Optimal Signal Constellations for Fading Space-Time Channels

Alfred O. Hero

University of Michigan - Ann Arbor

Outline

1. Space-time channels
2. Random coding exponent and cutoff rate
3. Discrete K-dimensional constellations
4. Bound on minimum distance
5. Low dimensional constellations
6. Conclusions

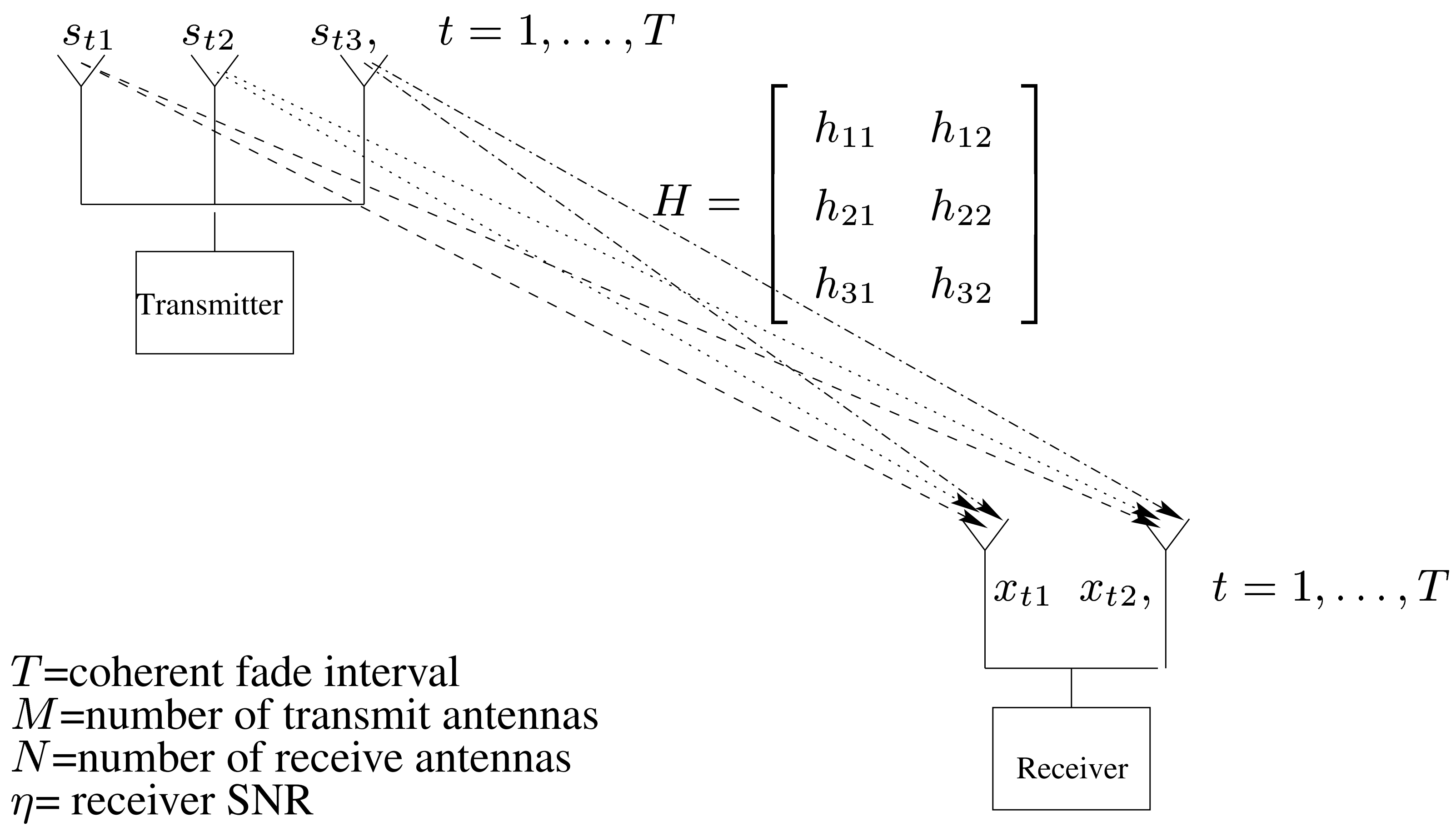


Figure 1. Narrowband space time channel for $M = 3$, $N = 2$

Received signal in l-th frame $(t = 1, \ldots, T)$

$$[x_{t1}^l, \ldots, x_{tn}^l] = \sqrt{\eta}[s_{t1}^l, \ldots, s_{tm}^l] \begin{bmatrix} h_{11}^l & \cdots & h_{1n}^l \\ \vdots & \vdots & \vdots \\ h_{m1}^l & \cdots & h_{mn}^l \end{bmatrix} + [w_{t1}^l, \ldots, w_{tn}^l],$$

or, equivalently

$$X^l = \sqrt{\eta} S^l H^l + W^l$$

- X^l: $T \times N$ received signal matrices
- S^l: $T \times M$ transmitted signal matrices
- H^l: i.i.d. $M \times N$ channel matrices $\sim \mathcal{CN}(0, I_M \bigotimes I_N)$
- W^l: i.i.d. $T \times N$ noise matrices $\sim \mathcal{CN}(0, I_T \bigotimes I_N)$

Block coding over L frames produces blocks of L symbols

$$[S^1, \ldots, S^L]$$

where $S = S^l$ is selected from a symbol alphabet $\mathcal{S}$

Random Block Coding: select S^l at random from $\mathcal{S}$ according to probability distribution $P \in \mathcal{P}$.

- Objective: Find optimal distribution $P(S)$ over $\mathcal{P}$
- Optimality criteria: capacity, outage capacity, random coding error exponent, cut-off rate
- Transmitter constraints:
  - average power constraint: $E[\|S\|^2] = \int \|S\|^2 dP \leq TM$
  - peak power constraint: $\|S\|^2 \leq TM$, for all $S \in \mathcal{S}$

  where

$$\|S\|^2 = \mathrm{tr}\{S^H S\}$$

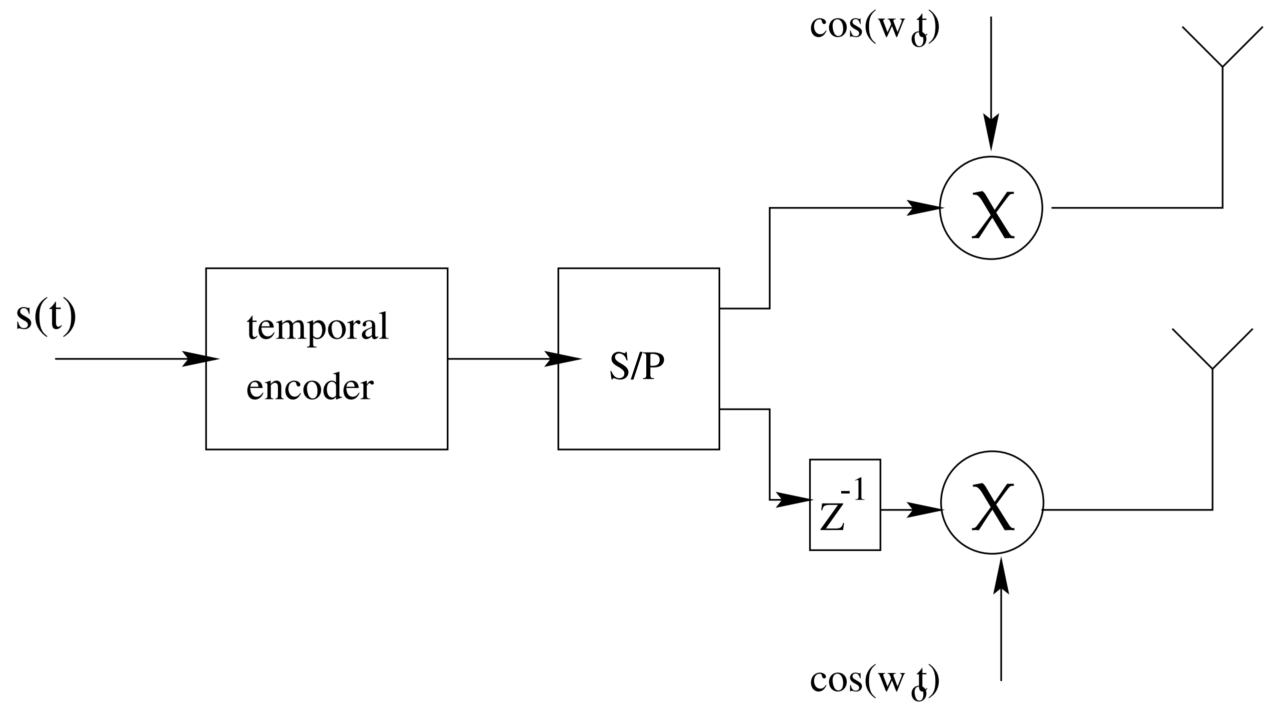


Figure 2. First generation space-time coding: Seshadri&Winters FDD transmitter with delay diversity (1994)

Capacity Results: (avg. power constraint - Telatar, BLTM 95):

1. **Channel H^l Known to Txmt and Rcv**:

Capacity: (*bits/sec/hz*) or ($\frac{bits/channel-use}{T}$)

$$C = \max_{P(S|H)} E[I(S,X|H)] = \max_{P(S|H)} E[\mathcal{H}(X|H) - \mathcal{H}(X|S,H)]$$

α**-Outage Capacity**: $C = \{C_o : P(C(H) > C_o) = \alpha\}$

Since

$$\mathcal{H}(X|H) \leq \ln(|I_N + \eta H^H R_s H|), \quad \text{and} \quad \mathcal{H}(X|S,H) = \mathcal{H}(N)$$

$$\begin{aligned} C(H) &= \max_{R_s \,:\, \mathrm{tr}\{R_s\} \leq M} \ln(|I_N + \eta H R_s H^H|) \\ &= \ln(|I + \eta H R_s^o H|) \end{aligned}$$

where, for $H = UDV$

$$R_s^o = V^H \mathrm{diag}\left(\mu - \frac{1}{\eta d_i^2}\right)^+ V$$

and μ is such that (water-filling)

$$\mathrm{tr}\{R_s^o\} = M$$

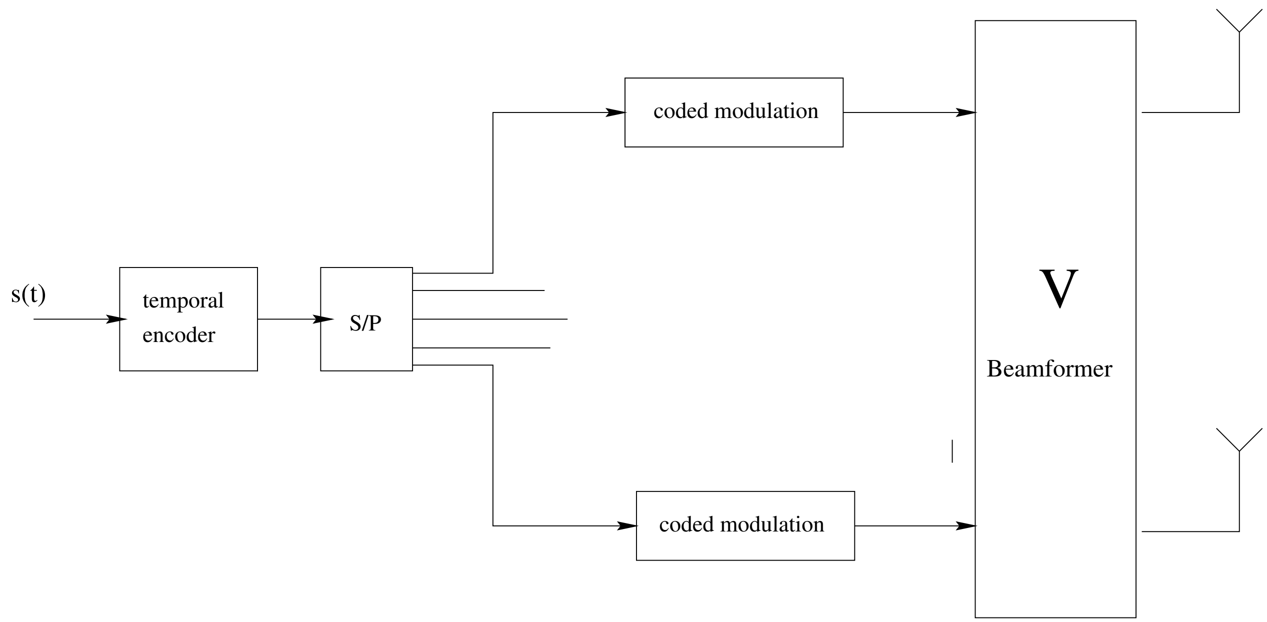


Figure 3. Second generation space-time coding with beamforming

2. **Channel H^l known only to Rcv**:

$$C = \max_{P(S)} E[I(X, S|H)] = \max_{P(S)} E[\mathcal{H}(X|H) - \mathcal{H}(X|S, H)]$$

$\Rightarrow C = E\left[\log\left|I_N + \frac{\eta}{M}\ H^H H\right|\right]$

Capacity achieving distribution:

• S Gaussian with orthogonal rows and columns of identical energy

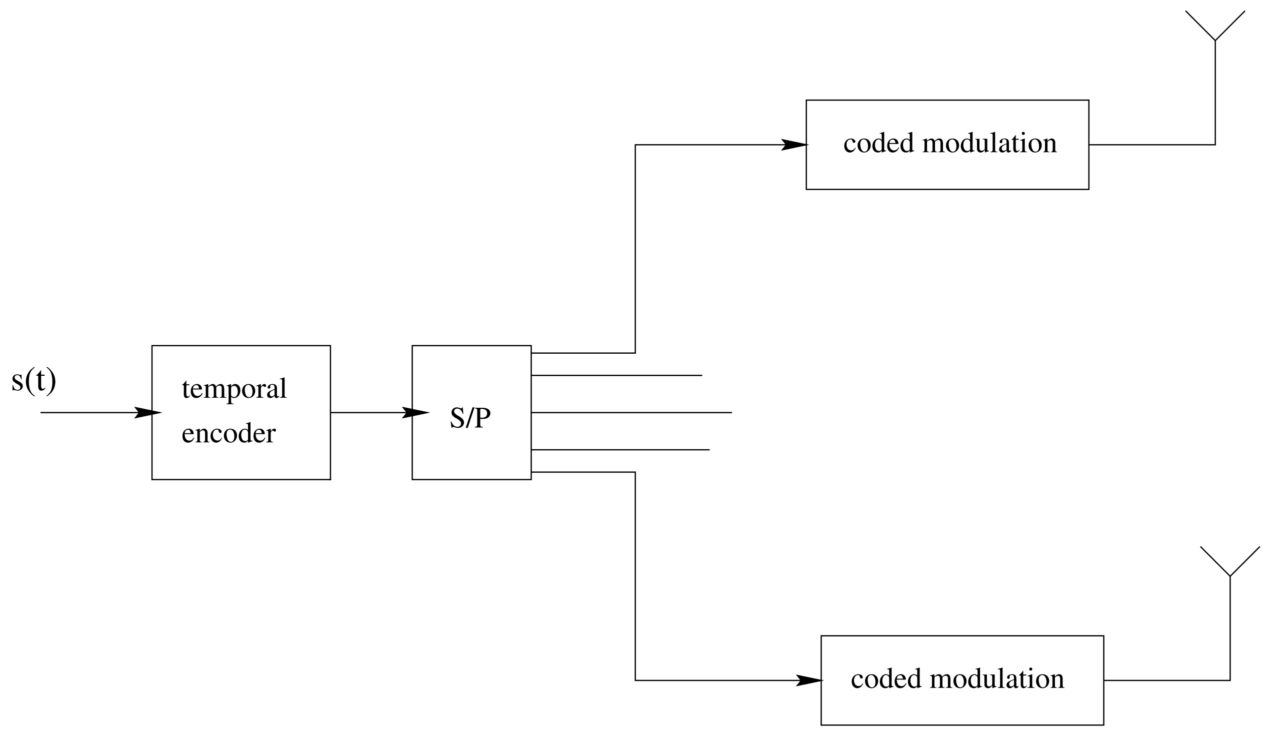
$\Rightarrow$ BLAST (Foschini, BLTJ 1996)

$\Rightarrow$ Space time 4-PSK/4-TCM (Tarokh&etal IT 98, Tarokh&etal COM 99)

In practice must transmit training within each frame to learn H

Capacity bounds: (Hochwald&Marzetta SPIE99, Driesen&Foschini COM99)

$$\underbrace{\log(1+\eta MN)}_{\text{``$=$''when rank}(H)=1} \le C(H) \le \underbrace{\min(M,N)\log\left(1+\frac{\eta MN}{\min(M,N)}\right)}_{\text{``$=$''when rank}(H)=\min(M,N)}$$



- 4-PSK/4-TCM: 2 bits/sec/Hz (simulation), M=N=2
- BLAST: 1.2 Mbps over 30kHz (40 bits/sec/Hz) in 800MHz band, M=8, N=12

Figure 4. Second generation space-time coding

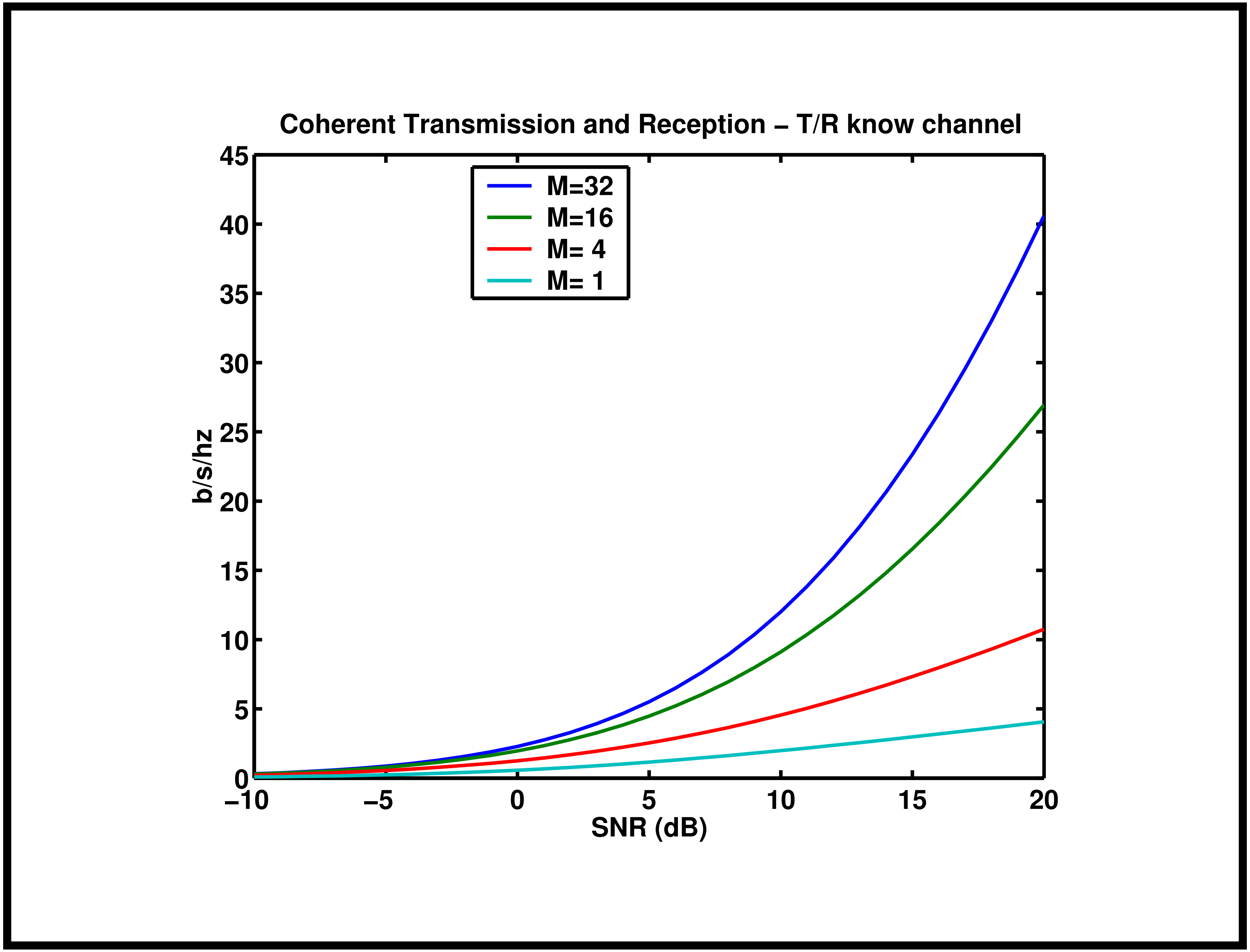


Figure 5. Capacity with perfect channel information at transmitter and receiver.

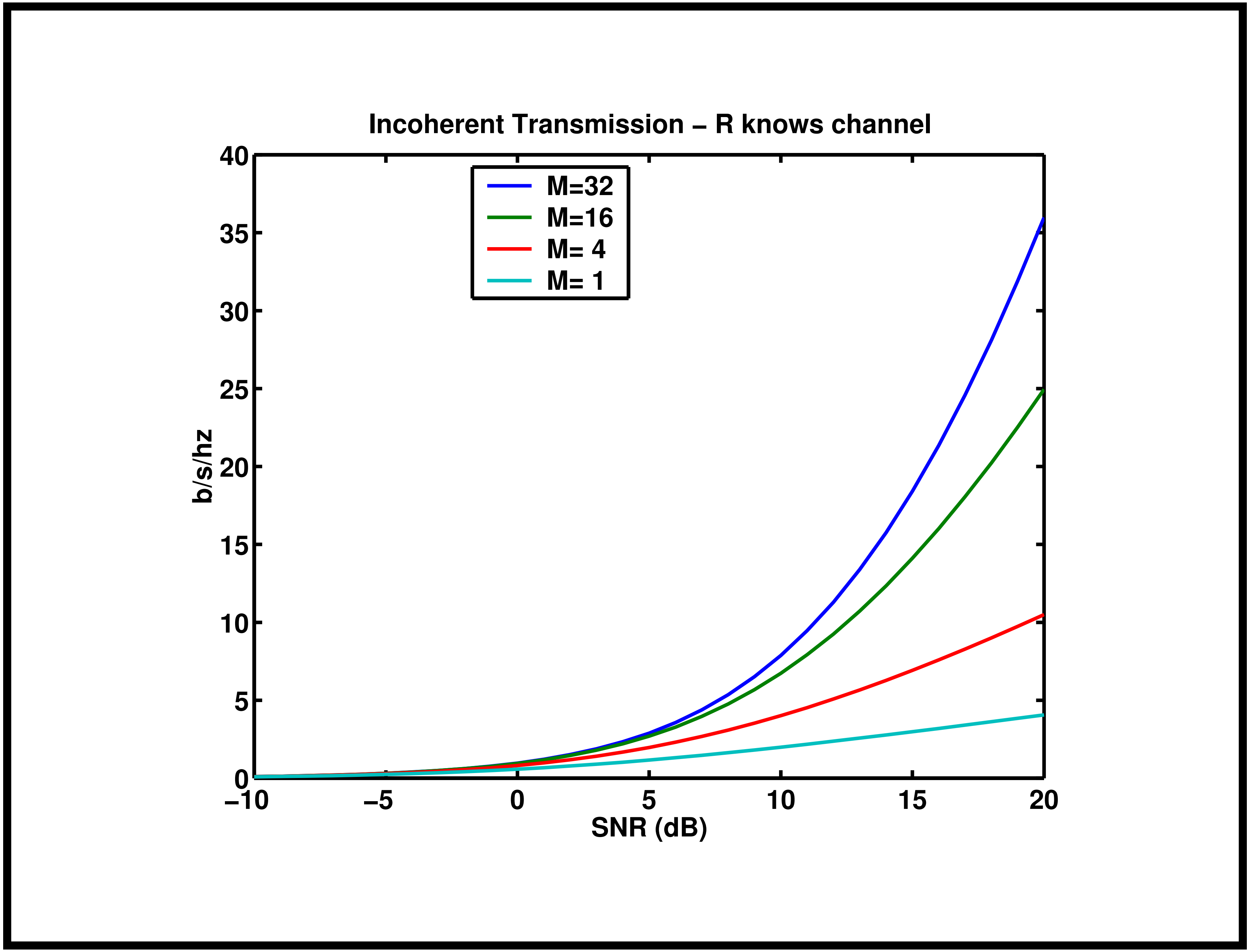


Figure 6. Capacity with incoherent transmission and perfect channel information at receiver.

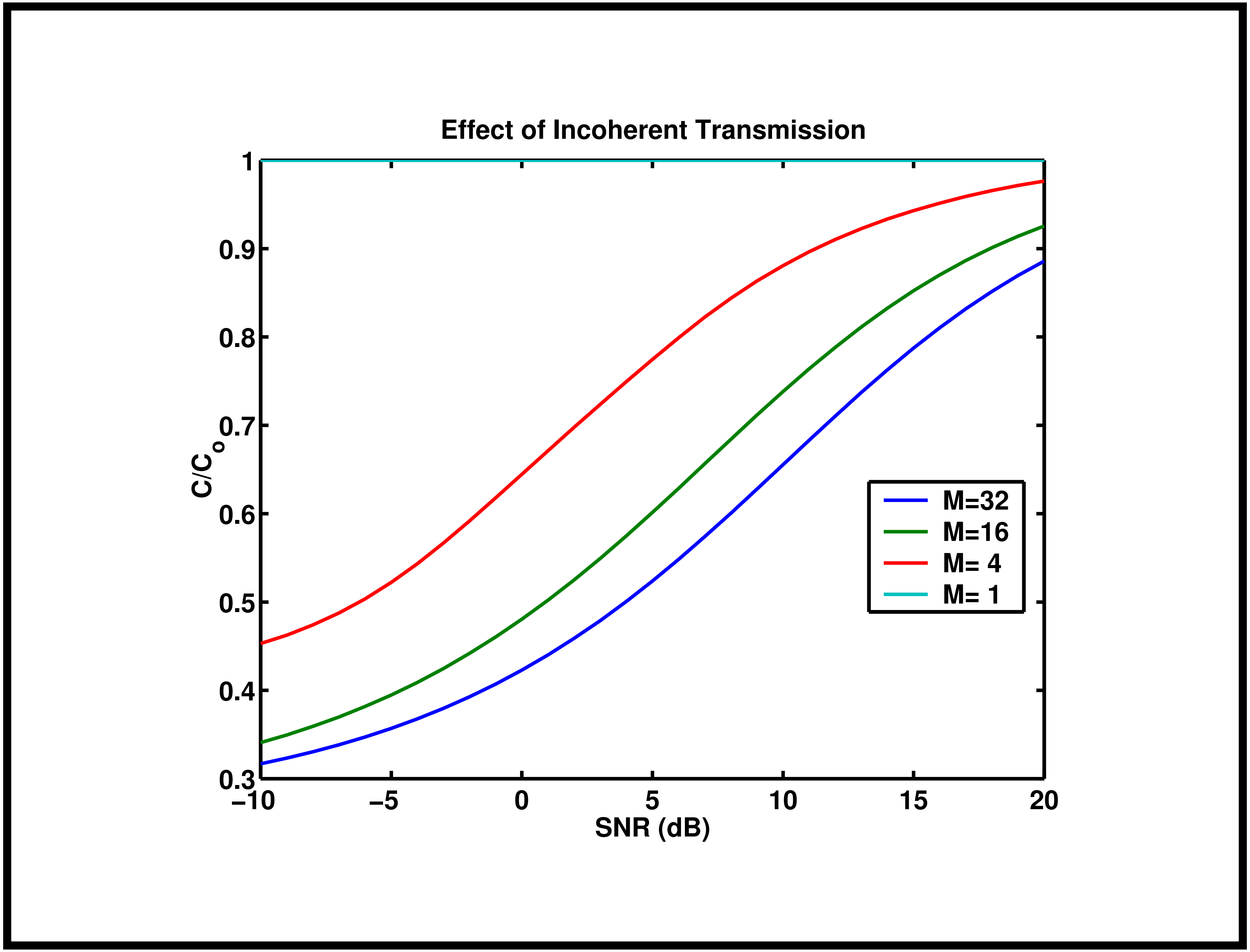


Figure 7. Capacity loss due to incoherent transmission.

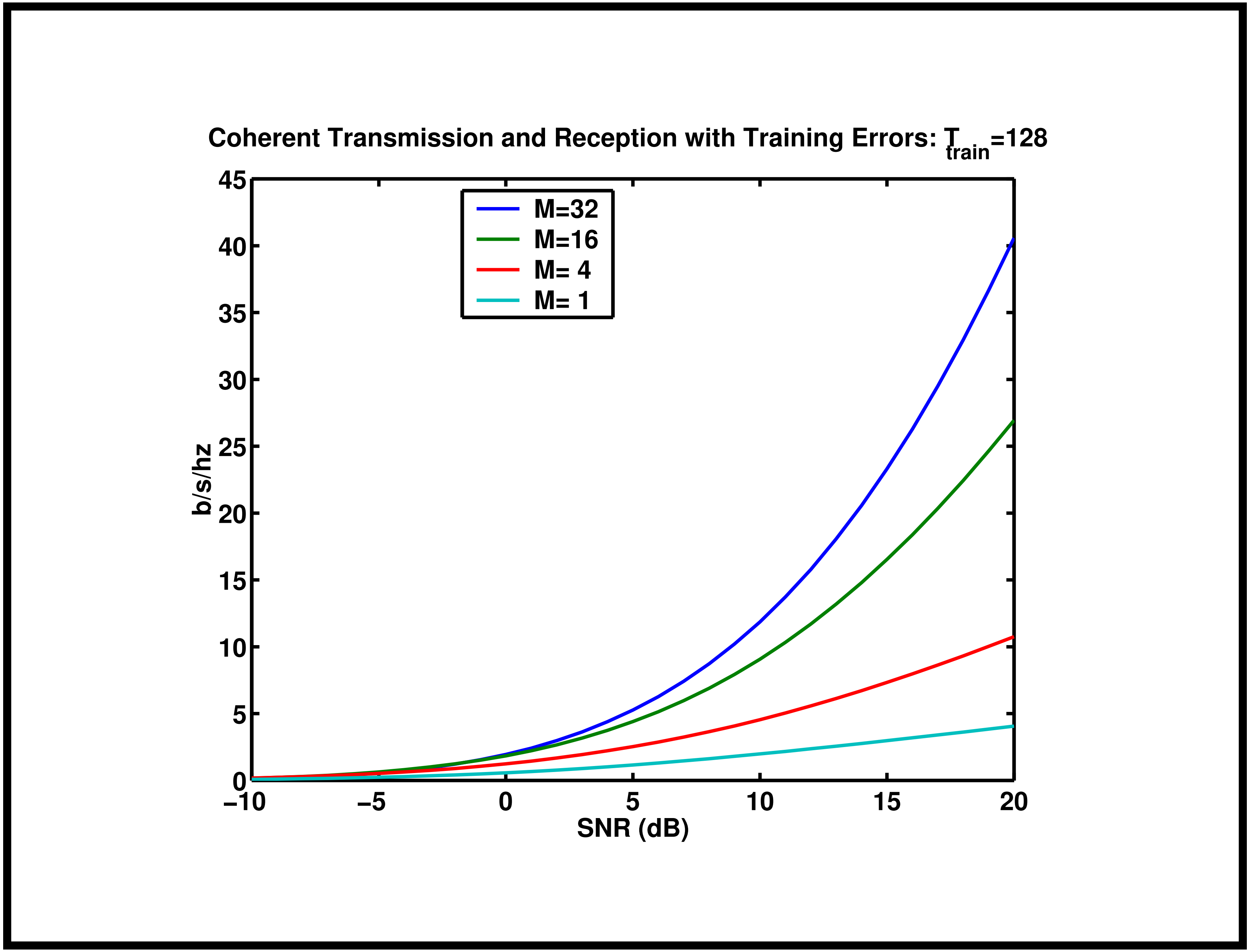


Figure 8. Capacity with coherent transmission and channel errors.

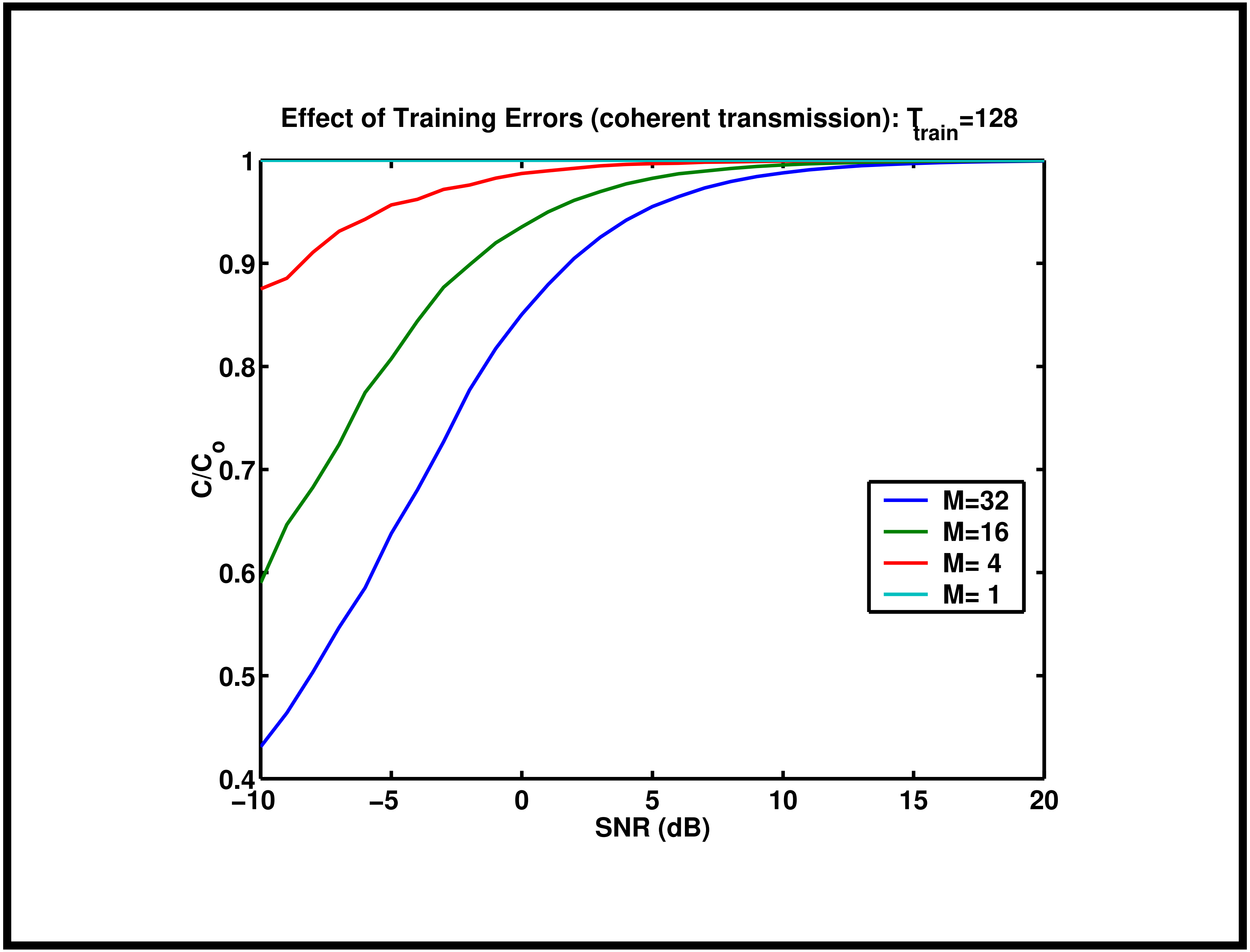


Figure 9. Capacity loss due to channel estimation errors at transmitter and receiver.

3. **Slow fading Rayleigh channel:** H **unknown**

Capacity? (Marzetta&Hochwald, BL TM 98, IT 99)

• $C = E\left[\log P(X|S)/P(X)\right]$ (bits/channel-use)

Capacity achieving distribution?

• $S = \Phi V$

where Φ and V are mutually independent matrices

- Φ: $T \times M$ unitary: $\Phi^H \Phi = I_M$
- V: $M \times M$ real diagonal

$$V \to cI_M \text{ as } \eta \to \infty \text{ or } T \to \infty.$$

$\Rightarrow$ Unitary space-time modulation (Hochwald&etal BL TM 1998)

$\Rightarrow$ Differential space-time modulation (Hochwald&Sweldens COM99)

$\Rightarrow$ Space-time group codes (Hughes SAM 00, Hassibi&etal BLTM 00)

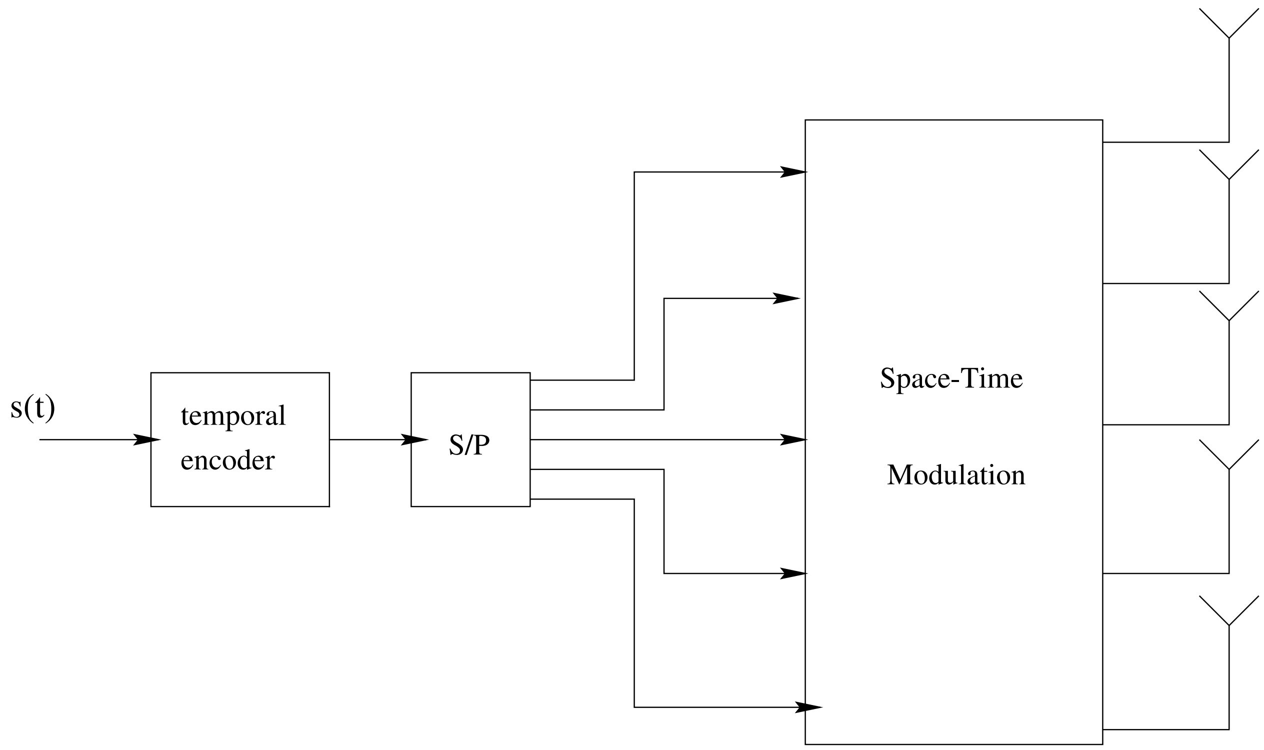


Figure 10. Third generation space-time coding.

Example: Unitary space-time constellation (Hochwald&etal BLTM 98)

- $T = 8$, $M = 3$, $K = 256$ unitary signal matrices

$$\mathcal{S} = \{\Phi_1, \ldots, \Phi_K\}, \qquad \Phi_k = \Theta^k \Phi_1$$

$$\Phi_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{8}5} & e^{j\frac{2\pi}{8}6} \\ 1 & e^{j\frac{2\pi}{8}2} & e^{j\frac{2\pi}{8}4} \\ 1 & e^{j\frac{2\pi}{8}7} & e^{j\frac{2\pi}{8}2} \\ 1 & e^{j\frac{2\pi}{8}4} & 1 \\ 1 & e^{j\frac{2\pi}{8}1} & e^{j\frac{2\pi}{8}6} \\ 1 & e^{j\frac{2\pi}{8}6} & e^{j\frac{2\pi}{8}4} \\ 1 & e^{j\frac{2\pi}{8}3} & e^{j\frac{2\pi}{8}2} \end{bmatrix}, \qquad \Theta = \begin{bmatrix} e^{j\frac{2\pi}{8}(0)} & 0 & \cdots \\ 0 & \ddots & \\ 0 & \cdots & e^{j\frac{2\pi}{8}(7)} \end{bmatrix}$$

1 Random Coding Error Exponent

The minimum error probability of any decoder of a block code over L frames satisfies (Fano 61)

$$\min P_e \leq e^{-LE_U(R)}, \quad R < C$$

where

- R: symbol rate (nats/symbol)
- C: channel capacity (nats/symbol)
- $E_U(R)$: error exponent

$$E_U(R) =$$
$$\max_{\mu\in[0,1]} \max_{P\in\mathcal{P}} \left\{ -\mu R - \ln \int_{X\in\mathcal{X}} \left[\int_{S\in\mathcal{S}} \left(p(X|S)\right)^{1/(1+\mu)} \, dP(S) \right]^{1+\mu} dX \right\}, \quad \text{nats/sym},$$

$E_U(R)$ has been studied under *avg. power constraint* for

$\Rightarrow$ Known H (Telatar BL TM 96)

$\Rightarrow$ Unknown H (Abou-Faycal & Hochwald BL TM 99)

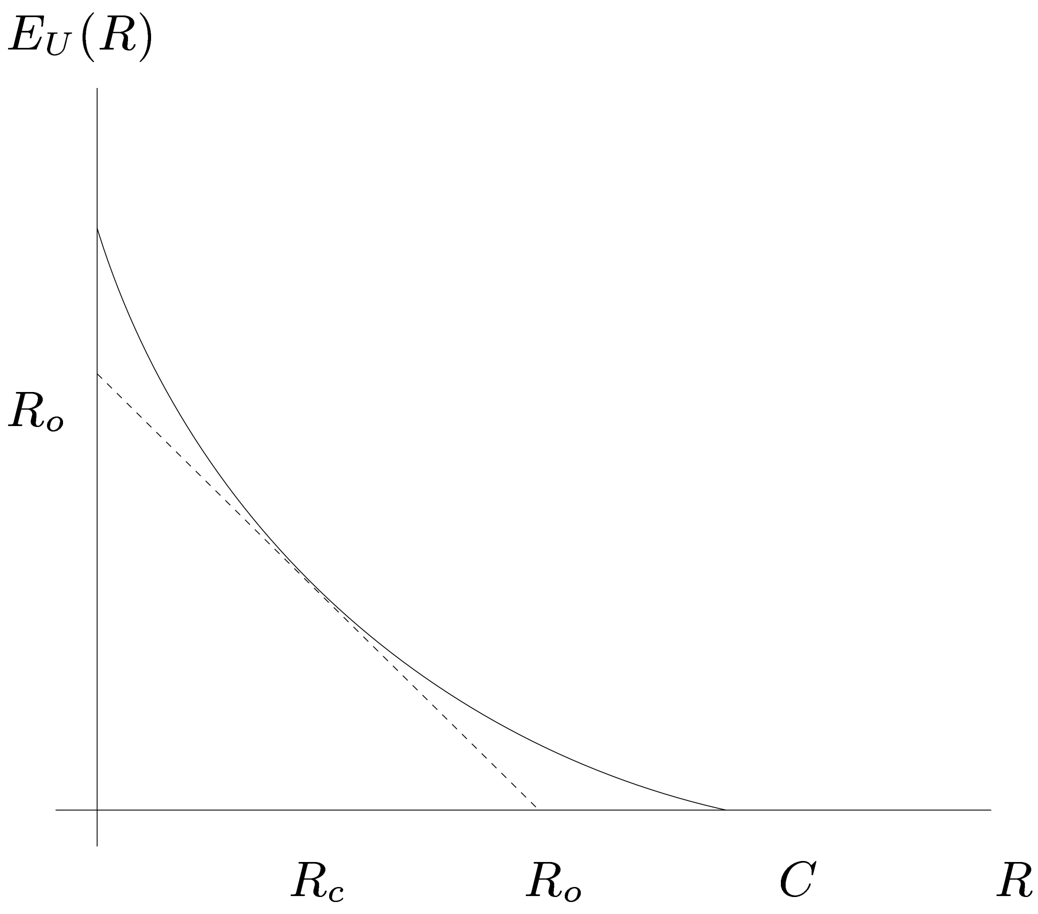


Figure 11. Error exponent $E_U(R)$ and cut-off rate bound $y(R) = R_o - R$.

R_o ***computational cut-off rate*** **lower bound** (Gallager IT 64)

$$E_U(R) \geq R_o - R, \quad R \leq R_o$$

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} \sqrt{p(X|S)} dP(S) \right]^2 dX, \quad \text{nats/symbol}$$

where $\mathcal{P}$ are suitably constrained distributions over $\mathbb{C}^{T \times M}$

$\Rightarrow$ Cut-off rate analysis has been used to evaluate

- practical coding limits (Wang&Costello COM 95, Hagenauer&etal IT 96)
- different coding and modulation schemes (Massey 74)
- signal design for optical fiber links (Snyder&Rhodes IT 80)
- signaling over multiple access channels (Narayan&Snyder IT81)

FACTS:

- $R_o \leq C$
- R_o is highest practical rate for sequential decoders (Savage 65)
- $E_U(R) \approx R_o - R$ when $R \approx R_c$, the critical rate
- R_o specifies upper bound on optimal decoder error

$$P_e \leq e^{-L(R_o - R)}, \quad R \leq R_o$$

2 Integral Representation for R_o

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{S_1 \in \mathcal{S}} dP(S_1) \int_{S_2 \in \mathcal{S}} dP(S_2)\ e^{-N D(S_1 \| S_2)}.$$

where

$$D(S_1 \| S_2) \quad \stackrel{\text{def}}{=} \tfrac{1}{2} \ln \frac{\left| I_T + \frac{\eta}{2}(S_1 S_1^H + S_2 S_2^H) \right|^2}{\left| I_T + \eta S_1 S_1^H \right| \left| I_T + \eta S_2 S_2^H \right|}.$$

Low SNR approximation:

$D(S_1 \| S_2) = \eta^2 / 8 \| S_1 S_1^H - S_2 S_2^H \|^2 + o(\eta^2)$

The following parallels Theorems 1 and 2 of Marzetta&Hochwald IT 99

Proposition 1 *Assume that the transmitted signal S is constrained to satisfy the peak power constraint $\|S\|^2 \leq MT$. There is no advantage to using $M > T$ transmit antennas. Furthermore, for $M \leq T$ the signal matrices achieving R_o can be expressed as*

$$S = \begin{bmatrix} \\ \Phi \\ \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix}$$

where

- Φ *is* $T \times M$ *unitary matrix* $V^H V = I_M$
- Λ *is* $M \times M$ *non-negative diagonal matrix.*

3 Case of Discrete K-dimensional Constellations

Specialize $\mathcal{P}$ to the discrete distributions over $\mathbb{C}^{T\times M}$

Then $R_o = \tilde{R}_o(K)$ is given by

$$\max_{\{P_i,S_i\}_{i=1}^K} -\ln \sum_{i=1}^K P_i \sum_{j=1}^K P_j \; e^{-ND(S_i\|S_j)} = -\ln \min_{\{P_i,S_i\}_{i=1}^K} \underline{P}^T E_K \underline{P}$$

where

- $E_k = ((D(S_i|S_j))_{i,j=1}^K$: dissimilarity (distance) matrix
- $\underline{P} = [P_1, \ldots, P_K]^T$

Under peak power constraint, $\|S_i\| \leq TM$,

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^{K}} \left(\min_{\{P_i\}_{i=1}^{K}} \underline{P}^T E_K \underline{P} \right)$$

Inner maximization:

$$\min_{\underline{P}>0 \,:\, \underline{1}_K^T \underline{P}=1} \{\underline{P}^T E_K \underline{P}\}$$

Lagrangian

$$J(\underline{P}) = \underline{P}^T E_K \underline{P} - 2c(\underline{1}_K^T \underline{P} - 1)$$

minimized for *equalizer probability* $\underline{P} = \underline{P}^*$

$$E_K \underline{P}^* = c\underline{1}_K \quad \Rightarrow \quad \sum_{j=1}^{K} P_j \, e^{-ND(S_i \| S_j)} = c$$

Fact: optimal constellation satisfies $E_K^{-1}\underline{1}_K \geq 0$

$$\underline{P}^* = cE_K^{-1}\underline{1}_K, \quad \text{and} \quad c = \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K}$$

and

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K} \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K} = \max_{\{S_i\}_{i=1}^K} \ln \left(\underline{1}_K^T E_K^{-1} \underline{1}_K \right)$$

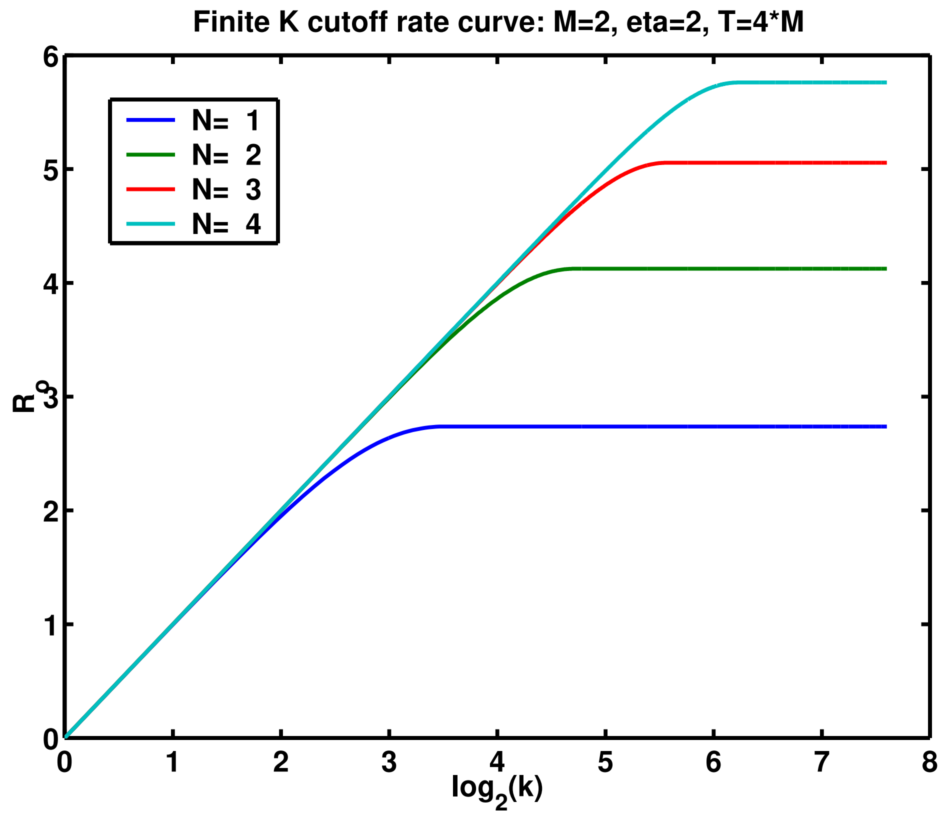
4 Bound on minimum distance

To $o(\eta^2)$ we have bounds

$$D^{**}_{\min} = \max_{\{S_i\}_{i=1}^{K}} \min_{i \neq j} D(S_i \| S_j) \geq \frac{\eta^2}{8} \frac{(TM)^2}{(2^p - 1)^2} > \frac{\eta^2 (TM)^2}{128} K^{-2/T}$$

(a) (b)

Figure 12. *Constellations of signal matrix singular values*


Finite K cutoff rate curve: M=2, eta=2, T=4*M
N= 1
N= 2
N= 3
N= 4
R_0
$\log_2(k)$
0
1
2
3
4
5
6
7
8
0
1
2
3
4
5
6

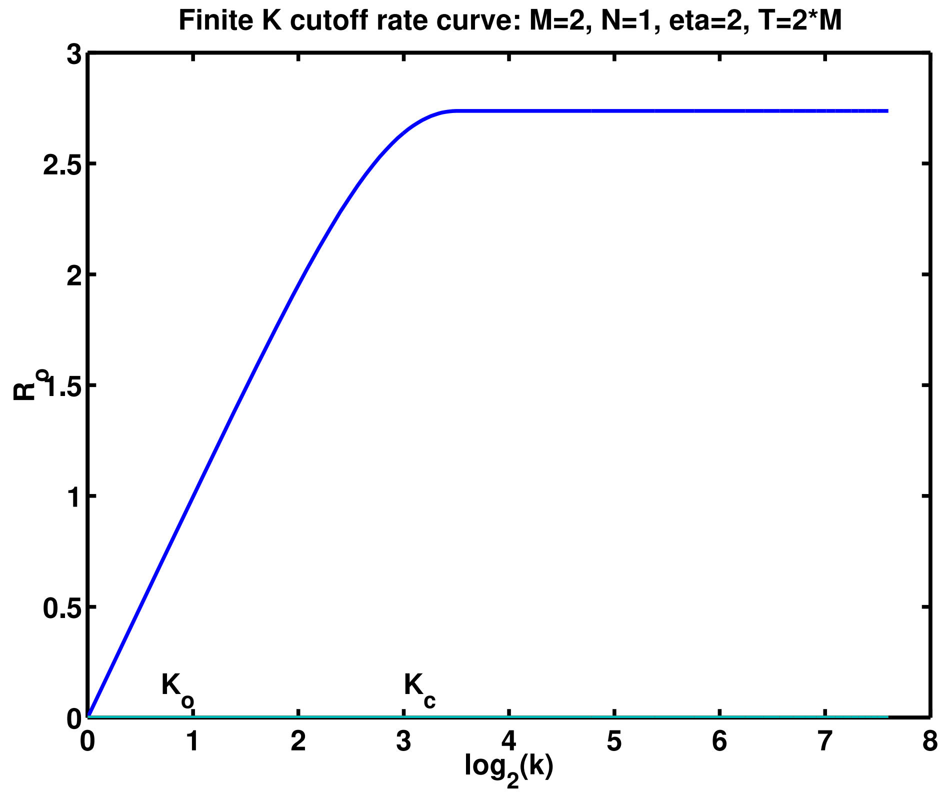


Figure 13. Cutoff-rate curve as function of size K.

Define:

$K_o = \lfloor T/M \rfloor$: "orthogonal size"

= max value K for which closed form expression $\tilde{R}_o$ exists

and

K_c: "logK" transition point

= knee of $\tilde{R}_o$

$\Rightarrow$ diminished returns by increasing K beyond K_c

5 Bound on logK transition point of constellation

Pick "test constellation" $\{S_i\}_{i=1}^{K}$ for which

$$D_{\min} = \min_{i \neq j} D(S_i \| S_j) > \gamma K^{-2/T}$$

$\gamma = \frac{\eta^2 (TM)^2}{128}$.

$$\begin{aligned}
\tilde{R}_o(K) \qquad & \geq \max_{\{P_i\}} \log \left(\frac{1}{\sum_{i,j} P_i P_j e^{-ND(S_i \| S_j)}} \right) \\
& \geq \max_{\{P_i\}} \log \left(\frac{1}{\sum_{i,j} P_i P_j + \sum_{i \neq j} P_i P_j e^{-ND_{\min}}} \right) \\
& = \log \left(\frac{1}{1/K + (K-1)/K \ e^{-ND_{\min}}} \right)
\end{aligned}$$

$$
\begin{aligned}
&> \log\left(\frac{1}{1/K + (K-1)/K \; e^{-N\gamma K^{-2/T}}}\right) \\
&= \log(K) - \log\left(1 + (K-1)\; e^{-N\gamma K^{-2/T}}\right) \\
&\approx \log(K), \qquad (K-1)\; e^{-N\gamma K^{-2/T}} \leq 1
\end{aligned}
$$

This gives lower bound on K_o

$$
K_o \geq \left\{K : K^{2/T} \ln K = \gamma N\right\}
$$

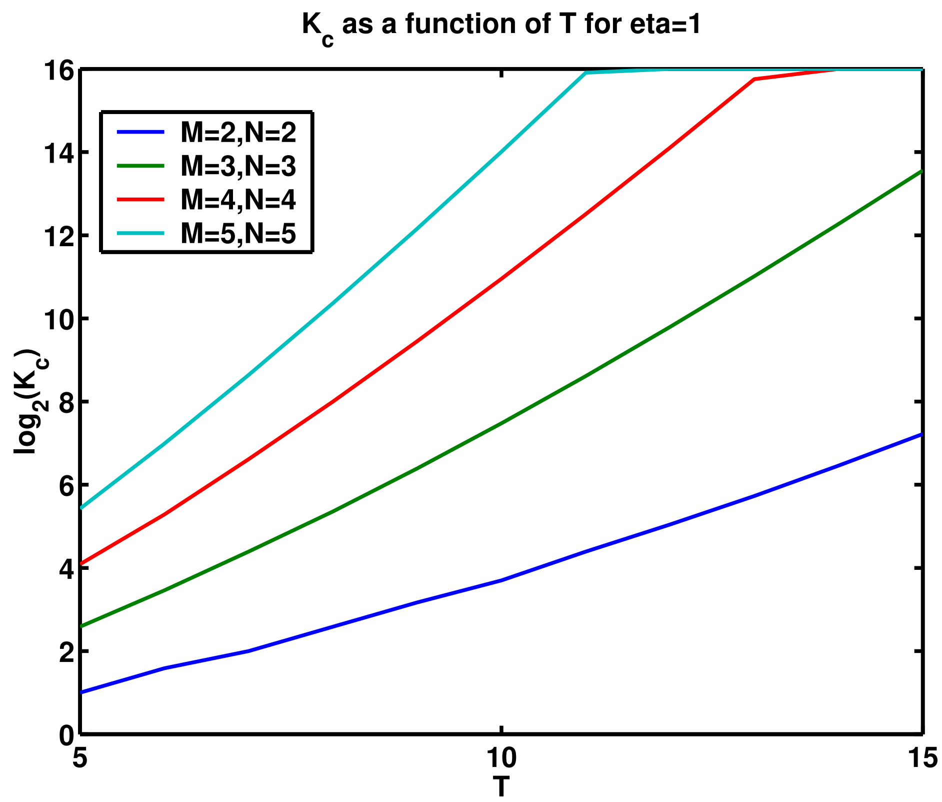


Figure 14. Corner sizes for equal M and N.

6 Low Dimensional Constellations $K \leq K_o$

For given η, T and M define the integer M_o

$$M_o = \text{argmax}_{m \in \{1,\ldots,M\}} \left\{ m \ln \frac{(1 + \eta TM/(2m))^2}{1 + \eta TM/m} \right\}.$$

First a result on max attainable distance under peak power constraint

Proposition 2 *Let* $2M \leq T$*. Then*

$$D_{\max} \stackrel{\text{def}}{=} \max_{S_1, S_2 \in \mathcal{S}^K_{\text{peak}}} D(S_1 \| S_2) = M_o \ln \frac{(1 + \eta TM/(2M_o))^2}{1 + \eta TM/M_o}. \qquad (1)$$

Furthermore, the optimal signal matrices which attain $D_{\max}$ *can be taken as scaled rank* M_o *mutually orthogonal unitary* $T \times M$ *matrices of the form*

$$S_1 = \eta \text{TM}\ \Phi_1, \qquad S_2 = \eta \text{TM}\ \Phi_2$$

where, for $j = 1, 2$,

$$\Phi_j^H \Phi_j = I_{M_o}, \quad \text{and} \quad \Phi_i^H \Phi_j = 0, \;\; i \neq j.$$

Proof is based on alternative representation for $D(S_1 \| S_2)$

$$D(S_1 \| S_2) = \quad \tfrac{1}{2} \ln \frac{\left| I_M + \frac{\eta}{2} S_1^H S_1 \right|^2 \left| I_M + \frac{\eta}{2} S_2^H S_2 \right|^2}{\left| I_M + \eta S_1^H S_1 \right| \left| I_M + \eta S_2^H S_2 \right|} \left| I_M - \kappa^H \kappa \right|^2,$$

where κ is a $M \times M$ multiple signal correlation matrix

$$\kappa = \tilde{S}_2^H \tilde{S}_1$$

$$\tilde{S}_i = \frac{\eta}{2} S_i \left[I_M + \frac{\eta}{2} S_i^H S_i \right]^{-1}$$

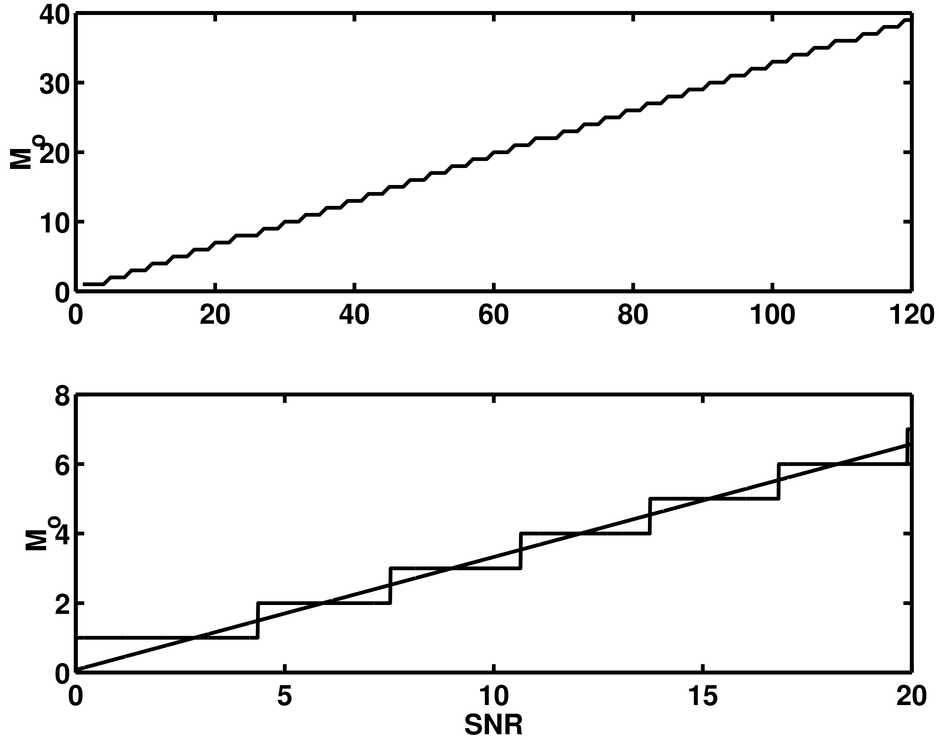


Figure 15. Top panel: M_o as a function of the SNR parameter ηTM. Bottom panel: blow up of first panel over a reduced range of SNR.

Proposition 3 *Let* $2M \leq T$ *and let* M_o *be as defined in (1). Suppose that* $M_o \leq \min\{M, T/K\}$*. Then the peak constrained* K *dimensional cut-off rate is*

$$\tilde{R}_o(K) = \ln\left(\frac{K}{1 + (K-1)e^{-ND_{\max}}}\right)$$

and $D_{\max}$ *is given by (1). Furthermore, the optimal constellation attaining* $\tilde{R}_o(K)$ *is the set of* K *rank* M_o *mutually orthogonal unitary matrices and the optimal probability assignment is uniform:* $P_i^* = 1/K$*,* $i = 1, \ldots, K$*.*

Example constellations for $T \times M = 4 \times 2$

- $M_o = 1$, $K = 4$: $(\eta^2 TM < 4.8)$

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

- $M_o = 2$, $K = 2$: $(\eta^2 TM \geq 4.8)$

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

7 Conclusions

- Peak power contrained cut-off rate reduces to minimizing Q-form
- optimal constellation equalizes the decoder error rates
- Average distance for optimal K-dim constellation decreases at most by $K^{-2/T}$
- Optimal low rate constellation is a set of scaled mutually orthogonal unitary matrices.
- Rank of the unitary signal matrices decreases in SNR
- For very low SNR, no diversity advantage: apply power to a single antenna element at a time.

References

[1] A. O. Hero and T. L. Marzetta, "On computational cut-off rate for space time coding," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 2000.

[2] I. Abou-Faycal and B. M. Hochwald, "Coding requirements for multiple-antenna channels with unknown Rayleigh fading," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 1999.

[3] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh fading," *IEEE Trans. on Inform. Theory*, vol. IT-45, pp. 139–158, Jan. 1999.