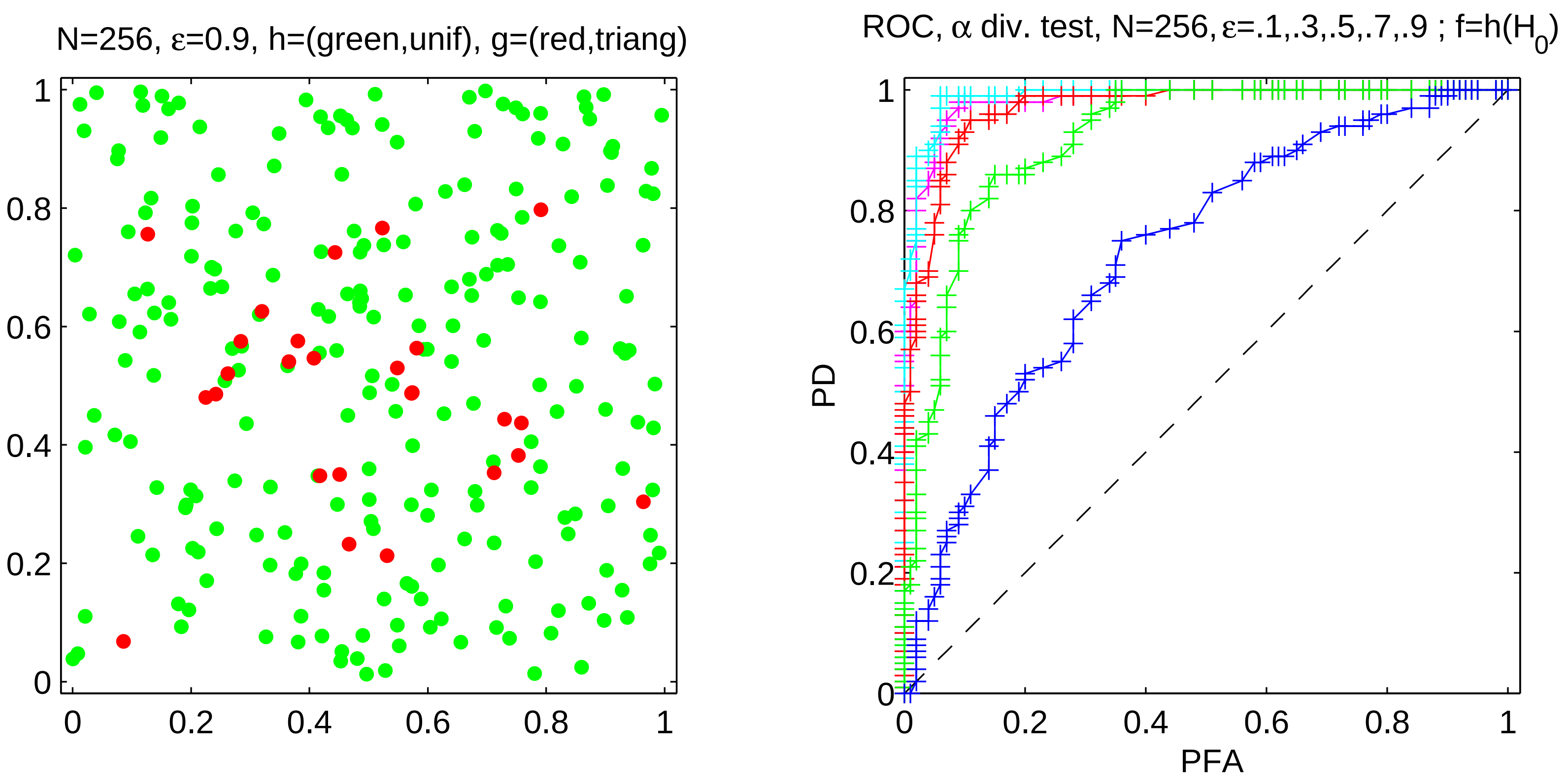


Figure 10: *Left: A sample from triangle-uniform mixture density with* $\epsilon = 0.9$ *in the transformed domain* Y_n*. Right: ROC curves of thresholded* α*-divergence test for deviation from g. Curves are decreasing in* ϵ *over the range* $\epsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$

Clustering Illustration

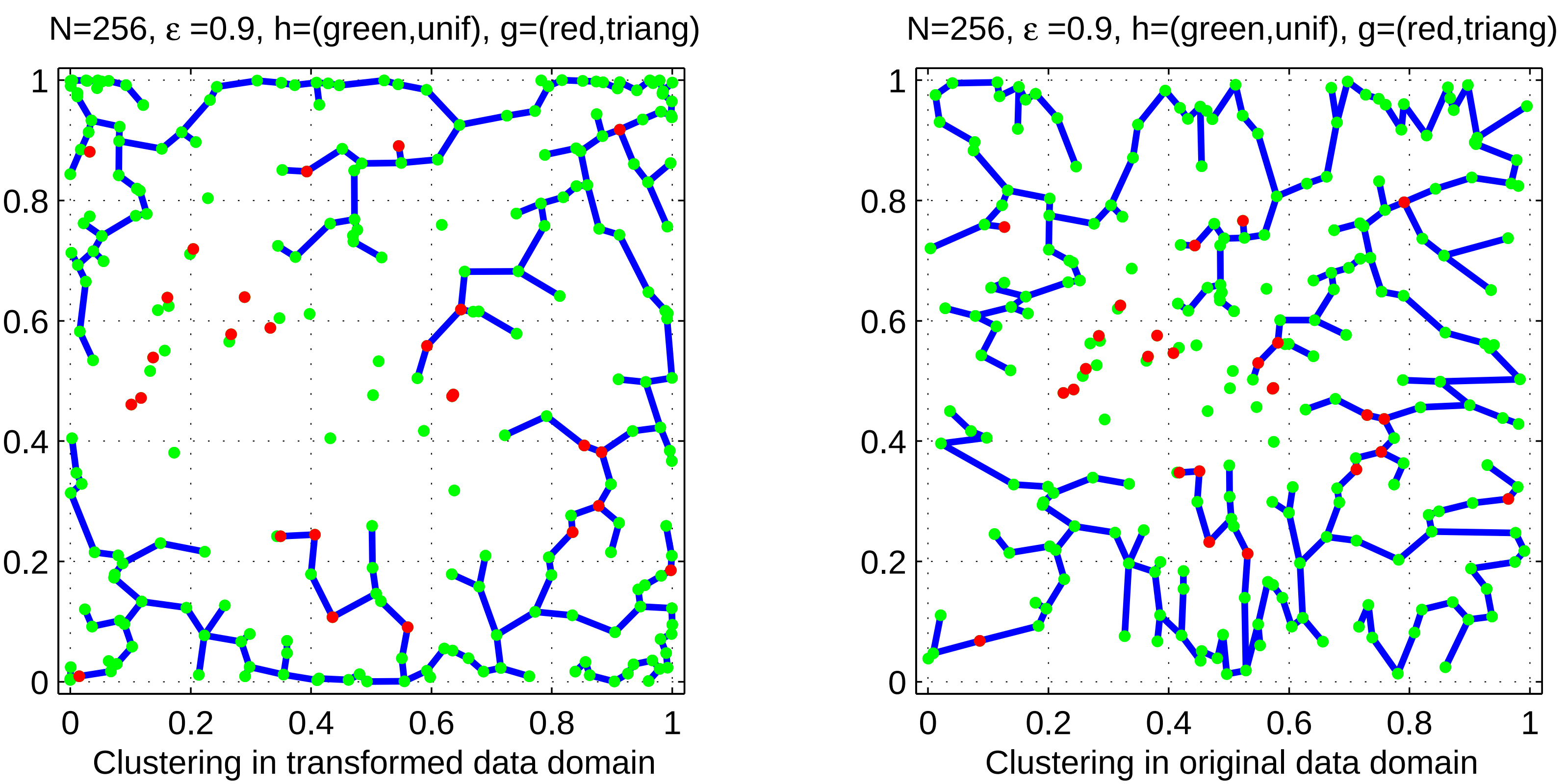


Figure 11: *Left: the K-MST implemented on the transformed scatterplot Y_n with $k = 230$. Right: same K-MST displayed in the original data domain.*

Bounds on Minimax Convergence Rate

Theorem 2 (Hero,Costa&Ma 2001) *Let $d \geq 2$ and $1 \leq \gamma \leq d-1$. Assume $X_1,\ldots,X_n$ are i.i.d. random vectors over $[0,1]^d$ with density $f \in \Sigma_d(\beta,l)$, $\beta, l > 0$, having support $S \subset [0,1]^d$. Assume also that $f^{\frac{1}{2}-\frac{\gamma}{d}}$ is integrable. Then,*

$$O\left(n^{-r_1(d,\beta)}\right) \leq \sup_{f\in\Sigma_d(\beta,l)} E\left[\left|L_\gamma(X_1,\ldots,X_n)/n^{(d-\gamma)/d} - \beta_{L_\gamma,d}\int_S f^{(d-\gamma)/d}(x)\mathrm{d}x\right|^p\right]^{1/p} \leq O\left(n^{-r_2(d,\gamma)}\right),$$

where

$$r_1(d,\beta) = \min\{\frac{4\beta}{4\beta+d}, 1/2\} \quad r_2(d,\gamma) = \frac{\alpha}{\alpha+1}\,\frac{1}{d}$$

and $\alpha = \frac{d-\gamma}{d}$.

Extension to Partition Approximations

$$L_\gamma^m(X_n) = \sum_{i=1}^{m^d} L_\gamma(X_n \cap Q_i) + b(m),$$

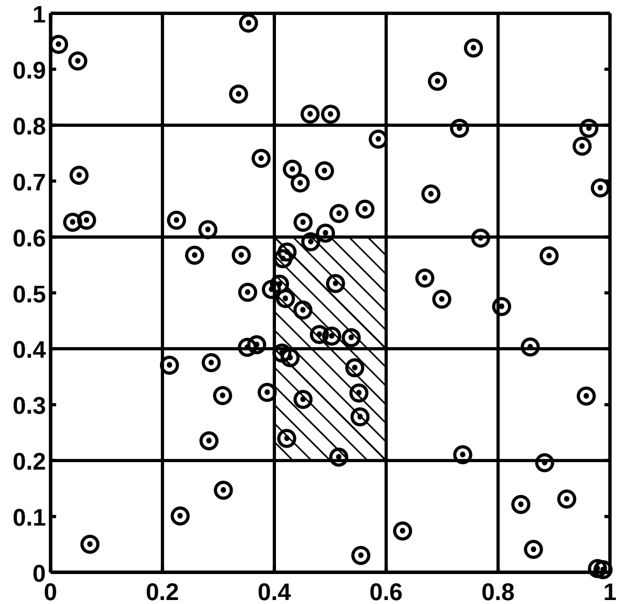


Figure 12: *Partition approximation.*

Theorem 3 (Hero,Costa&Ma 2001) *Let $L_\gamma^m(X_n)$ be a partition approximation to $L_\gamma(X_n)$. Under the same hypotheses as in the previous proposition, if* $b(m) = O(m^{d-\gamma})$

$$O\left(n^{-r_1(d,\beta)}\right) \leq \sup_{f \in \Sigma_d(\beta,l)} E\left[\left|L_\gamma^{m(n)}(X_1,\ldots,X_n)/n^{(d-\gamma)/d} - \beta_{L_\gamma,d}\int_S f^{(d-\gamma)/d}(x)\mathrm{d}x\right|^p\right]^{1/p} \leq O\left(n^{-r_3(d,\gamma)}\right),$$

where

$$r_3(d,\gamma) = \frac{\alpha}{\frac{d-1}{\gamma}\,\alpha+1}\,\frac{1}{d}\,.$$

This bound is attained by choosing the progressive-resolution sequence $m = m(n) = n^{1/[d(\frac{d-1}{\gamma}\,\alpha+1)]}$.

Application: Registration of Breast Images

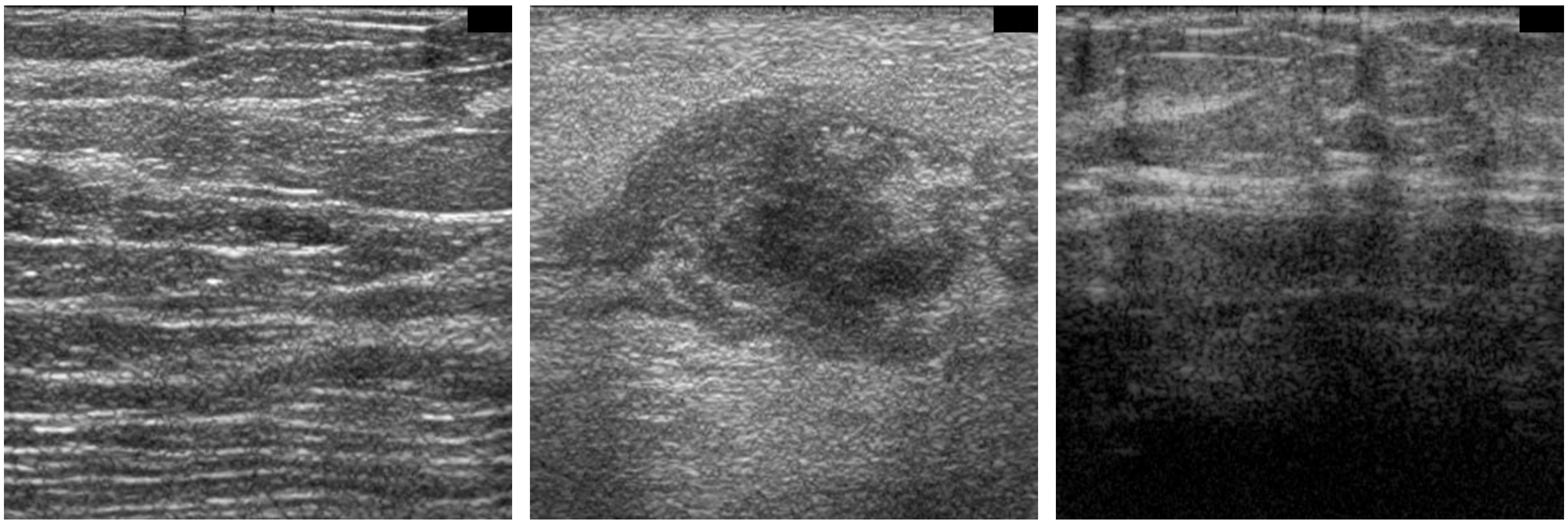


Figure 13: Three ultrasound breast scans. From top to bottom are: case 151, case 142 and case 162.

MI Registration of Gray Levels (Viola&Wells:ICCV95)

- X: a $N \times N$ image (lexicographically ordered)
- $X(k)$: image gray level at pixel location k
- X_0 and X_1: primary and secondary images to be registered

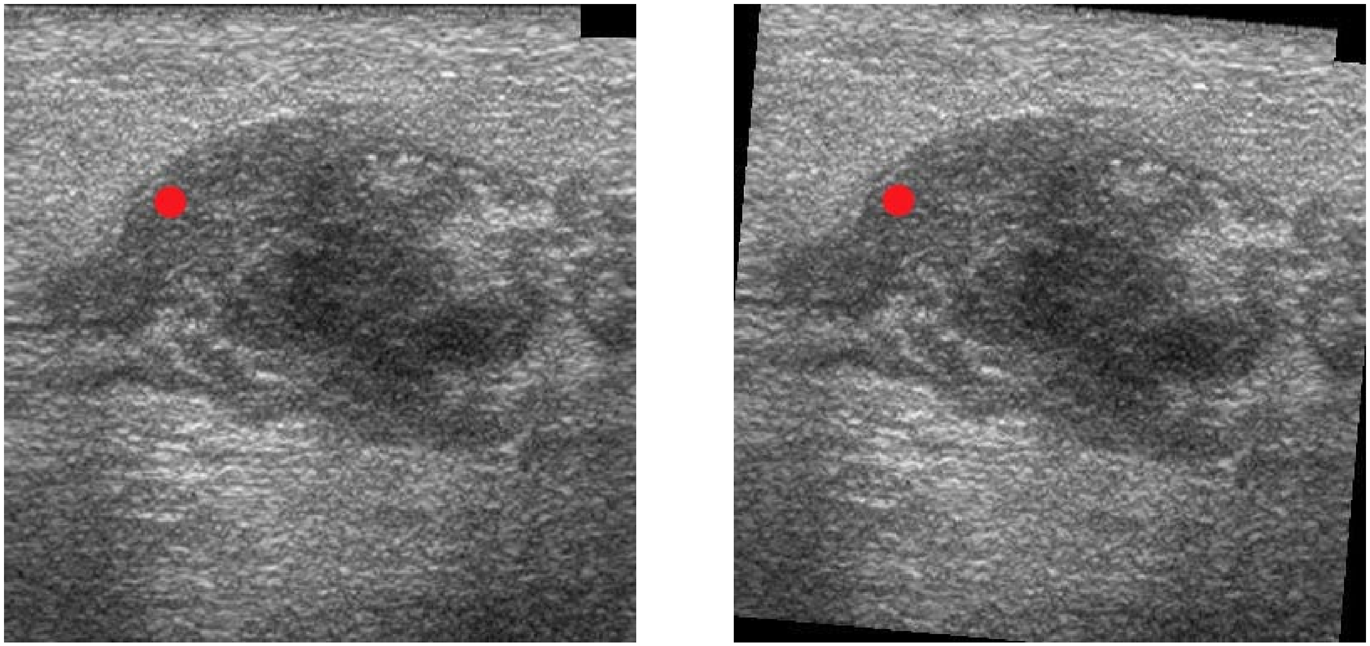
Hypothesis: $\{(X_0(k), X_i(k)\}_{k=1}^{N^2}$ are i.i.d. r.v.'s with j.p.d.f

$$f_{0,i}(x_0, x_1), \quad x_0, x_1 \in \{0, 1, \ldots, 255\}$$

Mutual Information (MI) criterion: $\mathrm{T} = \mathrm{argmax}_{\mathrm{T}_i} \hat{\mathrm{MI}}$

where $\hat{\mathrm{MI}}$ is an estimate of

$$\mathrm{MI}(f_{0,i}) = \int \int f_{0,i}(x_0, x_1) \ln f_{0,i}(x_0, x_1) / (f_0(x_0) f_i(x_1)) dx_1 dx_0. \qquad (1)$$



(a) Image I^R (b) Image I^T

Figure 14: Single Pixel Coincidences (Left and right: reference image I^R at 0^o and rotated image I^T at 8^o)

Single-Pixel Scatterplot $(Z_j^R, Z_j^T)_{j=1}^p$

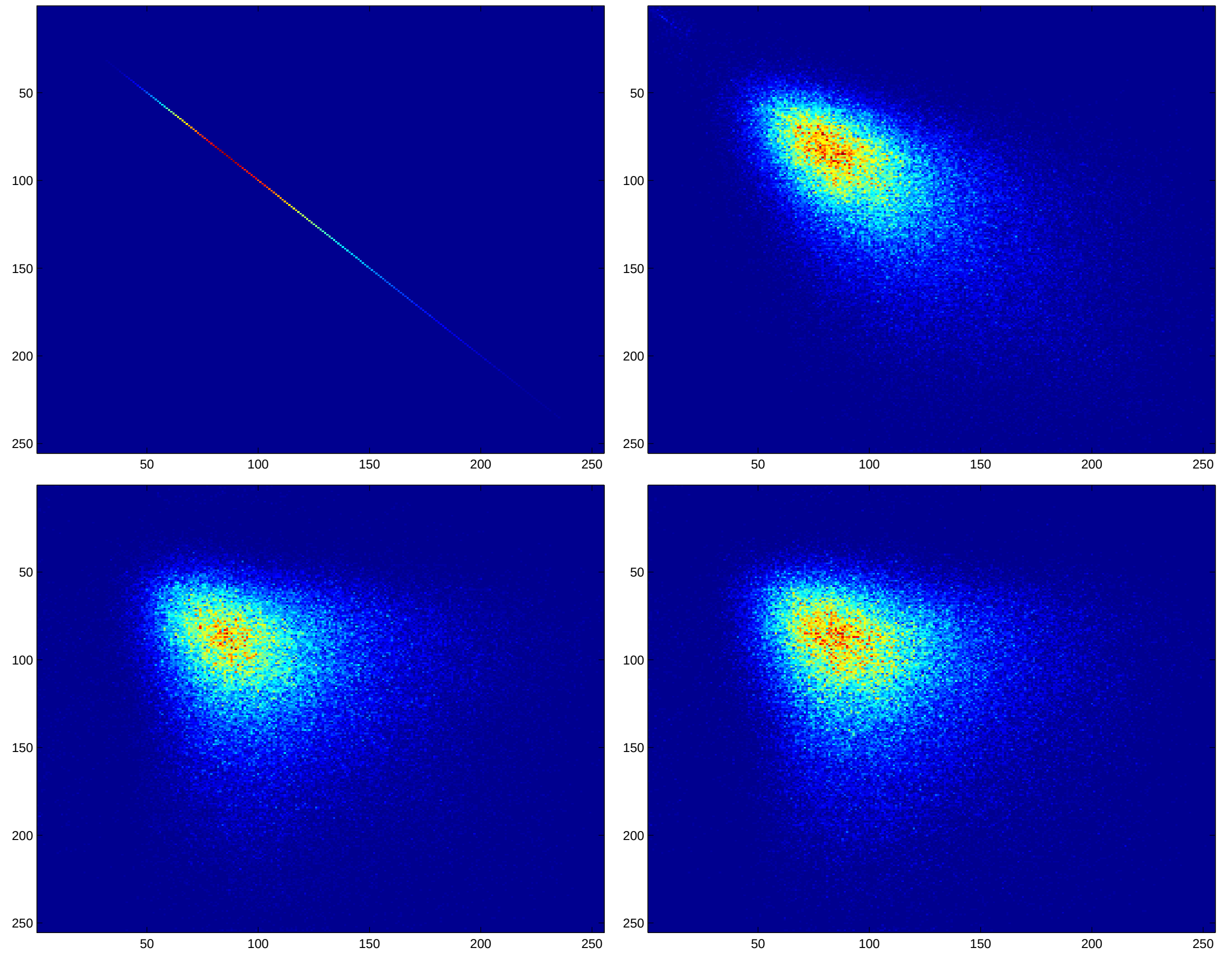


Figure 15: Grey level scatterplots. 1st Col: target=reference slice. 2nd Col: target = reference+1 slice.

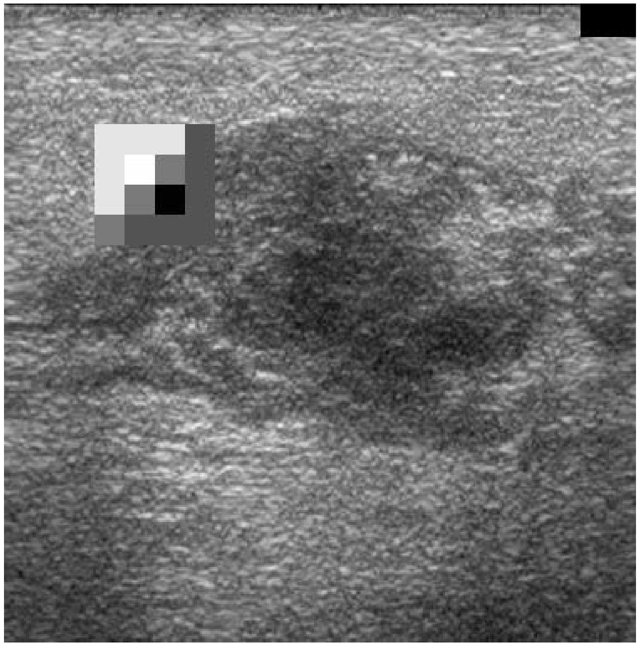
Higher Level Features

Disadvantages of single-pixel features:

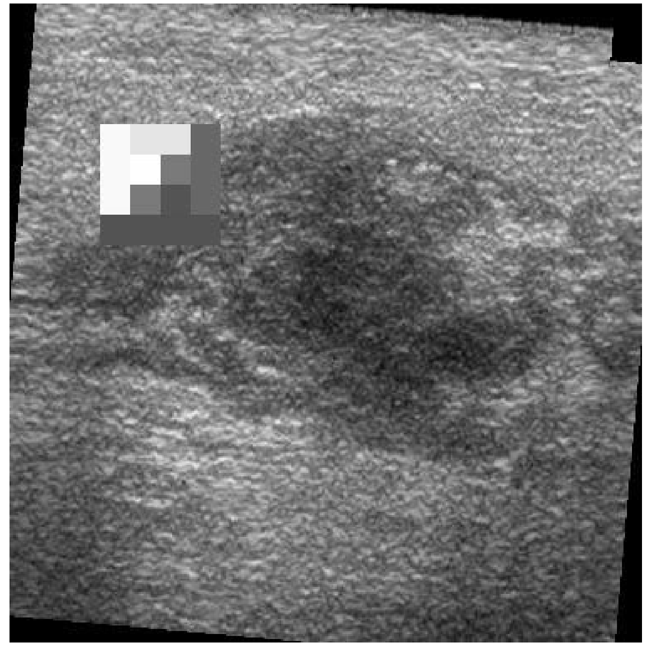
- Only depends on histogram of single pixel pairs
- Insensitive to spatial reording of pixels in each image
- Difficult to select out grey level anomalies (shadows, speckle)
- Spatial discriminants fall outside of single pixel domain

- **Alternative**: Aggregate spatial features

Local Tags



(a) Image I^R



(b) Image I^T

Figure 16: Local Tag Coincidences

Generalization: α-MI Registration of Coincident Features

- X: a $N \times N$ US image (lexicographically ordered)
- $Z = Z(X)$: a general image feature vector in a d-dimensional feature space

Let $\{Z_0(k)\}_{k=1}^K$ and $\{Z_i(k)\}_{k=1}^K$ be features extracted from X_0 and X_i at K pairs of identical spatial locations

α-MI coincident-feature criterion

$$\mathrm{T} = \mathrm{argmax}_{\mathrm{T}_i} \hat{\mathrm{MI}}_\alpha$$

where $\hat{\mathrm{MI}}_\alpha$ is an estimate of

$$\mathrm{MI}_\alpha(f_{0,i}) = \frac{1}{\alpha - 1} \log \int \int f_{0,i}^{\alpha}(z_0, z_1) f_0^{1-\alpha}(z_0) f_i^{1-\alpha}(z_1) dz_1 dz_0. \qquad (2)$$

α-MI and Decision Theoretic Error Exponents

$$
\begin{aligned}
H_0 &: Z_0(k), Z_i(k) \text{ independent} \\
H_1 &: Z_0(k), Z_i(k) \text{ o.w.}
\end{aligned}
$$

Bayes probability of error

$$
P_e(n) = \beta(n)P(H_1) + \alpha(n)P(H_0)
$$

Chernoff bound

$$
\liminf_{n\to\infty} \frac{1}{n} \log P_e(n) = -\sup_{\alpha\in[0,1]} \{(1-\alpha)\mathrm{MI}_\alpha(f_{0,i})\}.
$$

ICA Features

Decomposition of $M \times M$ tag images $Y(k)$ acquired at $k = 1, \ldots, K$ spatial locations

$$Y(k) = \sum_{p=1}^{P} a_{kp} S_p$$

- $\{S_k\}_{k=1}^{P}$: statistically independent components
- a_{kp}: projection coefficients of tag $Y(k)$ onto component S_p
- $\{S_k\}_{k=1}^{P}$ and P: selected via FastICA
- Feature vector for coincidence processing:

$$Z(k) = [a_{k1}, \ldots, a_{kP}]^T$$

ICA feature basis for US breast images

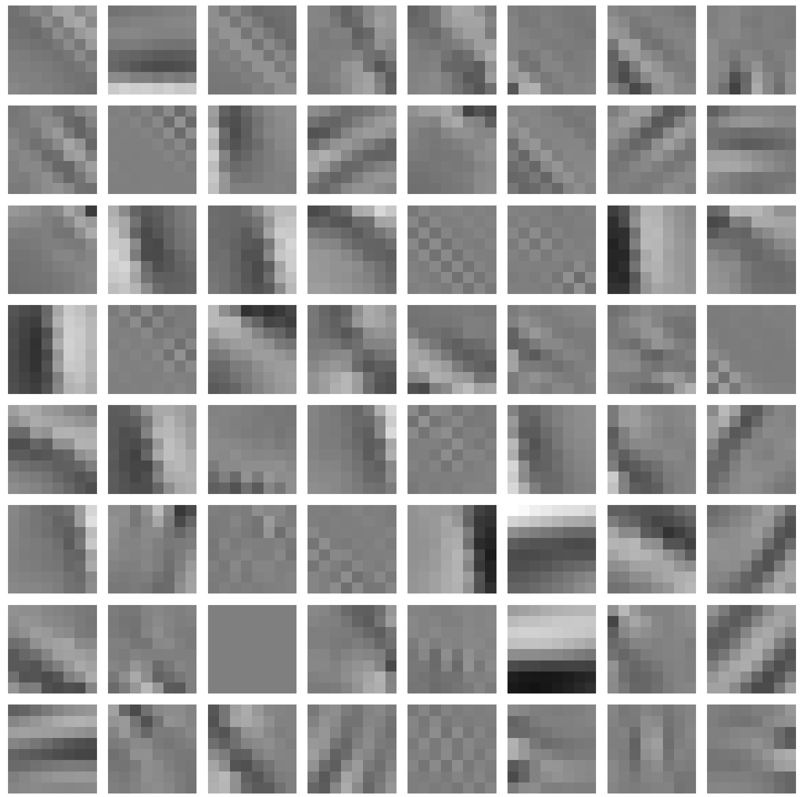


Figure 17: *Estimated ICA basis set using FastICA*

Simpler Objective Function: α-Jensen Difference

1. Extract features from reference and transformed target images:

$$X_m = \{X_i\}_{i=1}^m \quad and \quad Y_n = \{Y_i\}_{i=1}^n$$

2. Construct following MST function on X_m and Y_n

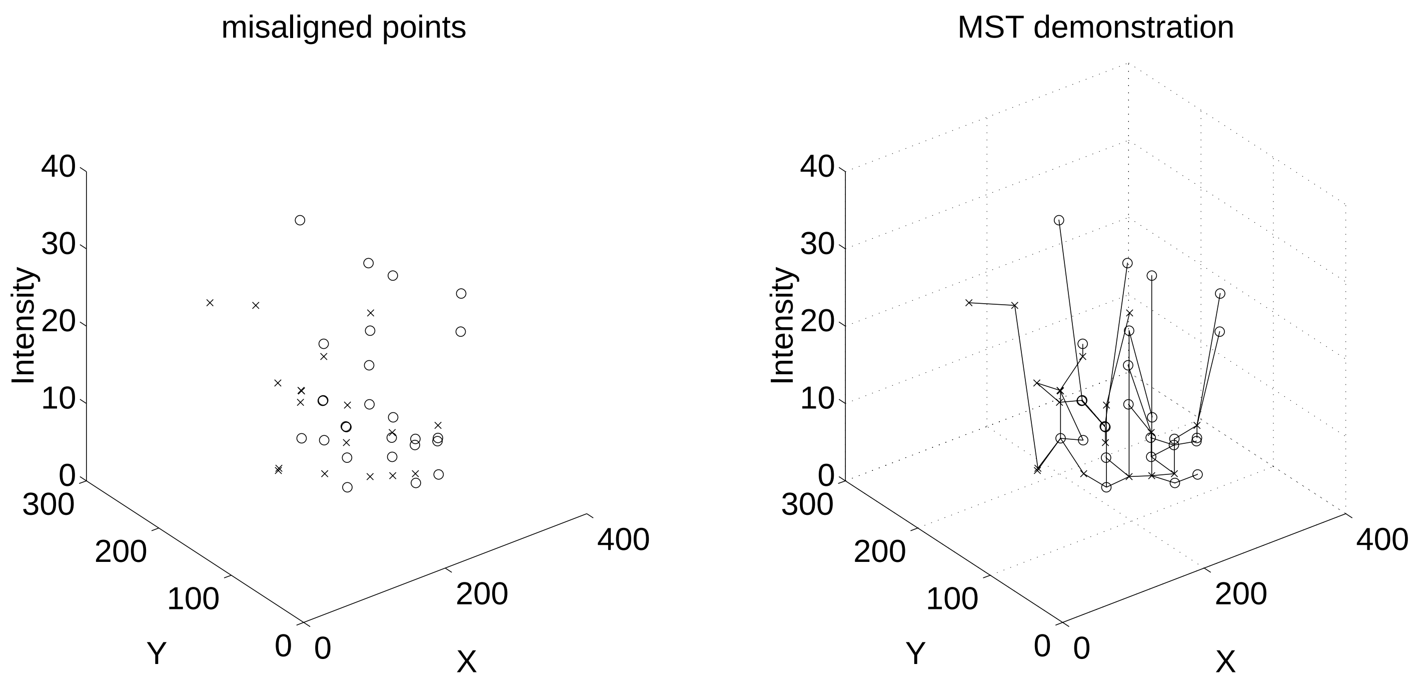
$$\Delta L = \ln L_\gamma(X_m \cup Y_n)/(n+m)^\alpha - \frac{m}{n+m} \ln L_\gamma(X_m)/m^\alpha - \frac{n}{n+m} \ln L_\gamma(Y_n)/n^\alpha$$

3. Minimize ΔL_γ over transformations producing Y_n.

$$(1-\alpha)^{-1}\Delta L \quad \rightarrow \quad H_\alpha(\varepsilon f_x + (1-\varepsilon) f_y) - \varepsilon H_\alpha(f_x) - (1-\varepsilon) H_\alpha(f_y)$$

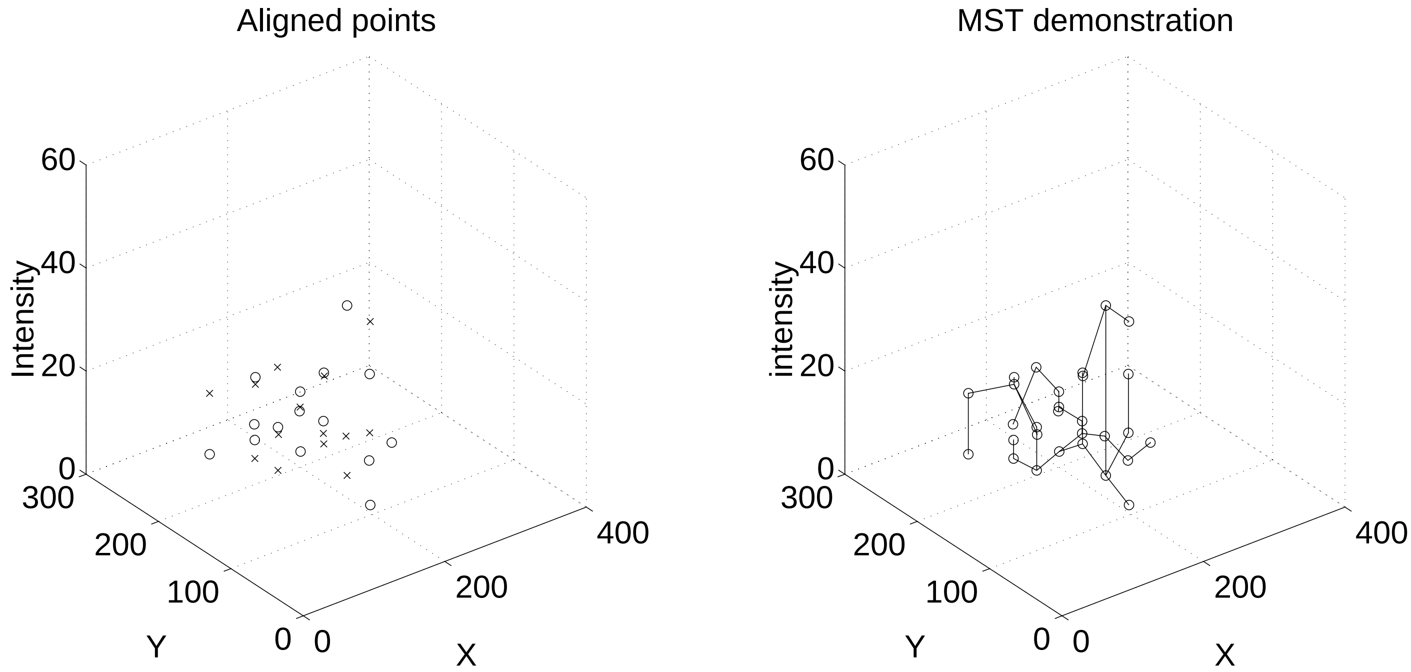
where $\varepsilon = \frac{m}{m+n}$

<u>Illustration</u>



(a) (b)

Figure 18: MST demonstration for misaligned images



(a) (b)

Figure 19: MST for aligned images. “x” denotes reference while “o” denotes a candidate image in the DEM database.

Quantitative Performance Comparisons for US Registration

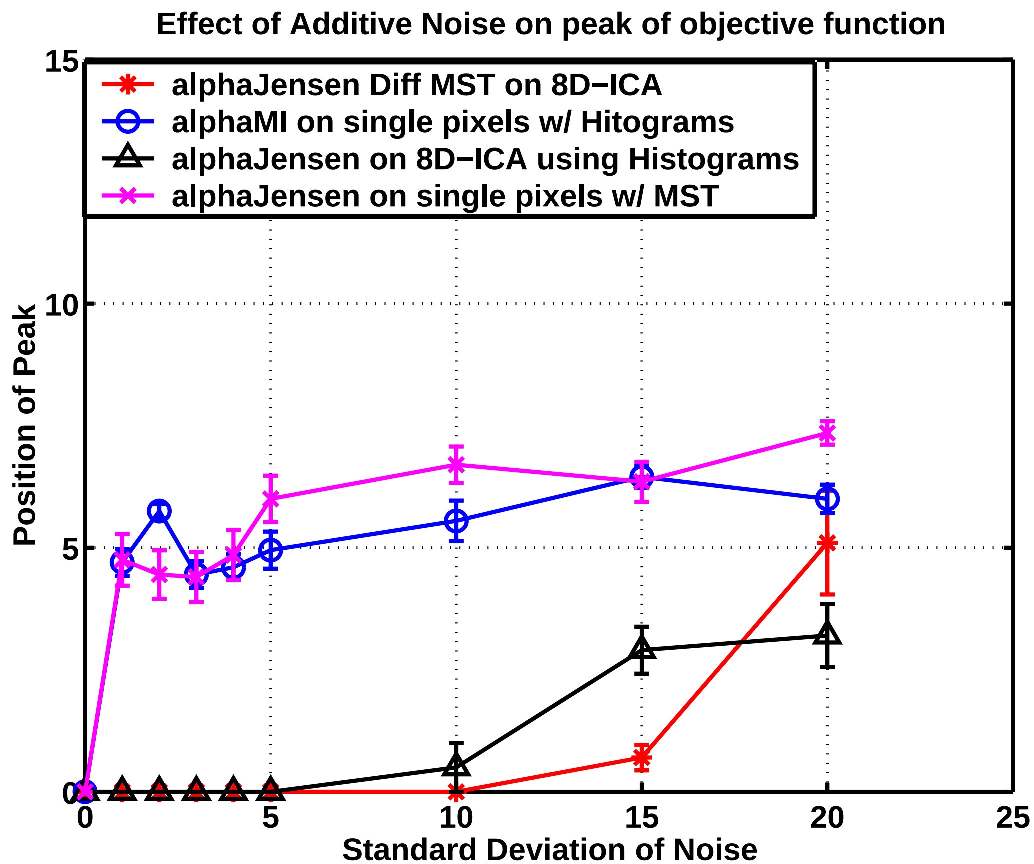


Figure 20: *US registration MSE comparisons.*

Conclusions

1. Entropic graphs can be used to estimate α-entropy and α-divergence
2. MST and k-NN applied to high dimensional feature-based image registration
3. Clustering using entropic K-point graphs
4. Extensions to larger class of continuous quasi-additive graphs (Yukich)