STABILITY ANALYSIS OF THE SEQUENTIAL PARTIAL UPDATE LMS ALGORITHM

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ABSTRACT

Partial updating of LMS filter coefficients is an effective method for reducing the computational load and the power consumption in adaptive filter implementations. The Sequential Partial Update LMS algorithm is one popular algorithm in this category. In [5] a first order stability analysis of this algorithm was performed on wide sense stationary signals under the restrictive assumption of small step size parameter μ. The necessary and sufficient condition derived on μ for convergence in the mean was indentical to the one for guaranteeing stability in the mean of LMS. In [7] first order sufficient conditions were derived for stability without the aforementioned small μ assumption. The sufficient region of convergence derived was smaller than that of regular LMS. In this paper, we establish that for stationary signals the sequential algorithm converges in mean for the same values of the step size parameter μ for which the regular LMS does. In other words, we show that the conclusion drawn in [5] holds without the restrictive assumption of small μ. We also derive sufficient conditions for stability on μ for cyclo-stationary signals.

1. INTRODUCTION

Partial updating of the LMS adaptive filter has been proposed to reduce computational costs and power consumption [2, 3, 4] which is quite attractive in the area of of mobile computing and communications. Partial update algorithms have application in many fields including adaptive beamforming, channel equalization in communications and space-time modulation/coding. Sequential Partial Update LMS algorithm is one such algorithm. However, for this algorithm theoretical performance predictions on convergence rate and steady state tracking error are more difficult to derive than for standard full update LMS. Accurate theoretical predictions are important as it has been observed that for the non-stationary signal scenario the standard LMS conditions on the step size parameter fail to ensure convergence of S-LMS.

In [5], under the assumption of small step-size parameter (μ), the bounds derived turned out to be the same as those for the standard LMS algorithm. We were interested in investigating the Sequential Partial Update LMS Algorithm (S-LMS) under large μ conditions for which faster convergence rates could be achieved. In [7], we derived a less restrictive sufficient condition on μ for stationary signals. However, we were unable to find an example of a stationary signal for which regular LMS was stable but S-LMS unstable. We also point out that the motivating example in Section 2 of [7] is incorrect.

This has led us to look for more accurate bounds on μ which hold for stationary signals and arbitrary fixed sequence of partial updates. We prove conclusively that for stationary signals first order stability of LMS implies first order stability of S-LMS without any assumptions on μ (Theorem 2). We also extend the analysis in [7] to cyclo-stationary signals. We show that for the cyclo-stationary case there exists some μ such that use of it in S-LMS could lead to divergence even if full update LMS converges for this μ.

The organization of the paper is as follows. First in Section 2, a brief description of the sequential partial update algorithm is given. The algorithm with arbitrary sequence of updates is analyzed for the case of stationary signals in Section 3. This is followed by the analysis of algorithm with the special case of alternate even and odd coefficient updates for cyclo-stationary signals in Section 4. In Section 5 an example is given to illustrate the usefulness of the bounds on step-size derived in Section 4. Finally, conclusions and directions for future work are indicated in Section 6.

2. ALGORITHM DESCRIPTION

The block diagram of S-LMS for a N-tap LMS filter with alternating even and odd coefficient updates is shown in Figure 1

It is assumed that the LMS filter is a standard FIR filter of even length, N. For convenience, we start with some definitions. Let $\{x_k\}$ be the input sequence and let $\{w_{i,k}\}$ denote the coefficients of the adaptive filter. Define

$$\begin{aligned} W_k &= [w_{1,k}\, w_{2,k}\, \ldots\, w_{N,k}]^T \\ X_k &= [x_k\, x_{k-1}\, x_{k-2}\, \ldots\, x_{k-N+1}]^T \end{aligned}$$

where the terms defined above are for the instant k. In addition, Let d_k denote the desired response. In typical applications d_k is a known training signal which is transmitted over a noisy channel with unknown FIR transfer function.

In this paper we assume that d_k itself obeys an FIR model given by $d_k = W_{opt}^H X_k + n_k$ where W_{opt} are the coefficients of an FIR model given by $W_{opt} = [w_{1,opt}\, \ldots\, w_{N,opt}]^T$. Here $\{n_k\}$ is assumed to be a zero mean i.i.d sequence that is independent of the input sequence $\{x_k\}$.

For description purposes we will assume that the filter coefficients can be divided into P mutually exclusive subsets of equal size, i.e. the filter length N is a multiple of P. For convenience,

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define the index set $S = \{1, 2, \ldots, N\}$. Partition S into P mutually exclusive subsets of equal size, $S_1,\ S_2, \ldots, S_P$. Define I_i by zeroing out the j^{th} row of the identity matrix I if $j \notin S_i$. In that case, $I_i X_k$ will have precisely $\frac{N}{P}$ non-zero entries. Let the sentence "choosing S_i at iteration k" stand to mean "choosing the weights with their indices in S_i for update at iteration k".

The S-LMS algorithm is described as follows. At a given iteration, k, one of the sets S_i, $i = 1, \ldots, P$, is chosen in a predetermined fashion and the update is performed.

$$w_{k+1,j} = \begin{cases} w_{k,j} + \mu e_k^* x_{k,j} & \text{if } j \in S_i \\ w_{k,j} & \text{otherwise} \end{cases} \tag{1}$$

where $e_k = d_k - W_k^H X_k$. The above update equation can be written in a more compact form in the following manner

$$W_{k+1} = W_k + \mu e_k^* I_i X_k \tag{2}$$

In the special case of even and odd updates, $P = 2$ and S_1 consists of all even indices and S_2 of all odd indices as shown in Figure 1.

We also define the coefficient error vector as

$$V_k = W_k - W_{opt}$$

which leads to the following coefficient error vector update for S-LMS when k is odd

$$\begin{aligned} V_{k+2} &= (I - \mu I_2 X_{k+1} X_{k+1}^H)(I - \mu I_1 X_k X_k^H) V_k + \\ &\quad \mu (I - \mu I_2 X_{k+1} X_{k+1}^H) n_k I_1 X_k + \mu n_{k+1} I_2 X_{k+1} \end{aligned} \tag{3}$$

and the following when k is even

$$\begin{aligned} V_{k+2} &= (I - \mu I_1 X_{k+1} X_{k+1}^H)(I - \mu I_2 X_k X_k^H) V_k + \\ &\quad \mu (I - \mu I_1 X_{k+1} X_{k+1}^H) n_k I_2 X_k + \mu n_{k+1} I_1 X_{k+1} \end{aligned} \tag{4}$$

3. ANALYSIS: STATIONARY SIGNALS

Assuming that $\{x_k\}$ is a WSS random sequence, we analyze the convergence of the mean coefficient error vector $E\left[V_k\right]$. We make the standard assumptions that V_k and X_k are mutually uncorrelated and that X_k is independent of X_{k-1} [1]. These assumptions are somewhat restrictive but greatly simplify the analysis. For regular LMS algorithm the recursion for $E\left[V_k\right]$ is given by

$$E\left[V_{k+1}\right] = (I - \mu R) E\left[V_k\right] \tag{5}$$

where I is the N-dimensional identity matrix and $R = E\left[X_k X_k^H\right]$ is the input signal correlation matrix. The necessary and sufficient condition for stability of the recursion is given by

$$0 < \mu < 2/\lambda_{max} \tag{6}$$

where λ_{max} is the maximum eigen-value of the input signal correlation matrix R.

Taking expectations under the same assumptions as above, using the independence assumption on the sequences X_k, n_k, the mutual independence assumption on X_k and V_k, and simplifying we obtain for odd k when S-LMS is operating under the special case of alternate even and odd updates

$$E\left[V_{k+2}\right] = (I - \mu I_2 R)(I - \mu I_1 R) E[V_k] \tag{7}$$

and for even k

$$E\left[V_{k+2}\right] = (I - \mu I_1 R)(I - \mu I_2 R) E[V_k] \tag{8}$$

It can be shown that under the above assumptions on X_k, V_k and d_k, the convergence conditions for even and odd update equations are identical. We therefore focus on (7). Now to ensure stability of (8), the eigenvalues of $(I - \mu I_2 R)(I - \mu I_1 R)$ should lie inside the unit circle. We will show that if the eigenvalues of $I - \mu R$ lie inside the unit circle then so do the eigenvalues of $(I - \mu I_2 R)(I - \mu I_1 R)$.

Now, if instead of just two partitions of even and odd coefficients ($P = 2$) we have any number of arbitrary partitions ($P \geq 2$) then the update equations can be similarly written as above with $P > 2$. Namely,

$$E[V_{k+P}] = \prod_{i=1}^{P} (I - \mu I_i R) E[V_k] \tag{9}$$

We will show that for any arbitrary partition and any number of partitions ($P \geq 2$) S-LMS converges if LMS converges (Theorem 2). The case $P = 2$ follows as a special case.

We will show that if R is a positive definite matrix of dimension $N \times N$ with eigenvalues lying in the open interval $(0, 2)$ then $\prod_{i=1}^{P} (I - I_i R)$ has eigenvalues inside the unit circle. I_i, $i = 1, \ldots, P$ is obtained from I, the identity matrix of dimension $N \times N$, by zeroing out some rows in I such that $\sum_{i=1}^{M} I_i$ is positive definite.

The following theorem is used in proving the result.

Theorem 1 *[6, Prob. 16, page 410] Let B be an arbitrary $N \times N$ matrix. Then $\rho(B) < 1$ if and only if there exists some positive definite $N \times N$ matrix A such that $A - B^H A B$ is positive definite. $\rho(B)$ denotes the spectral radius of B ($\rho(B) = \max_{1,\ldots,N} |\lambda_i(B)|$).*

Theorem 2 *Let R be a positive definite matrix of dimension $N \times N$ with $\rho(R) = \lambda_{max}(R) < 2$ then $\rho(\prod_{i=1}^{P} (I - I_i R)) < 1$ where I_i, $i = 1, \ldots, P$ are obtained by zeroing out some rows in the identity matrix I such that $\sum_{i=1}^{P} I_i$ is positive definite. Thus S-LMS converges in the mean if LMS converges in the mean.*

Proof: Let $\mathbf{x}_0 \in \mathbb{C}^N$ be an arbitrary non-zero vector of length N. Let $\mathbf{x}_i = (I - I_i R)\mathbf{x}_{i-1}$. Also, let $\mathbf{P} = \prod_{i=1}^{P} (I - I_i R)$.

First we will show that $\mathbf{x}_i^H R \mathbf{x}_i \leq \mathbf{x}_{i-1}^H R \mathbf{x}_{i-1} - \alpha \mathbf{x}_{i-1}^H R I_i R \mathbf{x}_{i-1}$, where $\alpha = \frac{1}{2}(2 - \lambda_{max}(R)) > 0$.

$$\begin{aligned} \mathbf{x}_i^H R \mathbf{x}_i &= \mathbf{x}_{i-1}^H (I - R I_i) R (I - I_i R) \mathbf{x}_{i-1} \\ &= \mathbf{x}_{i-1}^H R \mathbf{x}_{i-1} - \alpha \mathbf{x}_{i-1}^H R I_i R \mathbf{x}_{i-1} - \\ &\quad \beta \mathbf{x}_{i-1}^H R I_i R \mathbf{x}_{i-1} + \mathbf{x}_{i-1}^H R I_i R I_i R \mathbf{x}_{i-1} \end{aligned}$$

where $\beta = 2 - \alpha$. If we can show $\beta R I_i R - R I_i R I_i R$ is positive semi-definite then we are done. Now

$$\beta R I_i R - R I_i R I_i R = \beta R I_i (I - \frac{1}{\beta} R) I_i R$$

Since $\beta = (1 + \lambda_{max}(R)/2) > \lambda_{max}(R)$ it is easy to see that $I - \frac{1}{\beta} R$ is positive definite. Therefore, $\beta R I_1 R - R I_1 R I_1 R$ is positive semi-definite and

$$\mathbf{x}_i^H R \mathbf{x}_i \leq \mathbf{x}_{i-1}^H R \mathbf{x}_{i-1} - \alpha \mathbf{x}_{i-1}^H R I_i R \mathbf{x}_{i-1}$$

Combining the above inequality for $i = 1, \ldots, P$, we note that $\mathbf{x}_P^H R\mathbf{x}_P < \mathbf{x}_0^H R\mathbf{x}_0$ if $\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} > 0$ for at least one i, $i = 1, \ldots, P$. We will show by contradiction that is indeed the case.

Suppose not, then $\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} = 0$ for all $i, i = 1, \ldots, P$. Since, $\mathbf{x}_0^H RI_1R\mathbf{x}_0 = 0$ this implies $I_1R\mathbf{x}_0 = \mathbf{0}$. Therefore, $\mathbf{x}_1 = (I - I_1R)\mathbf{x}_0 = \mathbf{x}_0$. Similarly, $\mathbf{x}_i = \mathbf{x}_0$ for all i, $i = 1, \ldots, P$. This in turn implies that $\mathbf{x}_0^H RI_iR\mathbf{x}_0 = 0$ for all i, $i = 1, \ldots, P$ which is a contradiction since $R(\sum_{i=1}^P I_i)R$ is a positive-definite matrix and $0 = \sum_{i=1}^P \mathbf{x}_0^H RI_iR\mathbf{x}_0 = \mathbf{x}_0^H R(\sum_{i=1}^P I_i)R\mathbf{x}_0 \neq 0$.

Finally, we conclude that

$$\begin{aligned} \mathbf{x}_0^H \mathbf{P}^H R\mathbf{P}\mathbf{x}_0 &= \mathbf{x}_P^H R\mathbf{x}_P \\ &< \mathbf{x}_0^H R\mathbf{x}_0 \end{aligned}$$

Since $\mathbf{x}_0$ is arbitrary we have $R - \mathbf{P}^H R\mathbf{P}$ to be positive definite so that applying Theorem 1 we conclude that $\rho(\mathbf{P}) < 1$.

Finally, if LMS converges in the mean we have $\rho(I - R) < 1$ or $\lambda_{max}(R) < 2$. Which from the above proof is sufficient for concluding that $\rho(\prod_{i=1}^P(I - I_iR)) < 1$. Therefore, S-LMS also converges in the mean. □

4. ANALYSIS: CYCLO-STATIONARY SIGNALS

Next, we consider the case when $\{x_k\}$ is cyclo-stationary. We limit our attention to S-LMS with alternate even and odd updates as shown in Figure 1. Let $\{x_k\}$ be a cyclo-stationary signal with period L. i.e, $R_{i+L} = R_i$. For simplicity, we will assume L is even. For the regular LMS algorithm we have the following L update equations

$$E[V_{k+L}] = \prod_{i=0}^{L-1}(I - \mu R_{i+d})E[V_k] \tag{10}$$

for $d = 1, 2, \ldots, L$, in which case we would obtain the following sufficient condition for convergence

$$0 < \mu < \min_i\{2/\lambda_{i,max}\} \tag{11}$$

where $\lambda_{i,max}$ is the largest eigenvalue of the matrix R_i.

Define $A_k = (I - \mu I_1R_k)$ and $B_k = (I - \mu I_2R_k)$ then for the partial update algorithm the $2L$ valid update equations are

$$E[V_{k+L}] = \left(\prod_{i=0}^{\frac{L-1}{2}} B_{2*i+1+d}A_{2*i+d}\right)E[V_k] \tag{12}$$

for $d = 1, 2, \ldots, L$ and

$$E[V_{k+L}] = \left(\prod_{i=0}^{\frac{L-1}{2}} A_{2*i+1+d}B_{2*i+d}\right)E[V_k] \tag{13}$$

for $d = 1, 2, \ldots, L$.

For ensuring the convergence of the iteration (12) and (13) a sufficient condition is

$$\|B_{i+1}A_i\| < 1 \quad \text{for } i = 1, 2, \ldots, L \tag{14}$$

and

$$\|A_{i+1}B_i\| < 1 \quad \text{for } i = 1, 2, \ldots, L. \tag{15}$$

Since we can write $B_{i+1}A_i$ as

$$B_{i+1}A_i = (I - \mu R_i) + \mu I_2(R_i - R_{i+1}) + \mu^2 I_2R_{i+1}I_1R_i \tag{16}$$

and $A_{i+1}B_i$ as

$$A_{i+1}B_i = (I - \mu R_i) + \mu I_1(R_i - R_{i+1}) + \mu^2 I_1R_{i+1}I_2R_i \tag{17}$$

we have the the following expression which upper bounds both $\|B_{i+1}A_i\|$ and $\|A_{i+1}B_i\|$

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\| \tag{18}$$

This tells us that the sufficient condition to ensure convergence of both (12) and (13) is

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\| < 1 \tag{19}$$

for $i = 1, \ldots, L$.

If we make the assumption that

$$\mu < \min_i\{\frac{2}{\lambda_{i,max} + \lambda_{i,min}}\}$$

and

$$\delta_i = \|R_{i+1} - R_i\| < \max\{\lambda_{i,min}, \lambda_{i+1,min}\} = \eta_i$$

for $i = 1, 2, \ldots, L$ then (19) translates to

$$1 - \mu\eta_i + \mu\delta_i + \mu^2\lambda_{i,max}\lambda_{i+1,max} < 1 \tag{20}$$

which gives

$$0 < \mu < \min_{i=1}^{L}\{\frac{\eta_i - \delta_i}{\lambda_{i,max}\lambda_{i+1,max}}\} \tag{21}$$

(21) is the sufficient condition for the convergence of S-LMS.

5. EXAMPLE

The usefulness of the bound on step-size for the cyclo-stationary case can be gauged from the following example. Consider a 2-tap filter and a cyclo-stationary $\{x_k\}$ with period 2 having the following auto-correlation matrices

$$\begin{aligned} R_1 &= \begin{bmatrix} 5.1354 & -0.5733 - 0.6381i \\ -0.5733 + 0.6381i & 3.8022 \end{bmatrix} \\ R_2 &= \begin{bmatrix} 3.8022 & 1.3533 + 0.3280i \\ 1.3533 - 0.3280i & 5.1354 \end{bmatrix} \end{aligned}$$

For this choice of R_1 and R_2, η_1 and η_2 turn out to be 3.38 and we have $\|R_1 - R_2\| = 2.5343 < 3.38$. Therefore, R_1 and R_2 satisfy the assumption made for analysis. Now, $\mu = 0.33$ satisfies the condition for the regular LMS algorithm but, the eigenvalues of B_2A_1 for this value of μ have magnitudes 1.0481 and 0.4605. Since one of the eigenvalues lies outside the unit circle (12) is unstable for this choice of μ. On the other hand (21) gives $\mu = 0.0254$. For this choice of μ the eigenvalues of B_2A_1 turn out to have magnitudes 0.8620 and 0.8773. Hence (12) is stable.

We have plotted the evolution trajectory of the 2-tap filter with input signal satisfying the above properties. We chose $W_{opt} = [0.4\ 0.5]$ in Figures 2 and 3. For Figure 2 μ was chosen according to be 0.33 and for Figure 3 μ was chosen to be 0.0254. For simulation purposes we set $d_k = W_{opt}^H S_k + n_k$ where $S_k = [s_k\ s_{k-1}]^T$ is a vector composed of the cyclo-stationary process $\{s_k\}$ with correlation matrices given as above, and $\{n_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$. We set $\{x_k\} = \{s_k\} + \{v_k\}$ where $\{v_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$.

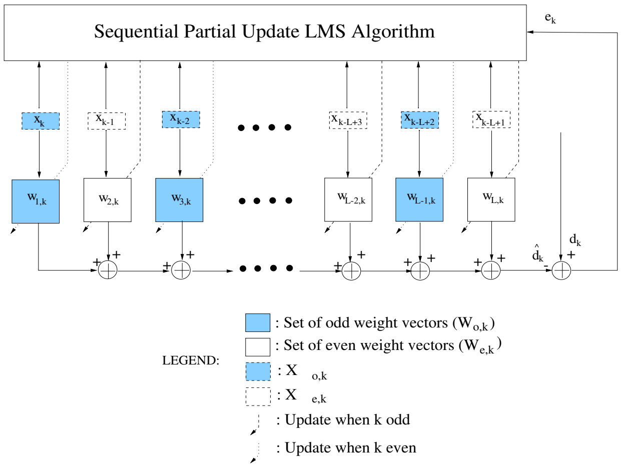

Fig. 1. Block diagram of S-LMS for the special case of alternating even/odd coefficient update

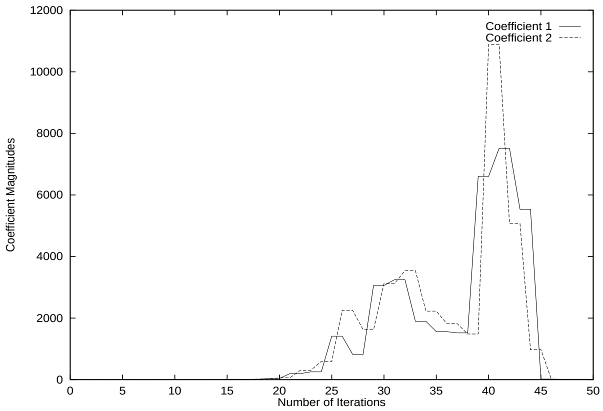

Fig. 2. Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.33$ for a 2-tap adaptive filter in a cyclo-stationary signal environment described in Section 5

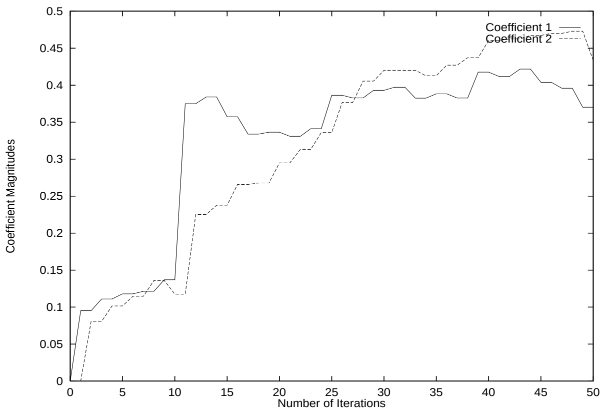

Fig. 3. Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.0254$ for the same adaptive system as shown in Figure 2

6. CONCLUSION

We have analyzed the alternating odd/even partial update LMS algorithm and we have derived stability bounds on step-size parameter μ for wide sense stationary and cyclo-stationary signals based on extremal properties of the matrix 2-norm. For the case of wide sense stationary signals we have shown that if the the regular LMS converges in mean then so does the sequential LMS for the general case of arbitrary fixed sequence of updates. For cyclo-stationary signals the bounds derived may not be the weakest possible bounds but they do provide the user with a useful sufficient condition on μ which ensures convergence in the mean. We believe the analysis undertaken in this paper is the first step towards deriving concrete bounds on step-size without undertaking simplifying assumptions. The analysis also leads directly to an estimate of mean convergence rate.

In the future, we would like to analyze the algorithm, in general all partial update algorithms, without the assumption of independent snapshots and also, if possible perform the second order analysis (mean square convergence) of the sequential LMS algorithm. S-LMS exhibits poor convergence in non-stationary signal scenarios [8] and it is necessary to develop a new partial update algorithm with better convergence properties. One such algorithm based on randomized updating of filter coefficients is described in [8].

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