LIMITATIONS ON DETECTION AND CLASSIFICATION FROM COMPRESSED IMAGES

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Outline of Presentation

- Overview of Vector Quantization (VQ)
- Motivation for Compression with Detection Criteria
- Optimal VQ for Detection vs. Estimation
- Examples
- Conclusions and Future Research

Scalar and Vector Quantization

Let X be a k-dimensional random process with density $q(x)$.

An N-point quantizer, (S, C) consists of:

- Cells $S = \{S_0, \ldots, S_{N-1}\}$, disjoint and exhaustive.
- Codebook $C = \{x_0, \ldots, x_{N-1}\}$, $x_i \in \mathbb{R}^k$.

Thus the quantization rule is:

$$\hat{x} = x_i, \text{ if } x \in S_i.$$

The MSE of the quantized source is

$$\mathcal{E} = \int (x - \hat{x})^2 q(x) dx = \sum_{i=0}^{N-1} \int_{S_i} (x - x_i)^2 q(x) dx$$

MSE-optimal (Lloyd-Max) quantizers have the following properties:

- x_i is the centroid of S_i:

$$x_i = \frac{\int_{S_i} x q(x) dx}{\int_{S_i} q(x) dx},$$

- Cells are partitioned such that quantizer is a nearest-neighbor mapping:

$$\text{if } x \in S_i,\ \|x - x_i\| \leq \|x - x_j\|, \forall j \neq i$$

For large N (using Bennet's integral),

$$\mathcal{E} \geq v(N,k) \int \frac{q(x)}{\zeta(x)^{2/k}} dx$$

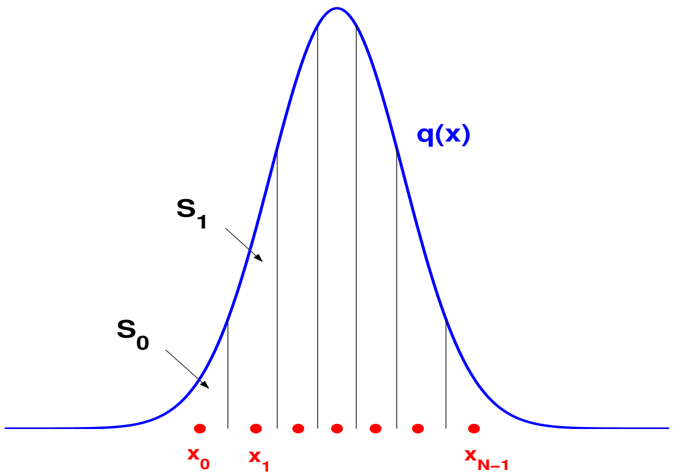
where $\zeta(x)$ is the "point density"

$$\zeta(x_i) \approx \frac{1}{NV(S_i)}, \qquad V(S_i) = \int_{S_i} dx.$$

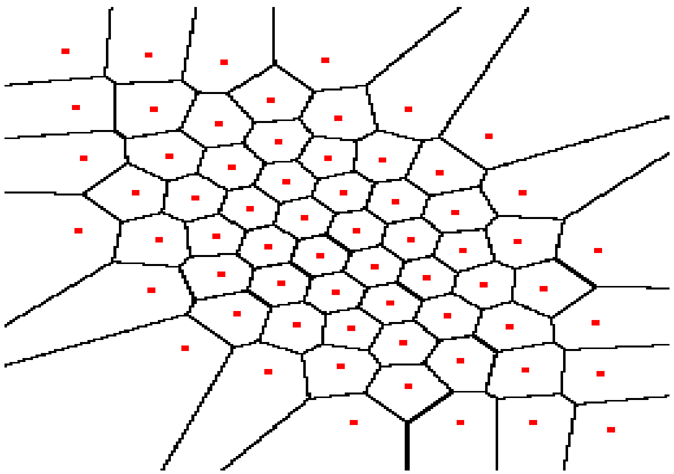
Hence, <u>quantizer can be designed by optimizing $\zeta(x)$</u>.

Examples of MSE-Optimal Quantizers for Gaussian Sources

Scalar Quantizer



Vector Quantizer



Hypothesis Testing (Detection)

Suppose the goal is to determine which of the two hypotheses is true:

$$\begin{aligned} H_0 &: X \sim q_0(x) \\ H_1 &: X \sim q_1(x). \end{aligned}$$

Define

$$\begin{aligned} \alpha &= P(\text{decide } H_1 | H_0) \quad \text{(False alarm probability)} \\ \beta &= P(\text{decide } H_0 | H_1) \quad \text{(Probability of miss).} \end{aligned}$$

The optimal (Neyman-Pearson) decision rule is

$$\begin{aligned} &\text{Decide } H_0 \;\text{ if } \Lambda(x) \geq T \\ &\text{Decide } H_1 \;\text{ if } \Lambda(x) < T \end{aligned}$$

where $\Lambda(x) = \log \frac{q_0(x)}{q_1(x)}$ is the log-likelihood ratio.

Discrimination between Sources

The (differential) discrimination (a.k.a. relative entropy, Kullback-Leibler distance) between $q_0(x)$ and $q_1(x)$ is defined as

$$L(q_0, q_1) = \int q_0(x) \log \frac{q_0(x)}{q_1(x)} dx.$$

Define the "tilted" density

$$q_\lambda(x) = \frac{q_0(x)^{1-\lambda} q_1(x)^\lambda}{\int q_0(x)^{1-\lambda} q_1(x)^\lambda dx}, \qquad \lambda \in [0, 1].$$

For large k (Blahut, 1987),

$$\alpha \approx e^{-kL(q_\lambda, q_0)} \qquad \beta \approx e^{-kL(q_\lambda, q_1)}.$$

The discrimination <u>can not increase</u> with processing (e.g. quantization).

Discrimination between Quantized Sources

Define

$$\Delta L_0 = L(q_\lambda, q_0) - L(\hat{q}_\lambda, \hat{q}_0), \qquad \Delta L_1 = L(q_\lambda, q_1) - L(\hat{q}_\lambda, \hat{q}_1)$$

where $\hat{q}_0$ and $\hat{q}_1$ are the densities of the quantized source under hypotheses H_0 and H_1, respectively, and $\hat{q}_\lambda$ is the corresponding tilted density. Also define

$$\begin{aligned} \hat{\alpha} &= \text{False alarm probability with quantized data} \\ \hat{\beta} &= \text{Probability of miss with quantized data.} \end{aligned}$$

Then:

$$\begin{aligned} \hat{\alpha} &= \alpha e^{\lambda k \Delta} \\ \hat{\beta} &= \beta e^{(1-\lambda) k \Delta} \end{aligned}$$

where

$$\Delta = \Delta L_0 + \Delta L_1$$

VQ Design for Maximum Discrimination

For a many-point VQ (large N), we find that

$$\Delta \;=\; \lim_{N\to\infty} \frac{1}{2} \sum_{i=0}^{N-1} q_\lambda(x_i) \int_{S_i} (x - x_i)^T \mathcal{F}_{x_i} (x - x_i) dx$$

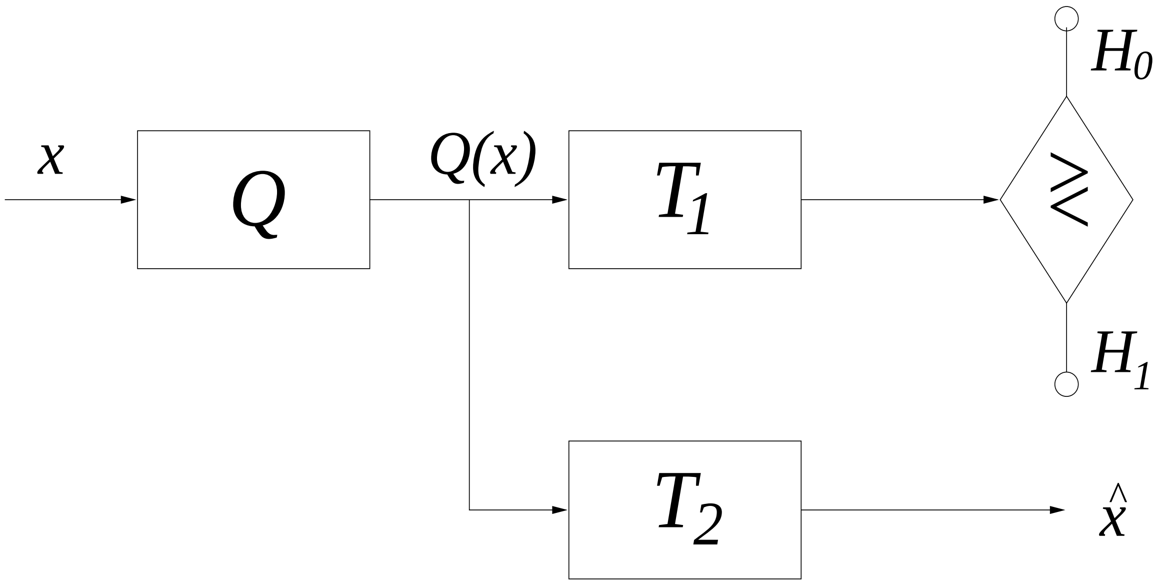
where

$$\mathcal{F}_y \;=\; \nabla\Lambda(y)\nabla\Lambda(y)^T, \qquad \Lambda(x) = \log \frac{q_0(x)}{q_1(x)}$$

$$\Rightarrow \Delta = u(N,k) \int \frac{q_\lambda(x)\|\nabla\Lambda(x)\|^2}{\zeta(x)^{2/k}} dx$$

Once again, <u>quantizer can be designed by optimizing $\zeta(x)$</u>.

Joint Detection / Estimation Design Criterion



Objective: Design Q to be optimal wrt detection and estimation performance simultaneously.

Quantizer Design Criterion 1: Average MSE

Optimal for <u>estimation</u>

$$\bar{\mathcal{E}} = \int (x - \hat{x})^2 \bar{q}(x) dx, \qquad \bar{q}(x) = P_0 q_0 + (1 - P_0) q_1$$

Optimal Point Density:

$$\zeta(x) = \frac{\bar{q}(x)^{\frac{k}{k+2}}}{\int \bar{q}(x)^{\frac{k}{k+2}} dx}$$

MSE Lower Bound:

$$\bar{\mathcal{E}}_1 \geq v(N, k) \int \bar{q}(x)^{\frac{2}{k+2}} dx \int \bar{q}(x)^{\frac{k}{k+2}} dx$$

Discrimination:

$$\Delta_1 = u(N, k) \left(\int \bar{q}(x)^{\frac{k}{k+2}} dx \right)^{2/k} \int \bar{q}(x)^{\frac{-2}{k+2}} q_\lambda(x) \|\nabla \Lambda(x)\|^2 dx$$

Quantizer Design Criterion 2: Tilted MSE

$$\mathcal{E}_\lambda = \int (x - \hat{x})^2 q_\lambda(x) dx$$

Optimal Point Density:

$$\zeta(x) = \frac{q_\lambda(x)^{\frac{k}{k+2}}}{\int q_\lambda(x)^{\frac{k}{k+2}} dx}$$

MSE Lower Bound:

$$\bar{\mathcal{E}}_2 \geq v(N,k) \left(\int q_\lambda(x)^{\frac{k}{k+2}} dx \right)^{2/k} \int \bar{q}(x) q_\lambda(x)^{\frac{-2}{k+2}} dx$$

Discrimination:

$$\Delta_2 = u(N,k) \left(\int q_\lambda(x)^{\frac{k}{k+2}} dx \right)^{2/k} \int q_\lambda(x)^{\frac{k}{k+2}} \|\nabla \Lambda(x)\|^2 dx$$

Quantizer Design Criterion 3: Discrimination

Optimal for <u>detection</u>

$$\Delta = \Delta L_0 + \Delta L_1$$

Optimal Point Density:

$$\zeta(x) = \frac{[q_\lambda(x)\|\nabla\Lambda(x)\|^2]^{\frac{k}{k+2}}}{\int [q_\lambda(x)\|\nabla\Lambda(x)\|^2]^{\frac{k}{k+2}} dx}$$

MSE Lower Bound:

$$\bar{\mathcal{E}}_3 \geq v(N,k) \left(\int [q_\lambda(x)\|\nabla\Lambda(x)\|^2]^{\frac{k}{k+2}} dx \right)^{2/k} \int \bar{q}(x)[q_\lambda(x)\|\nabla\Lambda(x)\|^2]^{\frac{-2}{k+2}} dx$$

Discrimination:

$$\Delta_3 = u(N,k) \left(\int [q_\lambda(x)\|\nabla\Lambda(x)\|^2]^{\frac{k}{k+2}} dx \right)^{\frac{k+2}{k}}$$

Quantizer Design Criterion 4: Combined $\mathcal{E}_\lambda$ and Δ

$$J = \rho\Delta + (1-\rho)\mathcal{E}_\lambda, \qquad \rho \in [0,1]$$

For large N,

$$J = ku(N,k)\int \frac{q_\lambda(x)\left(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2 + 1 - \rho\right)}{\zeta(x)^{2/k}}dx$$

Optimal Point Density:

$$\zeta(x) = \frac{[q_\lambda(x)(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2 + 1 - \rho)]^{\frac{k}{k+2}}}{\int[q_\lambda(x)(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2 + 1 - \rho)]^{\frac{k}{k+2}}dx}$$

Quantizer Design Criterion 4: Combined $\mathcal{E}_\lambda$ and Δ (Contd.)

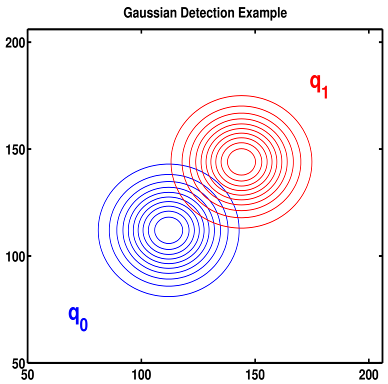
MSE Lower Bound:

$$\bar{\mathcal{E}}_4 \geq v(N,k)\left(\int \left[q_\lambda(x)\left(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2+1-\rho\right)\right]^{\frac{k}{k+2}}dx\right)^{2/k}\cdot \int \bar{q}(x)\left[q_\lambda(x)\left(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2+1-\rho\right)\right]^{\frac{-2}{k+2}}dx$$

Discrimination:

$$\Delta_4 = u(N,k)\left(\int \left[q_\lambda(x)\left(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2+1-\rho\right)\right]^{\frac{k}{k+2}}dx\right)^{\frac{2}{k}}\cdot \int q_\lambda(x)^{\frac{k}{k+2}}\|\nabla\Lambda(x)\|^2\left(\frac{\rho}{k}\|\nabla\Lambda(x)\|^2+1-\rho\right)^{\frac{-2}{k+2}}dx$$

Comparison of VQ's: Gaussian Hypotheses



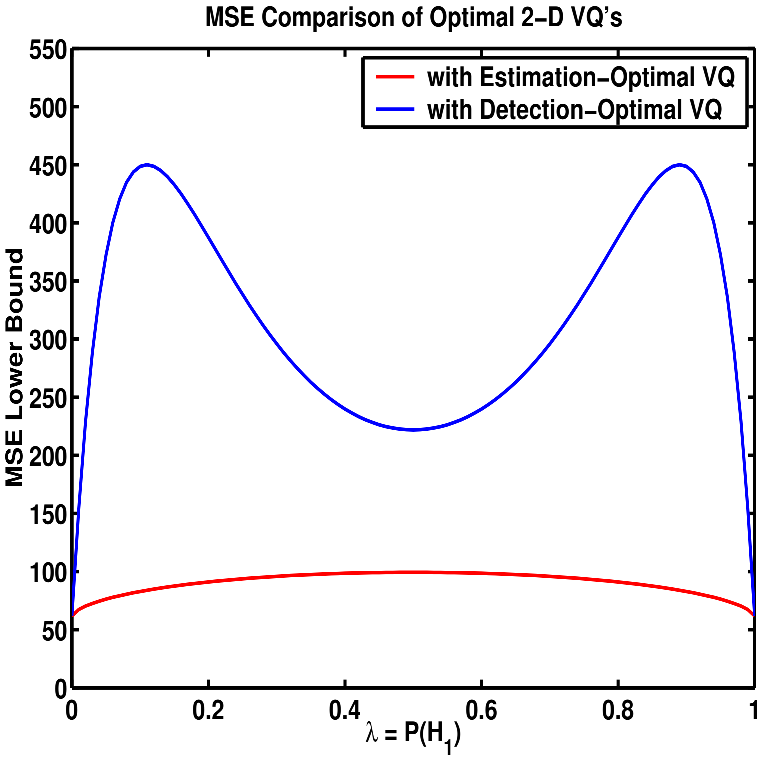
$$q_0 \sim N(\boldsymbol{\mu_0}, \mathbf{K_0}) \qquad q_1 \sim N(\boldsymbol{\mu_1}, \mathbf{K_1})$$

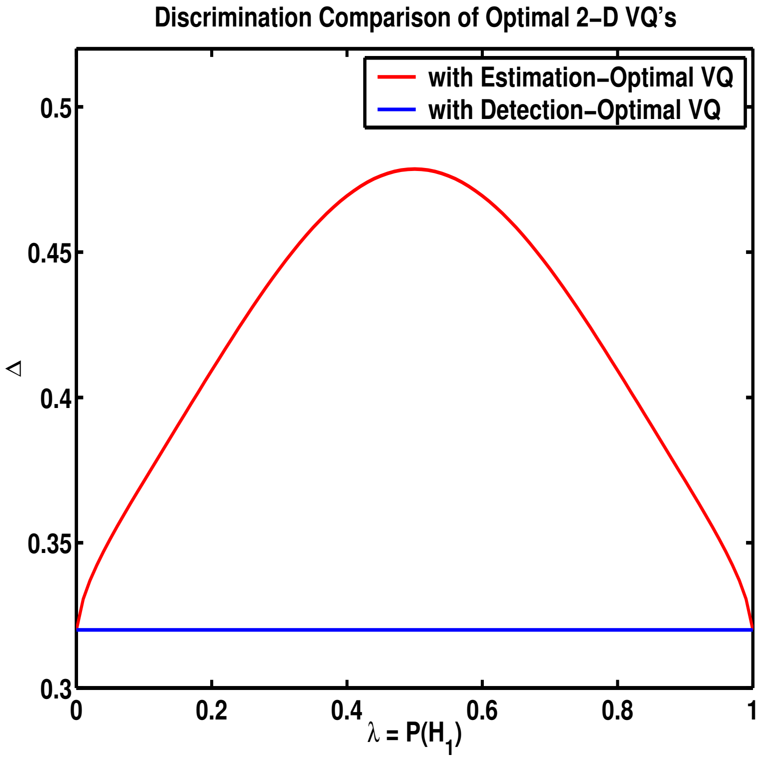
$$\boldsymbol{\mu_0} = [112, 112]^T \qquad \boldsymbol{\mu_1} = [144, 44]^T$$

$$\mathbf{K_0} = 200I \qquad \mathbf{K_1} = 200I$$

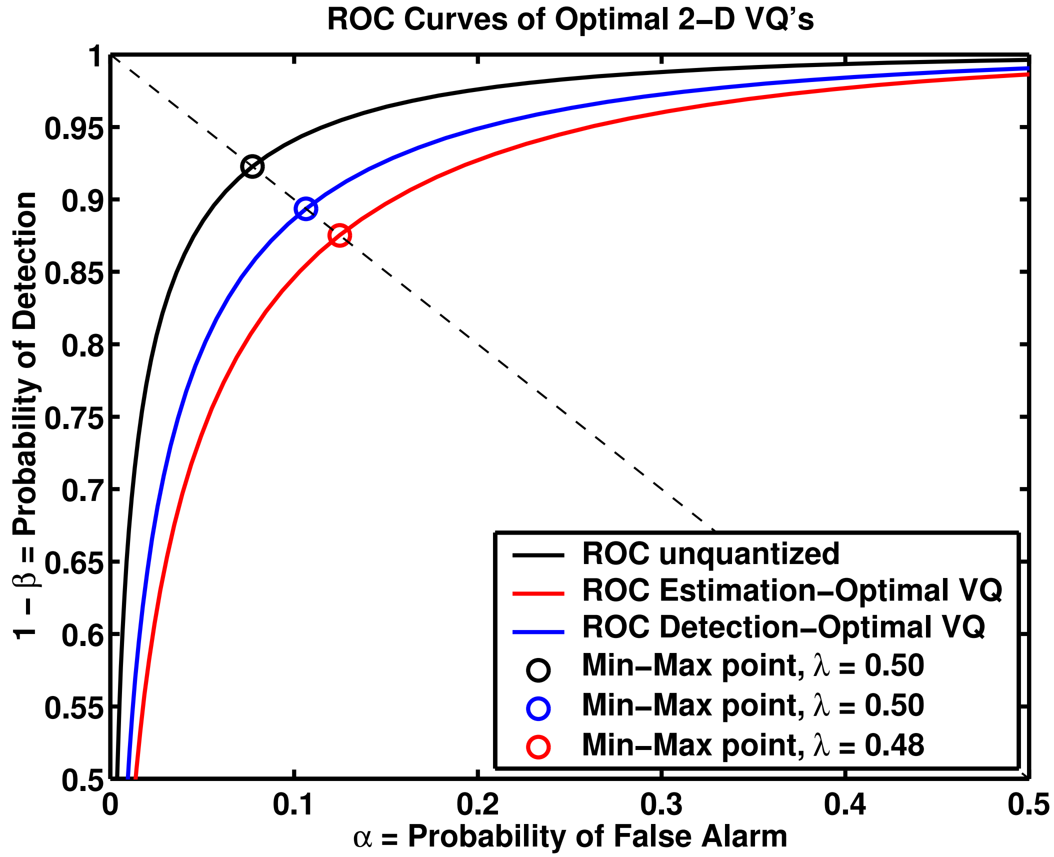
Consider optimal VQ's of dimension $k = 2$, with $N = 128$ cells.

Comparison of VQ's: MSE and Discrimination





Comparison of VQ's: ROC Curves



Implementation: Lloyd Algorithm

`Step 1:` Initialize codebook, C_1.

`Step 2:` Find optimal nearest-neighbor partition.

`Step 3:` Move codebook points to centroids (wrt pdf, $q(x)$) of their cells.

`Step 4:` Compute total distortion. If change in distortion is small enough, stop. Otherwise, go to `Step 2`.

- Results in MSE-Optimal quantizer (with correct initialization).
- If point density $\zeta(x)$ is known, can replace $q(x)$ with $\zeta(x)$ in Lloyd algorithm.

Gaussian Example

$$\mu_{\mathbf{0}} = 64, \quad \mu_{\mathbf{1}} = 192, \quad \mathbf{K_0} = 1000I, \quad \mathbf{K_1} = 500I, \quad N = 25, \quad \lambda = 1/2$$

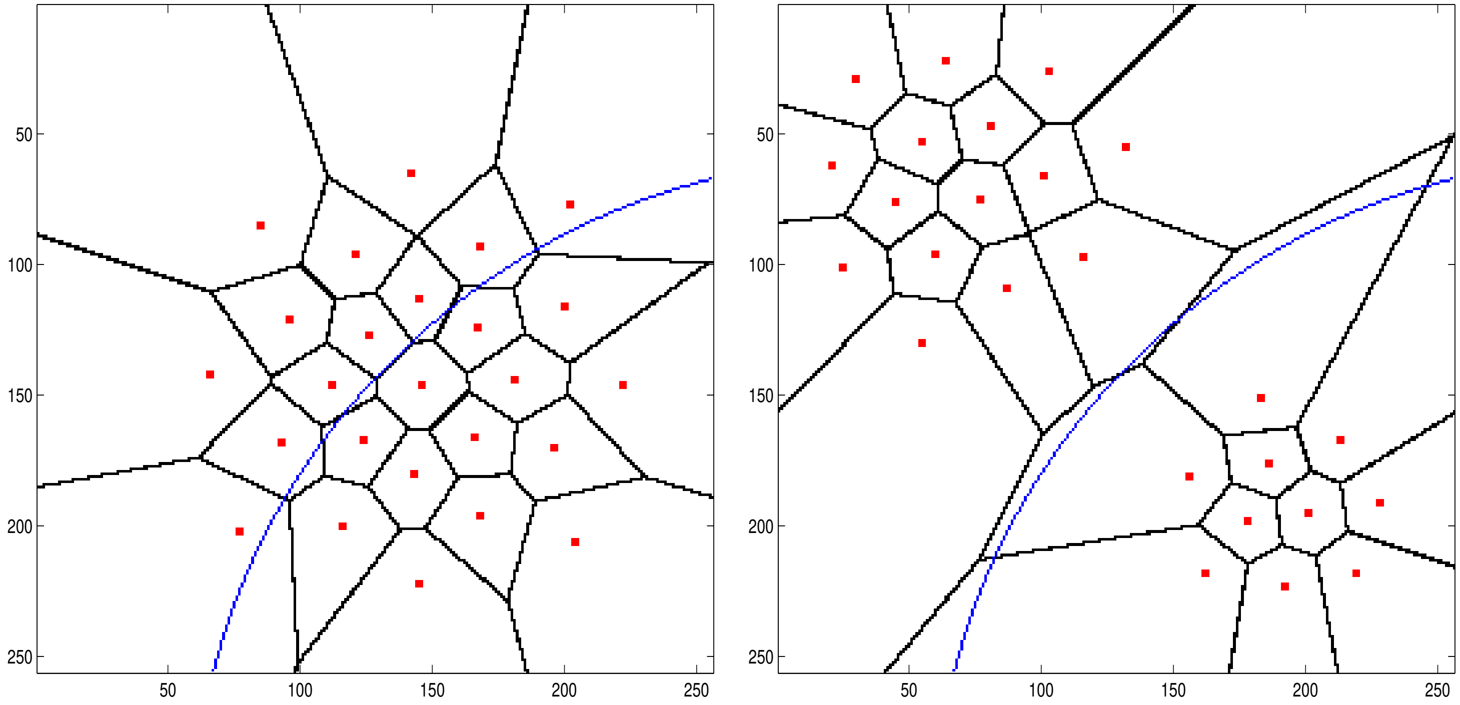
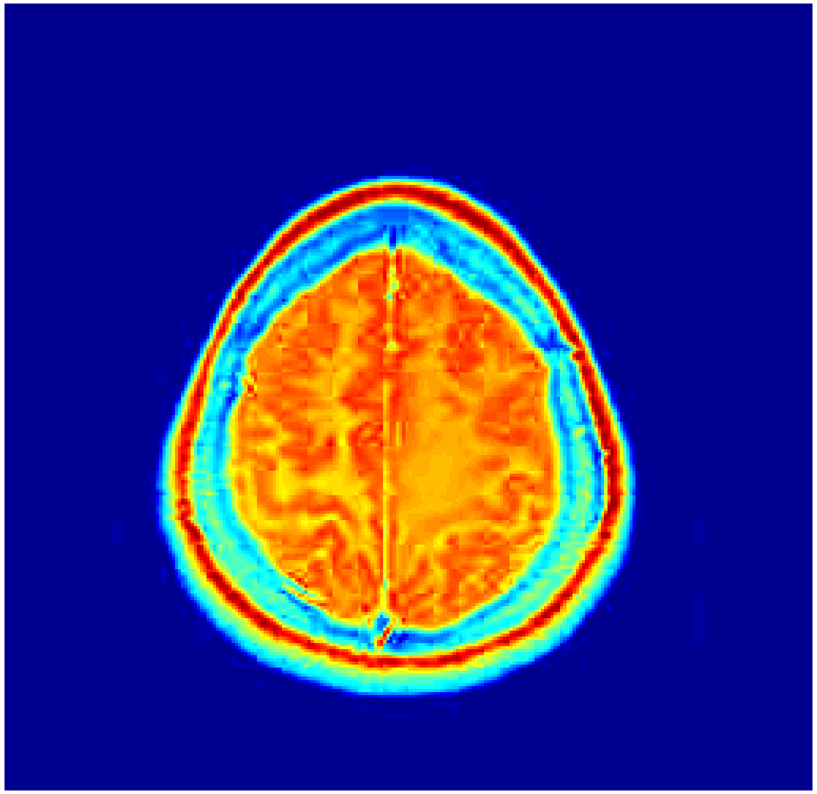
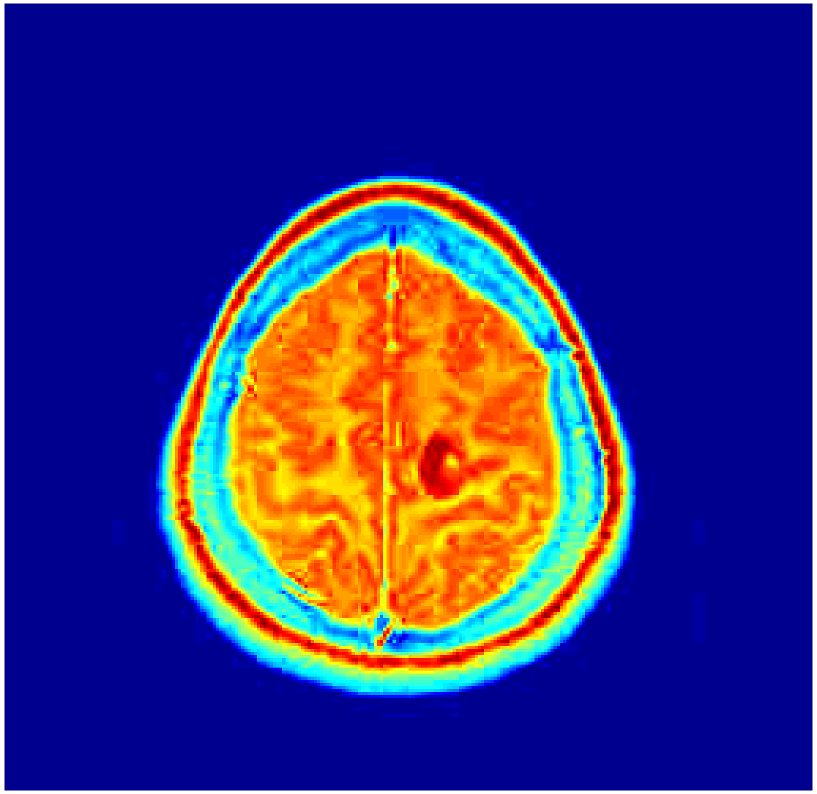
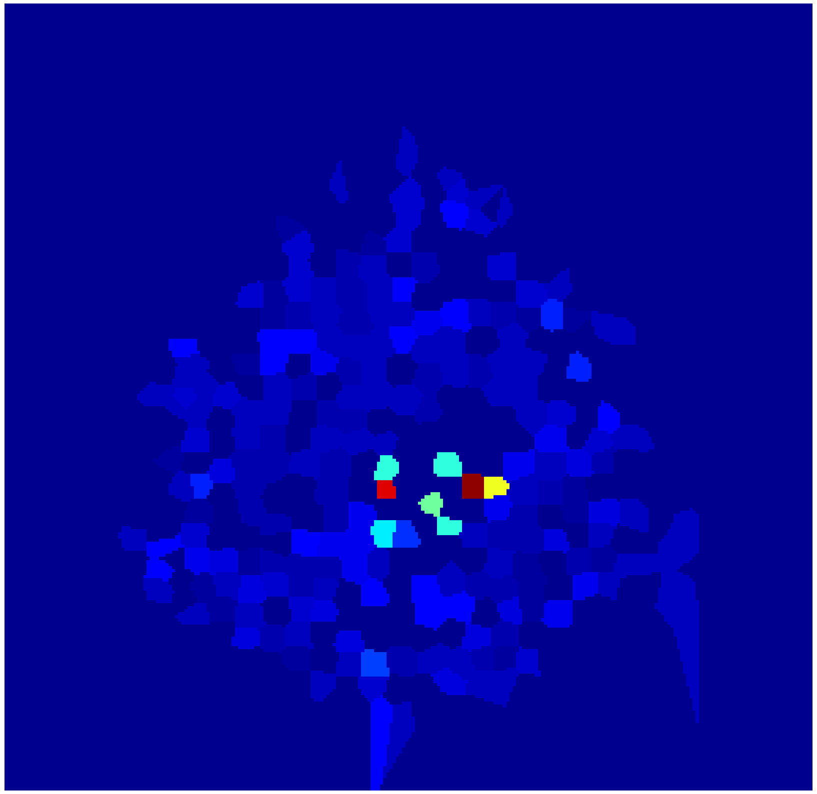


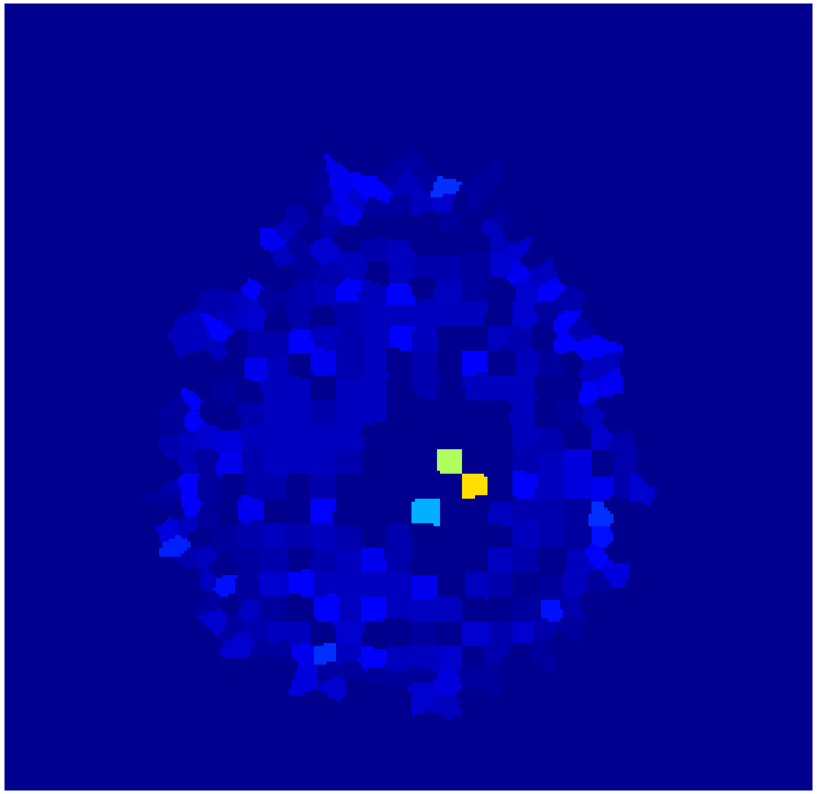
Image Compression Example



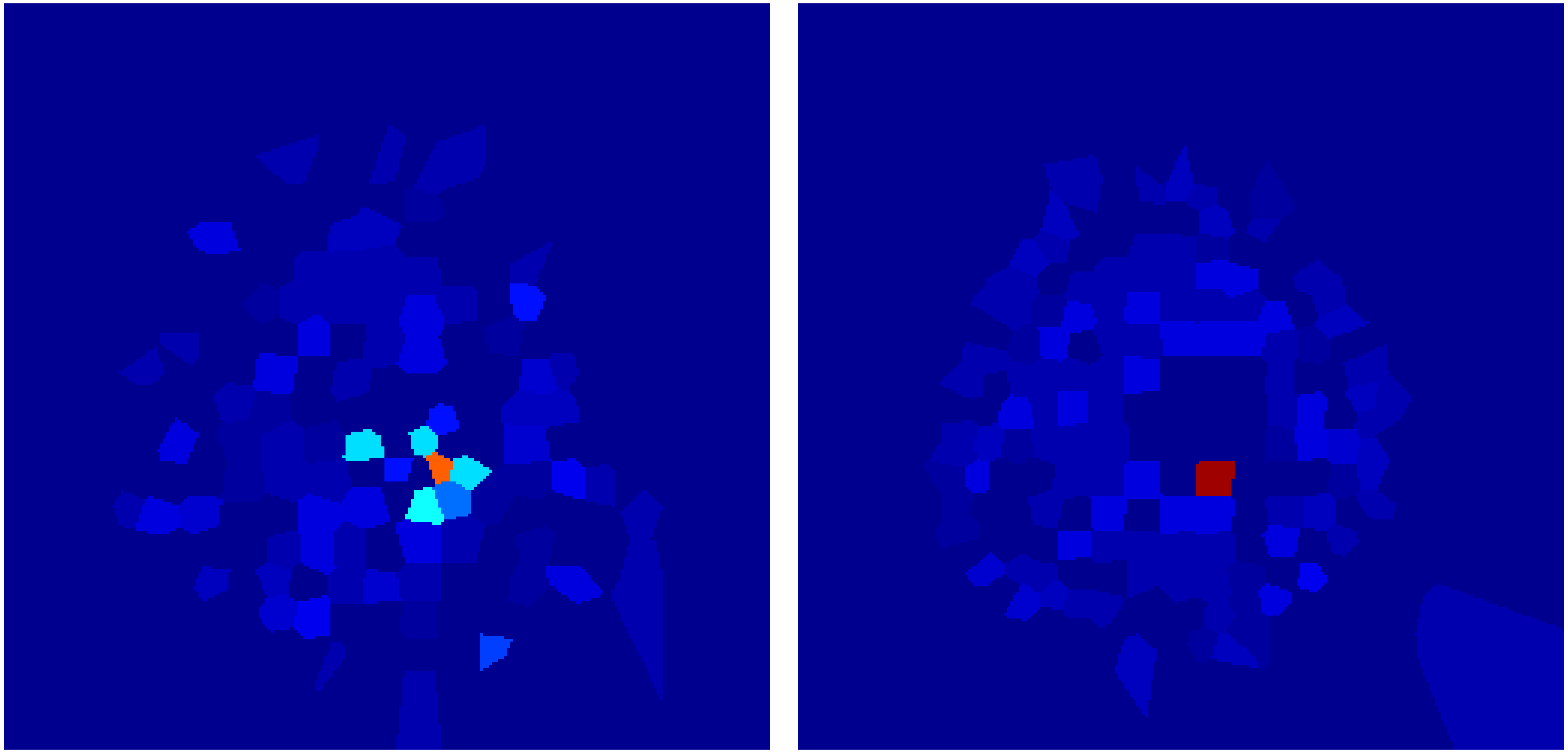


Difference Images with $N = 1024$ Cells, $\lambda = 1/2$





Difference Images with $N = 400$ Cells, $\lambda = 1/2$



Discrimination Distance

Recall that discrimination decrease is

$$\Delta = \lim_{N\to\infty} \frac{1}{2} \sum_{i=0}^{N-1} q_\lambda(x_i) \int_{S_i} (x - x_i)^T \mathcal{F}_{x_i} (x - x_i) dx.$$

Hence $D(x, x_i) = (x - x_i)^T \mathcal{F}_{x_i}(x - x_i)$ can be used as distortion metric (as opposed to Euclidean distance) in Lloyd algorithm.

For Gaussian i.i.d. hypotheses with $\mathbf{K_0} = \mathbf{K_1} = \sigma^2 I$:

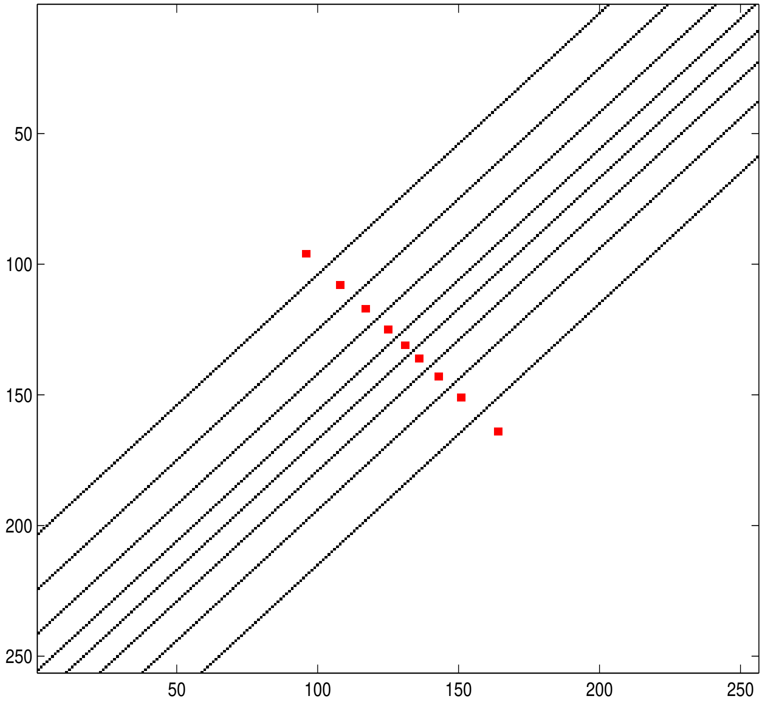
$$\mathcal{F}_x = \frac{(\mu_0 - \mu_1)^2}{\sigma^4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow D(x, x_i) = \frac{(\mu_0 - \mu_1)^2}{\sigma^4} [(x + y) - (x_i + y_i)]^2$$

$\Rightarrow$ Quantize optimally w.r.t. MSE of $z = x + y$ under density q_λ.

$\boxed{\text{Example with } \lambda = 0}$

$$q_\lambda(x) = q_0(x) \sim N(128, 500)$$



Conclusions

- Optimal VQ's for detection can differ greatly from MSE-optimal VQ's.
- Optimization method: point density
- Penalty for not using detection-optimal quantizer: significant loss in discrimination.
- For some cases, VQ is optimal over all prior probabilities, $P(H_0)$.

Future Research

- Application: Compression of large databases with Detection vs. Estimation criteria
- Non-asymptotic theory