Searching for Noncooperative Equilibria in Centralized Electricity Markets

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***Abstract*—The use of Nash equilibria as strategic bidding solutions for players competing in a centralized electricity market has been explored in the literature. In addition, the existence of multiple market equilibria, both in pure and in mixed strategies, when system constraints are taken into account in the market solution, has been shown for particular test cases. This paper addresses the difficult problem, due to its exponential order, of finding multiple equilibria in centralized electricity markets. A systematic procedure which allows the analysis of multimachine systems is employed. Moreover, some conclusions concerning the existence of multiple market equilibria in real networks are made possible by the method presented here. The paper carefully explains the models and market assumptions under discussion and details the algorithmic procedure for the search method. The IEEE 30-bus and 57-bus systems are used as test cases.**

***Index Terms*—Game theory, market equilibria, pure and mixed strategies, search algorithms.**

I. Introduction

SINCE competitive bulk power markets were established in various jurisdictions around the world, a key concern has been the study of market outcomes or solutions, given specific market rules. This study is justified from the standpoints of both the regulatory entities and the market participants. The former have the responsibility of designing and monitoring the markets to ensure true competitiveness, and hence, social efficiency. The latter have invested interest in the markets, and therefore, desire their investments be recovered at maximum return. By understanding market rules and how these rules are used and explored, participants and regulators have the opportunity to carry into effect their respective roles.

The strategic equilibria (Nash equilibria [1]) framework has been used as a tool for finding rational solutions for bidding strategies in centralized (Poolco) electricity markets-PJM Interconnection is a specific example of this market model implementation. It has been shown that there may be multiple strategic solutions for a single market period [2]–[4]. The model we consider assumes that market participants are rational and attempt to maximize their individual profits by untruthfully revealing their costs in their offer curves. They are assumed to play a static, noncooperative, continuous-kernel game under complete information. The solutions prescribed by this game are Nash equilibria. Further, a method to find multiple equilibria in pure (but not in mixed) strategies under this model has been considered [6].

The main problem addressed in this paper is that of finding those multiple equilibria, if they exist, in centralized electricity markets. Although their existence for special, small systems has been shown in [6], their existence in more realistic networks has yet to be established. This has been so because of the lack of a systematic and workable procedure to tackle the problem, which is of nonpolynomial order. The method for finding those multiple equilibria and the conclusions that we may derive from its application to multimachine systems, constitutes the core contribution of this paper.

The remainder of the paper is organized as follows. In Section II, we revisit the individual welfare maximization (IWM) algorithm, while formalizing the mathematical framework and providing a game theoretic perspective. Moreover, we extend the IWM algorithm to the problem of finding an equilibrium in mixed strategies. The procedure for searching for multiple equilibria is detailed in Section III, and its application is exemplified with multimachine systems in Section IV. Some conclusions are provided in Section V. Appendix B gives an example of an equilibrium in mixed strategies.

II. Individual Welfare Maximization Algorithm

A. Pure Strategies

The IWM algorithm assumes that the independent grid operator (IGO) runs a centralized economic dispatch subject to system constraints (OPF) [2], [4]. This OPF, which uses bids and offers freely submitted by the participants, sets the nodal prices (Lagrangian multipliers) that are used to charge consumption and/or pay generation on every node of the grid. Under this model, the participants may game the system by untruthfully revealing their costs/benefits on their offer/bid curves or schedules and they may do so by continuously changing one or more parameters of the marginal cost/benefit curves that they submit to the IGO. We assume that the players have reasonable estimates about all participant's true costs/benefits and we also assume that these estimates constitute common knowledge among players. Although we explore the complete information model in this paper, this is not necessary since the model can accommodate uncertainty [5]. The rational behavior of all players con-

Manuscript received January 24, 2003. This work was supported in part by the Science and Technology Foundation of the Portuguese Ministry of Science and Technology through Scholarship PRAXIS XXI/BD/13801/97, and in part by the U.S. National Science Foundation through Grant ECS 00-80279 and Grant DMI 01-09461.

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Digital Object Identifier 10.1109/TPWRS.2003.818692

stitutes common knowledge as well. The schedules submitted by the players are valid for one time period of the market, typically 1 h. Therefore, they need to optimize their bid/offer for a specific time period, which is viewed as a snapshot in time. For equilibria in mixed strategies, defined later in the paper, it is implicitly assumed that the conditions of a particular snapshot will repeat in time.

Each player in the market may find the equilibrium points through his or her own choice of the parameters of the reported schedule and by mimicking the other players' choices. This is possible from the common knowledge on rationality and information. Hence, the equilibria found by each player alone will match those found by the other players. This constitutes the main strength of the Nash equilibrium concept.

The IWM problem is cast as a nested optimization problem. The inner problem is the OPF, and the outer problem is the optimization of the individuals' utility functions subject to the OPF solutions. Moreover, each player p controls a vector of i_p reported variables $\boldsymbol{\phi}_p \in \boldsymbol{\Phi}_p$, where $\boldsymbol{\Phi}_p$ is the decision space for player p. The outer problem objective function may be called the players' decision rule [7], [8]. To facilitate the following definitions, we define $\boldsymbol{\Phi} = \boldsymbol{\Phi}_1 \times, \ldots, \times \boldsymbol{\Phi}_n$ and denote $\boldsymbol{\Phi}_{\bar{p}}$ as $\boldsymbol{\Phi}$ with $\boldsymbol{\Phi}_p$ removed. Also, $\boldsymbol{\phi}_{\bar{p}} \in \boldsymbol{\Phi}_{\bar{p}}$ denotes $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)$ with $\boldsymbol{\phi}_p$ removed.

Definition 1: A *decision rule* for player $p \in \mathcal{P} = \{1, \ldots, n\}$ is a correspondence $S_p(\cdot)$ from $\boldsymbol{\Phi}_{\bar{p}}$ to $\boldsymbol{\Phi}_p$, which associates the vector of multistrategies (or multidecision vector) $\boldsymbol{\phi}_{\bar{p}} \in \boldsymbol{\Phi}_{\bar{p}}$ of players $\bar{p}$ with the strategy (or decision) vector $\boldsymbol{\phi}_p = (\phi_p^1, \ldots, \phi_p^{i_p}) \in S_p(\boldsymbol{\phi}_{\bar{p}})$, which may be played by p when all other players $\bar{p}$ are playing $\boldsymbol{\phi}_{\bar{p}}$.

Definition 2: A multidecision vector $\boldsymbol{\phi}^* \in \boldsymbol{\Phi}$, which satisfies the static equilibrium condition $\boldsymbol{\phi}_p^* \in S_p(\boldsymbol{\phi}_{\bar{p}}^*), \forall\, p \in \mathcal{P}$, is called a *set of consistent multistrategies*.

The set of consistent multistrategies may be empty when the decision rules $S(\cdot)$ do not cross. On the other hand, it may be a large or a reduced number of multistrategies. It is dependent on the decision rules adopted by the players. This paper addresses, among other issues, the problem of finding multiple points of consistent multistrategies, given a specific decision rule. The decision rule adopted in the games developed here is profit maximization, as established by the IWM algorithm described later in (1). The vector of multistrategies represents all the control variables reported by all players to the bid-based OPF run by the IGO. These are the variables that affect the trading solution and consequently the players' revenues.

In terms of the IWM algorithm, the vector of generation controlled by player p is denoted by $\mathbf{P}_p$. The nodal prices applied to the generation controlled by player p are a byproduct of the OPF and appear as $\boldsymbol{\lambda}_p$. The cost of each generator is, in the IWM model, represented by a quadratic function

$$C_g(P_g) = a_{Pg} \cdot P_g^2 + b_{Pg} \cdot P_g + c_{Pg}, \quad \forall g \in \mathcal{G}$$

where $\mathcal{G}$ represents the set of generators. If the loads are elastic, they are represented by quadratic benefit functions

$$B_d(D_d) = a_{Dd} \cdot D_d^2 + b_{Dd} \cdot D_d + c_{D,d}, \quad \forall d \in \mathcal{D}$$

where $\mathcal{D}$ represents the set of elastic loads. These quadratic cost and benefit functions correspond to having piecewise linear price and value functions with only one segment.

Unlike the original IWM algorithm, we consider the case in which only the generators game. That is the situation faced by present-day electricity markets, where loads are still modeled as fixed forecast quantities whose consumers have no ability to game. For completeness though, Section IV gives examples with elastic load.

The game is played assuming that the players use supply-function competition [9], as opposed to Bertrand or Cournot competition, and need only to game with one of the parameters of their offer functions. For example, gaming may be achieved by replacing the true cost variable a_{Pg} by a decision variable ϕ_g in the reported price schedule. Furthermore, we associate each element of the vector of multistrategies $\boldsymbol{\phi}$ with an element of the vector of cost parameters $\mathbf{a}_P = [a_{P1}, a_{P2}, \ldots]^T$.

The individual welfare maximization problem, where only generators game and load is considered elastic, may be thus described as

$$\begin{aligned} \min_{\phi_p} \quad & f_p(\mathbf{P}_p, \boldsymbol{\lambda}_p), \quad \forall p \in \mathcal{P} \\ \text{s.t.} \quad & (\mathbf{P}_p, \boldsymbol{\lambda}_p) \text{ are determined by} \\ & \begin{pmatrix} \min\limits_{\mathbf{x},\mathbf{P},\mathbf{D}} & \sum_{g\in\mathcal{G}} C_g(P_g, \phi_g) - \sum_{d\in\mathcal{D}} B_d(D_d) \\ & \text{s.t.} \quad \mathbf{h}(\mathbf{x},\mathbf{P},\mathbf{D}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{x},\mathbf{P},\mathbf{D}) \le \mathbf{0} \end{pmatrix} \end{aligned} \tag{1}$$

where the equality and inequality constraints of the OPF are represented by $\mathbf{h}(\cdot)$ and $\mathbf{g}(\cdot)$, respectively, $\mathbf{P}$ denotes the vector of all generated power, $\mathbf{D}$ denotes the vector of all loads, and $\mathbf{x}$ represents the vector of state variables.

If the demand were to be considered fixed, then the benefit function $B_d(\cdot)$ would vanish and the vector of all load variables $\mathbf{D}$ would be substituted by a vector of fixed quantities $\mathbf{D}_{\text{fix}}$.

It is assumed in game theory that the players set their preferences according to some ordering function. We will call such a function the *loss function* f_p of player p such that $\boldsymbol{\phi}^a \in \boldsymbol{\Phi}$ is preferred to $\boldsymbol{\phi}^b \in \boldsymbol{\Phi}$ if and only if $f_p(\boldsymbol{\phi}^a) < f_p(\boldsymbol{\phi}^b)$. The loss function is the profit function negated $\pi_p(\boldsymbol{\phi}) = -f_p(\boldsymbol{\phi})$. In (1), the loss function of each player p is given by the sum of costs minus payments for his or her controlled generators. This may be written as

$$\begin{aligned} f_p(\mathbf{P}_p, \boldsymbol{\lambda}_p) &= \sum_{g\in\mathcal{G}_p} [C_g(P_g, a_{Pg}, b_{Pg}) - \lambda_g \cdot P_g] \\ &= \mathbf{P}_p^T \cdot \operatorname{diag}\{\mathbf{a}_{Pp}\} \cdot \mathbf{P}_p + \mathbf{b}_{Pp}^T \cdot \mathbf{P}_p - \boldsymbol{\lambda}_p^T \cdot \mathbf{P}_p \end{aligned} \tag{2}$$

where the fixed components have been dropped. The quadratic parameters are represented in matrix $\operatorname{diag}\{\mathbf{a}_{Pp}\}$ and the linear parameters appear in vector $\mathbf{b}_{Pp}$. The set of generators controlled by player p is denoted by $\mathcal{G}_p$.

Note that the cost function in the inner OPF loop uses the (untruthfully revealed) decision variables $\boldsymbol{\phi}$, whereas the loss function $f_p(\cdot)$ uses the true cost parameters $\mathbf{a}_p$.

The existence of a consistent set of multistrategies is a so-called fixed-point problem. We may denote the collection of decision rules S_p as $\mathbf{S}(\boldsymbol{\phi}) = S_1 \times \cdots \times S_n$. Therefore, $\mathbf{S}(\boldsymbol{\phi})$ provides a mapping of $\boldsymbol{\Phi}$ into itself. A consistent set of multistrategies may be written $\boldsymbol{\phi}^* \in \mathbf{S}(\boldsymbol{\phi}^*)$. Kakutani's fixed-point theorem addresses this existence issue [7].

Theorem 1: Suppose that the n strategy sets $\boldsymbol{\Phi}_p$ are convex, compact subsets (of $\mathbb{R}^{i_p}$) and that the n decision rules $S_p(\cdot)$ are upper semicontinuous with nonempty, convex, closed values. Then there exists a consistent multistrategy.

In real electricity markets, system constraints, such as transmission line limits, often influence operations. Therefore, the convexity of decision rules required by Theorem 1 is unlikely. However a way of proceeding is to divide the decision space $\boldsymbol{\Phi}$ into subsets (or *regions*), with the boundaries of these regions composed of points where constraints change status. Therefore, no constraint status changes occur within regions, only between regions.

Experience shows that (1) is generally well behaved. If the inner optimization problem uses a dc power flow formulation and linear constraints, then this inner problem is convex with respect to $\mathbf{x}$, for fixed $\boldsymbol{\phi}$, and is also convex with respect to $\boldsymbol{\phi}$, for fixed $\mathbf{x}$. But it is not convex overall. In addition, the outer objective function has a bilinear term $\boldsymbol{\lambda}_p^T \cdot \mathbf{P}_p$, which makes it neither convex nor concave. Notwithstanding, the searching method presented in Sections III and IV assumes good behavior (convexity) inside each of the multiple regions.

If the multistrategies of all players are chosen according to *canonical decision rules*, also known as *reaction curves*, $S_p^* : \boldsymbol{\Phi}_{\bar{p}} \rightarrow \boldsymbol{\Phi}_p$ such that

$$S_p^*(\phi_{\bar{p}}) := \left\{ \phi_p^* \in \boldsymbol{\Phi}_p \mid f_p(\phi_p^*, \phi_{\bar{p}}) = \inf_{\phi_p \in \boldsymbol{\Phi}_p} f_p(\phi_p, \phi_{\bar{p}}) \right\}$$

$\forall p \in \mathcal{P}$, then the definition of a Nash equilibrium arises naturally.

Definition 3: A decision vector $\boldsymbol{\phi}^* \in \boldsymbol{\Phi}$ is called a *noncooperative equilibrium in pure strategies* or *Nash equilibrium in pure strategies* if

$$f_p(\phi^*) = \inf_{\phi_p \in \boldsymbol{\phi}_p} f_p\left(\phi_p, \phi_{\bar{p}}^*\right), \quad \forall p \in \mathcal{P}.$$

A Nash equilibrium (or noncooperative equilibrium) is a solution that is an individual's best response to strategies actually played by his or her opponents. In other words, it has individual stability. Sufficient conditions for the existence of a Nash equilibrium are given by the following theorem.

Theorem 2: (**Nash**) Suppose the sets $\boldsymbol{\Phi}_p$ are convex and compact and the loss functions f_p are continuous and convex, $\forall p \in \mathcal{P}$. Then there exists a noncooperative equilibrium.

So in order to determine an equilibrium in pure strategies, assuming the conditions of theorem 2 , each player runs the optimization problem as cast in (1). It is run for his or her own strategy vector and for his or her opponents as well, by mimicking their optimization of their multistrategies. This is made possible by the common knowledge assumption. A stationary solution may be reached by all players independently, by iterative readjustment of $\boldsymbol{\phi}$ using Newton's method

$$\phi_p^{(k+1)} = \phi_p^{(k)} - \left(\nabla_{\phi_p}^2 f_p\right)^{-1}\Big|_k \cdot \left(\nabla_{\phi_p} f_p\right)\Big|_k, \quad \forall p \in \mathcal{P}. \quad (3)$$

If a stationary point is found, then it is stable with respect to the selected readjustment scheme and it constitutes, by definition, a Nash equilibrium.

B. Mixed Strategies

Sometimes, in the context of the IWM problem, inconsistent sets of strategies may be found. This is due to power system constraints, which give rise to discontinuities in the reaction curves [4]. As a consequence, the reaction curves might not cross. Even though equilibria in pure strategies do not exist, it may be possible to identify mixed strategies that cause the IWM algorithm to cycle back and forth across a given constraint (or set of constraints). This cyclical behavior describes an equilibrium state for a mixed strategy scenario.

Consider the compact subsets (regions) of the decision space $\boldsymbol{\Phi}$ bounded by the discontinuities. These regions, denoted by $r \in \mathcal{R} = \{1, \ldots, s\}$, are assumed to be convex. Equilibria in mixed strategies can be located by convexifying the strategy set by associating with each region a probability of playing a particular strategy in that region. In other words, each set of strategies $\boldsymbol{\upsilon}_p = (\boldsymbol{\phi}_p^1, \ldots, \boldsymbol{\phi}_p^s) \in \boldsymbol{\Upsilon}_p = \boldsymbol{\Phi}_p^1 \times, \ldots, \times \boldsymbol{\Phi}_p^s$ is associated with the $(s-1)$ simplex of $\mathbb{R}^s$

$$\mathbf{R}_p^s := \left\{ \boldsymbol{\rho}_p \in \mathbb{R}_+^s \mid \sum_{r \in \mathcal{R}} \rho_p^r = 1 \right\}$$

where ρ_p^r denotes the probability with which player p plays strategy $\boldsymbol{\phi}_p^r$. If the regions are compact and convex, then their convex combination is also compact and convex. Moreover, assuming the loss functions $f_p(\cdot)$ are continuous and convex, then their convex combination in each region is also continuous and convex.

Definition 4: A set of decision vectors $\boldsymbol{\upsilon}^* = (\boldsymbol{\upsilon}_1^*, \ldots, \boldsymbol{\upsilon}_n^*) \in \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1 \times \cdots \times \boldsymbol{\Upsilon}_n$ associated with $\mathbf{R}_p^{s*}$ is called a *noncooperative equilibrium in mixed strategies* or *Nash equilibrium in mixed strategies* if

$$e_p(\boldsymbol{\upsilon}^*, \boldsymbol{\rho}^*) = \inf_{\substack{\boldsymbol{\upsilon}_p \in \boldsymbol{\Upsilon}_p \\ \boldsymbol{\rho}_p \in \boldsymbol{R}_p^s}} \sum_{\bigotimes_{r,q \in \mathcal{R}}} \left[f_p\left(\boldsymbol{\phi}_p^r, \boldsymbol{\phi}_{\bar{p}}^{q*}\right) \cdot \rho_p^r \prod_{i \in \bar{\mathcal{P}}} \rho_i^{q*} \right] \forall p \in \mathcal{P} \quad (4)$$

where $\bigotimes_{r,q \in \mathcal{R}}$ denotes all combinations of $r, q \in \mathcal{R}$, and $\bar{\mathcal{P}}$ denotes $\mathcal{P}$ with p removed. [An example illustrating (4) is given in Appendix A.]

A solution for an equilibrium in mixed strategies is, therefore, one that maximizes the expected profit of all players by associating optimal strategies, and probabilities of playing those strategies, with each region.

It is important to note that the use of equilibria in mixed strategies is a departure from the static game. Since the definition involves the use of probabilistic (random) strategies, it is implicit that the game will be repeated. Therefore, when oscillations of the algorithm (1) across system constraints indicate the presence of an equilibrium in mixed strategies, the convex combination $e_p(\cdot)$ is substituted for $f_p(\cdot)$. As a consequence, the IWM problem assumes the form

$$\begin{aligned}\min_{\boldsymbol{v}_p}\quad & e_p(\boldsymbol{\Pi}_p, \boldsymbol{\Lambda}_p), \quad \forall p \in \mathcal{P}\\ \text{s.t.}\quad & (\boldsymbol{\Pi}_p, \boldsymbol{\Lambda}_p) \text{are determined } \forall r \in \mathcal{R} \text{ by}\\ & \begin{pmatrix} \min\limits_{\mathbf{x}^r, \mathbf{P}^r, \mathbf{D}^r} & \sum\limits_{g\in\mathcal{G}} C_g\left(P_g^r, \phi_g^r\right) - \sum\limits_{d\in\mathcal{D}} B_d\left(D_d^r\right)\\ \text{s.t.} & \mathbf{h}(\mathbf{x}^r, \mathbf{P}^r, \mathbf{D}^r) = \mathbf{0}\\ & \mathbf{g}(\mathbf{x}^r, \mathbf{P}^r, \mathbf{D}^r) \le \mathbf{0}\end{pmatrix}\end{aligned} \tag{5}$$

where $\boldsymbol{\Pi}_p = (\mathbf{P}_p^1, \ldots, \mathbf{P}_p^s)$ and $\boldsymbol{\Lambda}_p = (\boldsymbol{\lambda}_p^1, \ldots, \boldsymbol{\lambda}_p^s)$ are, respectively, the matrices of generated power and nodal prices for all regions spanned by the (mixed strategy) equilibrium. An equilibrium in mixed strategies is found using Newton's method, substituting $e_p(\cdot)$ for $f_p(\cdot)$ in (3).

A continuation process is generally required to locate the desired mixed strategy equilibrium $(\boldsymbol{v}^*, \boldsymbol{\rho}^*)$. The continuation algorithm gradually varies the value of a line limit from an initial (high) value chosen to ensure the existence of a pure strategy equilibrium. In other words, at initialization, the line limit has no influence on market behavior. As the line limit is reduced from that initial value to its final true value (where the constraint induces mixed strategy behavior), the solution of (5) evolves smoothly along the continuation path to the desired solution $(\boldsymbol{v}^*, \boldsymbol{\rho}^*)$.

The rationale for such a method is the fact that the loss functions in mixed strategies $e_p(\cdot)$ must have continuous first-derivatives and can be assumed convex only inside a small decision subspace where the constraints do not cause discontinuities. The high sensitivity of the loss functions with respect to the offer parameters and probabilities makes it extremely difficult to guess an initial solution inside that convex region. Appendix B provides a simple example to illustrate this continuation process.

III. Searching for Multiple Equilibria

The strategy sets $\boldsymbol{\Phi}_p$ for all players are convex but, as shown earlier, the loss functions $f_p(\boldsymbol{\phi}_p, \boldsymbol{\phi}_{\bar{p}})$ are generally not convex. The functions are continuous and differentiable if the set of system constraints does not change. When some system constraint changes status though, the first derivatives of the loss functions suffer a discontinuity and the players' reaction curves $\mathbf{S}(\boldsymbol{\phi})$ may also exhibit a discontinuity. Nevertheless, we assume that the loss functions $\mathbf{f}(\boldsymbol{\phi})$ are convex inside each of the regions given by different combinations of system constraints. Therefore, it is necessary only to look for a single equilibrium inside each of those regions.

If we were only concerned with equilibria in pure strategies, we could run the IWM algorithm for pure strategies for each region of the bidding space by forcing the system to operate in that particular region. We would only have to enforce the system constraints defining that region. Those constraints would be made equality constraints while the remaining ones would be relaxed. If an equilibrium was found for this modified problem, then the IWM algorithm for pure strategies would be run for the original problem (1), using the constrained equilibrium as a starting point, to check its feasibility.

A drawback of this procedure is the need to check all possible regions given by all combinations of system constraints. The problem is of combinatorial nature and the number of possibilities may render a nonmanageable problem. In addition, by extending (in the modified problem) each region beyond its true boundaries, equilibria in mixed strategies are not detectable.

Consequently, any method of searching for multiple equilibria, both in pure and in mixed strategies, has to address two problems. First, the number of regions to be tested for equilibria must be dramatically reduced. Second, the algorithm should be given the opportunity to start each search inside each of these different regions without artificially enforcing or relaxing any of the system constraints.

Reducing the number of regions may be viewed as a problem of filtering power flow solutions to determine those that are feasible for specified regions while subject to the OPF run by the IGO.

A. Feasible Regions

The initial step in locating multiple equilibria is to search for feasible power flow solutions that correspond to specified regions. A feasible power flow inside a specified region, if it exists, may be found from a particular base case power flow via a linear programming (LP) problem, when the power flows are described by dc power flow equations. If regions are characterized by line constraints but not load limits, then for each region $r \in \mathcal{R}$, the problem has the form

$$\begin{aligned}\min\quad & \sum_{\forall g\in\mathcal{G},\forall d\in\mathcal{D}} \Delta P_g^+ + \Delta P_g^- + \Delta D_d^+ + \Delta D_d^-\\ \text{s.t.}\quad & \sum_{\forall g\in\mathcal{G},d\in\mathcal{D}} \Delta P_g^+ - \Delta P_g^- - \Delta D_d^+ + \Delta D_d^- = 0\\ & \mathbf{J}_l \cdot (\Delta P - \Delta D) = c_l^r \cdot P_l^{\max} - P_l^{\text{base}}, \quad \forall l \in \mathcal{L}^r\\ & \mathbf{J}_l \cdot (\Delta P - \Delta D) < P_l^{\max} - P_l^{\text{base}}, \quad \forall l \notin \mathcal{L}^r\\ & -\mathbf{J}_l \cdot (\Delta P - \Delta D) < P_l^{\max} + P_l^{\text{base}}, \quad \forall l \notin \mathcal{L}^r\\ & 0 \le \Delta P_g^+ \le P_g^{\max} - P_g^{\text{base}}\\ & 0 \le \Delta P_g^- \le P_g^{\text{base}}\\ & 0 \le \Delta D_d^+ \le D_d^{\max} - D_d^{\text{base}}\\ & 0 \le \Delta D_d^- \le D_d^{\text{base}}\end{aligned} \tag{6}$$

where ΔP_g^+ and ΔP_g^- represent the positive and negative changes of active power generated by generator g, ΔD_d^+ and ΔD_d^- are the positive and negative changes of active power consumed by load d (for fixed load this variation is zero), $\mathbf{J}_l$ denotes the row vector of sensitivities of the flow on link l to changes in the controls, and ΔP and ΔD denote the vectors of all active power generation and load, respectively.

We denote by $\mathcal{L}^r$ the set of lines defining a particular region r. Moreover, we associate each region r with a case $\mathcal{C}_r = \{c_1^r, \ldots, c_m^r\}$, where c_l^r denotes the state of constraint l for that specific region. In this case, where we take into consideration only the active power flow line limits m will denote the number of lines in the system and $c_l^r \in \{0, 1, -1\}$ represents the possible states of the active power flow on line l for region r. Values of 1 or -1 indicate the line is congested with the active power flow in the corresponding (forward or reverse) direction. A value of $c_l^r = 0$ indicates that $l \notin \mathcal{L}^r$.[1] The bidirectional limit on active power flow on line l is given by $P_l^{\max}$, while the limits on active power generation of generator g and active power consumption of load d are given by $P_g^{\max}$ and $D_d^{\max}$. The base case line flows, generation, and consumption are denoted by P_l^{base}, P_g^{base} and D_d^{base}, respectively.

If a full power flow was substituted for the dc equations, a sequential method could be used. In this case, matrix $\mathbf{J}$ would be replaced by the sensitivities obtained from the Jacobian.

B. Feasible, Optimal Regions

The second step is the screening of the feasible power flow solutions to determine feasible, optimal power flow solutions. This further filters the number of regions that have to be tested for equilibria and, in addition, provides the initial solutions $\boldsymbol{\phi}^{r,0}$ to be used in the IWM algorithm for the surviving regions. Because these initial solutions are inside specified regions, they are appropriate for finding equilibria in mixed strategies.

A feasible, optimal power flow inside a region $r \in \mathcal{R}$ may be found, if it exists, by means of a modified version of the IWM algorithm

$$\begin{aligned}\min_{\phi} \quad & \sum_{l\in\mathcal{L}^r} (P_l - c_l^r P_l^{\max})^2 + \sum_{l\notin\mathcal{L}^r} |s_l| \cdot (P_l - s_l P_l^{\max})^2 \\ & \text{where } s_l = \begin{cases} -1: P_l < -P_l^{\max} \\ +1: P_l > P_l^{\max} \\ 0, \text{otherwise} \end{cases} \\ \text{s.t.} \quad & \mathbf{P}_l \text{ is determined by} \\ & \begin{pmatrix} \min_{\mathbf{x},\mathbf{P},\mathbf{D}} & \sum\limits_{g\in\mathcal{G}} C_g(P_g, \phi_g) & - \sum\limits_{d\in\mathcal{D}} B_d(D_d) \\ \text{s.t.} & \mathbf{h}(\mathbf{x},\mathbf{P},\mathbf{D}) = \mathbf{0} & \\ & \tilde{\mathbf{g}}(\mathbf{x},\mathbf{P},\mathbf{D}) \le \mathbf{0} & \end{pmatrix}\end{aligned} \tag{7}$$

where the IWM objective function is substituted by a penalty function penalizing the active power flow on lines where limits are violated. In this formulation, $\tilde{\mathbf{g}}(\cdot)$ denotes $\mathbf{g}(\cdot)$ with the removal of all active power flow limits on lines.

The outer penalty function is consistent with determining whether the players have the ability to drive solutions to specified regions by changing their decision vector. If the regions were imposed in the inner problem, through equality constraints, then the IGO would be specifying the regions in advance and the solutions could differ from those in the original problem. In summary, the existence of a solution in the original problem (1) implies the existence of a solution in the modified problem (7) but the converse is not true.

[1]The line l may or may not be congested, as determined by the associated inequalities of (6).

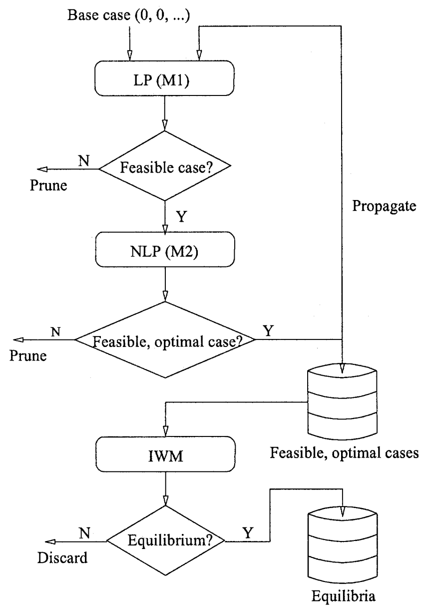

Fig. 1. Procedure for equilibria search.

Problem (7) is solved using diagonally scaled steepest descent with stepsize selection using the golden section method [10]. One crucial point while searching iteratively for a decision vector $\boldsymbol{\phi}$ that makes the system operate inside a specific region is to keep all the generators controllable. For this reason, every time a generator hits its active power generation limit, the corresponding control variable ϕ_g must be kept at the kink. In other words, the control variable is updated such that the generation is as close as possible to the limit, but not at the limit. The reason for doing this is to keep every generator from prematurely "giving up" contributing for a feasible, optimal solution inside a specified region during the steepest descent method.

The IWM algorithm will, in the end, be run for only the feasible, optimal regions that correspond to a database of feasible, optimal cases. Fig. 1 summarizes the entire procedure, from the base case to obtaining a database of equilibria. Because of the region-based initial solutions provided by the filtering procedure, the IWM algorithm (1) will be given the opportunity to search in every feasible, optimal region for an equilibrium either in pure or in mixed strategies. For the former, the algorithm will provide a specific solution, whereas for the latter it will exhibit oscillations implying that the algorithm in (5) must be employed.

C. Propagation and Pruning

The algorithm described in Sections III-A and B is of nonpolynomial order, being of order $\mathcal{O}(3^m)$ if only the line constraints are taken into account. Therefore, it is impossible to manage any real life example in the absence of a method for carefully propagating and pruning the cases. Fortunately, the two-step filtering process, together with careful propagation and pruning, allow the elimination of a substantial number of cases. The tree of Fig. 2 illustrates the sequence in which the cases

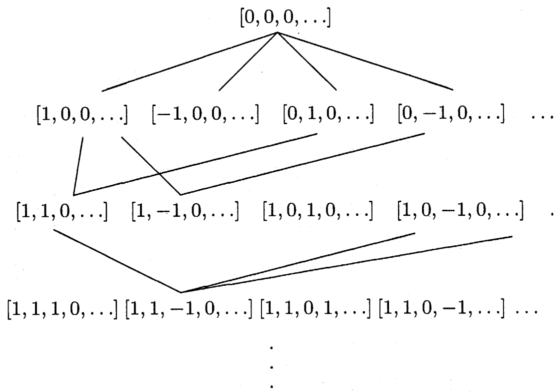

Fig. 2. Propagation of cases.

must be generated in order to maximize pruning.[2] Every new child case is generated from the parent cases that survive the feasibility and optimality tests of the previous level. They are formed by including more congested lines further to the right, until the last represented line is reached. This order of generation avoids repetition or formation of cases that have no parents on the previous level.

On each level, every case represents an equal number of constraints; one for level one, two for level two, and so on. The addition of new congested lines from left to right (it could be from right to left) requires the existence of all necessary parents in the previous level. If any of the parents are not present, the new case is not generated. The rationale is that if there is no solution for a specific set of constraints then there is no solution for that same set of constraints with additional constraints added. Checking the existence of parents is achieved using a binary search on the set of feasible, optimal cases of the previous level. The binary search must be able to compute the correct order of cases on each level, as shown from left to right in Fig. 2. This simple procedure avoids repetition of cases.

For real systems where players have the ability to congest a reduced number of lines, this method proves very efficient. Moreover, the method may be implemented with high parallelism given the fact that cases within one level are independent. Besides, this search has to be performed only once for a given load profile. Once the feasible, optimal regions are known for that load profile, it is possible to try a different number of players and/or different cost parameters.

IV. Example

We chose a version of the IEEE 30-bus, six-generator test system and a version of the IEEE 57-bus, seven-generator test system for our examples [11]. These systems yield a sufficiently high number of possible regions to make them interesting.

The two systems were run for both fixed load and elastic load, served entirely by the players. For the experiments with elastic loads, we considered no upper limits for these loads. Furthermore, two scenarios of generator ownership were considered; one where each generator belonged to a different player, another where the ownership was divided among three players. The line limits were fixed close to the flows established by the solution for fixed load, maximum number of players, and truthful revelation of costs. However, the line limits were not chosen too close to those flows, so the generators were given some room to game the system. The power flows on lines were determined using dc power flow equations.

TABLE I
Feasible, Optimal Regions and Equilibria

Test System	Fixed load	Elastic load
IEEE-30 (6 players)	(5),1	(9),1
IEEE-30 (3 players)	(5),1	(9),1
IEEE-57 (7 players)	(8),2	(4),1
IEEE-57 (3 players)	(8),1	(4),1

The results for eight different experiments are summarized in Table I. The number of feasible, optimal regions found for each test system is shown in parenthesis, along with the number of equilibria for that particular experiment.[3] Naturally, these results could vary for changes in the line limits, costs, and benefits.

In these experiments, all cases resulted in equilibria in pure strategies without binding line constraints. No general conclusions can be drawn from that observation though. One of the cases, the IEEE 57-bus system with seven players and fixed load, produced a second equilibrium in pure strategies with one binding line constraint. This second equilibrium dominates strategically the first equilibrium, which makes it the only relevant equilibrium in terms of the players' strategic game. From a strategic point of view, equilibria that are dominated may be ignored or eliminated, because none of the players will gain anything by playing it.

Definition 5: In the game $\mathbf{f}\colon \mathbf{\Phi} \to \mathbb{R}^n$, let $\boldsymbol{\phi}_p^a \in \mathbf{\Phi}_p$ and $\boldsymbol{\phi}_p^b \in \mathbf{\Phi}_p$ be strategies for player p. Strategy $\boldsymbol{\phi}_p^b$ is *strictly dominated* by strategy $\boldsymbol{\phi}_p^a$ if, for all vectors of multistrategies $\boldsymbol{\phi}_{\bar{p}} \in \mathbf{\Phi}_{\bar{p}}$, p's loss from playing $\boldsymbol{\phi}_p^a$ is strictly less than p's loss from playing $\boldsymbol{\phi}_p^b$. That is, $f_p(\boldsymbol{\phi}_p^a, \boldsymbol{\phi}_{\bar{p}}) < f_p(\boldsymbol{\phi}_p^b, \boldsymbol{\phi}_{\bar{p}})$, $\forall \boldsymbol{\phi}_{\bar{p}} \in \mathbf{\Phi}_{\bar{p}}$.

An example of a possible variation of the experiments presented here would be to decrease some line limits to the point where there is no equilibrium without binding constraints. One could then possibly see the appearance of an equilibrium in mixed strategies spanning two regions. However, this again would most certainly be either a unique equilibrium, or one of very few equilibria for that system.

V. Conclusion

This paper addresses the problem of finding multiple market equilibria for multimachine systems that potentially yield an unmanageable number of search regions. The method proposed eliminates the majority of those regions through successive filtering from power flow solutions to feasible, optimal power flow solutions.

Although the existence of multiple equilibria in realistic networks is possible, both in theory and in practice, they are not to

[2]Fig. 2 shows $\mathcal{C}_r$ for each case, with entries c_l^r determining line congestion status.

[3]A maximum of 50 iterations were allowed before assuming nonconvergence.

TABLE II
PRICE SCHEDULE PARAMETERS

Generator	a_P (\$/MW^{2}h)	b_P (\$/MWh)
1	0.01	10.0
2	0.01	10.0

TABLE III
VALUE SCHEDULE PARAMETERS

Load	a_D (\$/MW^{2}h)	b_D (\$/MWh)
1	-0.04	30.0

be expected in large numbers. This is illustrated by the examples in the paper. Even if the realistic parameters used were changed, we expect similar results in terms of the number of equilibria attained.

In addition, this paper formally defines equilibria in multiple strategies in the context of power systems and in the presence of system constraints. It shows, through a simple example, how to find these equilibria by means of a continuation method. Once the spanned regions are known, this method establishes a path from an initial solution to the desired equilibrium.

APPENDIX A

ILLUSTRATION OF e_p IN DEFINITION 4

Consider a situation where there are three players and two regions. Then $\mathcal{P} = \{1,2,3\}$, $\mathcal{R} = \{1,2\}$, and

$$\begin{aligned} e_1(\boldsymbol{v},\boldsymbol{\rho}) = &\underbrace{f_1\left(\phi_1^1,\phi_2^1,\phi_3^1\right)\rho_1^1\rho_2^1\rho_3^1}_{r=1,q=[1,1]} + \underbrace{f_1\left(\phi_1^1,\phi_2^2,\phi_3^1\right)\rho_1^1\rho_2^2\rho_3^1}_{r=1,q=[2,1]} \\ &+ \underbrace{f_1\left(\phi_1^1,\phi_2^1,\phi_3^2\right)\rho_1^1\rho_2^1\rho_3^2}_{r=1,q=[1,2]} + \underbrace{f_1\left(\phi_1^1,\phi_2^2,\phi_3^2\right)\rho_1^1\rho_2^2\rho_3^2}_{r=1,q=[2,2]} \\ &+ \underbrace{f_1\left(\phi_1^2,\phi_2^1,\phi_3^1\right)\rho_1^2\rho_2^1\rho_3^1}_{r=2,q=[1,1]} + 3 \text{ more terms,} \end{aligned}$$

with e_2 and e_3 following a similar pattern. Because there are only two regions in this case, $\rho_p^2 = (1-\rho_p^1)$.

APPENDIX B

TWO-BUS EXAMPLE

The IWM algorithm was run for a two-bus, one-line system $(x = 0.0485 \text{ p.u.})$ with two generators and one load, as specified in Tables II and III. Choosing a sufficiently high line flow limit resulted in convergence to the equilibrium in pure strategies given by $\{\phi_1,\phi_2\} = \{0.0256, 0.0256\}$. Fig. 3 shows the reaction curves for the two players for a line limit of $P_{12}^{\max} = 115 \text{ MW}$. The equilibrium corresponds to the intersection of the reaction curves $S_1^*(\phi_2)$ and $S_2^*(\phi_1)$.

However, upon lowering the line limit to $P_{12}^{\max} = 80 \text{ MW}$, the reaction curves of the two players no longer intersect. This is shown in Fig. 4. Instead of intersecting, the end-points of each

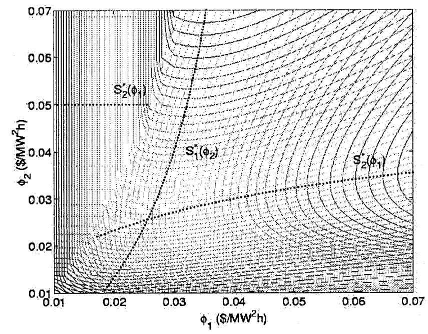

Fig. 3. Reaction curves for $P_l^{\max} = 115 \text{ MW}$.

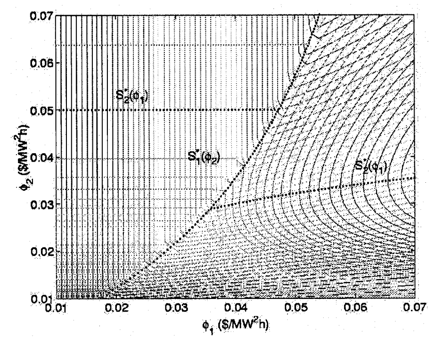

Fig. 4. Reaction curves $P_l^{\max} = 80 \text{ MW}$.

section of the discontinuous reaction curve $S_2^*(\phi_1)$ touch the continuous reaction curve $S_1^*(\phi_2)$.[4] As a consequence, the pure strategy algorithm cycled between the congested and noncongested regions, instead of converging to an equilibrium solution.

For this small system, the expressions for the expected loss (5), when the two regions spanned by the discontinuity are considered, become

$$\begin{aligned} e_1 &= \rho_2^1 \cdot f_1\left(\phi_1,\phi_2^1\right) + \left(1-\rho_2^1\right)\cdot f_1\left(\phi_1,\phi_2^2\right) \\ e_2 &= \rho_2^1 \cdot f_2\left(\phi_1,\phi_2^1\right) + \left(1-\rho_2^1\right)\cdot f_2\left(\phi_1,\phi_2^2\right) \\ \rho_2^1 &\in [0,1]. \end{aligned}$$

This has a simplified form because only player 2 encounters multiple regions, and hence, presents a discontinuity in his or her reaction curve $S_2^*(\phi_1)$. In fact, player 1 exhibits a continuous reaction curve $S_1^*(\phi_2)$, implying that his or her strategy is independent of regions. This is implemented by setting $\rho_1^1 = 1$, giving $\rho_1^2 = 0$, and so forcing player 1 to always play the same strategy.

The continuation algorithm proposed in Section II was used to obtain the mixed strategy equilibrium for the situation depicted in Fig. 4. An initial value of $P_{12}^{\max} = 107 \text{ MW}$ was chosen, and the line limit was subsequently decreased in steps of 0.01 MW. Initial values for the optimization (strategy and probability) variables $\phi_1, \phi_2^1, \phi_2^2$, and ρ_2^1 were obtained as follows. The value for ρ_2^1 was chosen to equal 1 since this corresponds to an equilibrium in pure strategies in the absence of the line constraint. Therefore, the values for ϕ_1 and ϕ_2^1 were also set to

[4]The right section of S_2^* was obtained by initializing the IWM algorithm in the noncongested region, whilst the left section corresponds to the congested region.

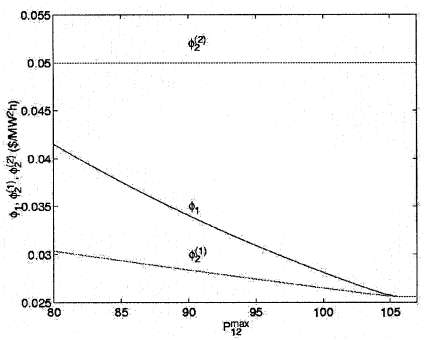

Fig. 5. Offer parameters in the continuation algorithm.

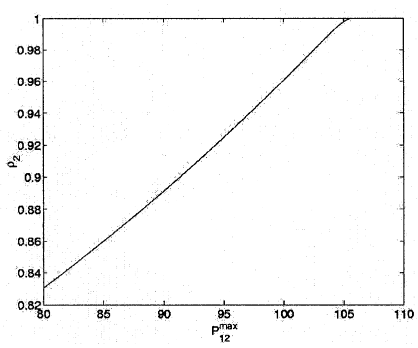

Fig. 6. Player 2's strategies in the continuation algorithm.

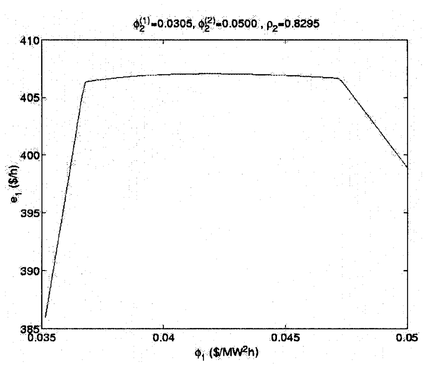

Fig. 7. Player 1's expected gains in mixed strategies.

their values at the equilibrium in pure strategies. Only the initial value for ϕ_2^2 deserved some care. This variable has meaning only after the pure strategy equilibrium disappears. Its value under those conditions must be predicted. By running a binary search, it was discovered that the equilibrium disappears at a line limit of $P_{12}^{\max} = 94.68$ MW and the reaction of player 2 at that point was $\phi_2^2 = 0.0500$.

The values for the optimization variables along the continuation path are shown in Figs. 5 and 6. The leftmost points on the curves give the solution for the desired line limit $P_{12}^{\max} = 80$ MW, $\{\phi_1, \phi_2^1, \phi_2^2, \rho_2^1\} = \{0.0419, 0.0305, 0.0500, 0.8295\}$.

The nonsmoothness of the function representing the expected gains for player 1, given a fixed strategy by player 2, may be appreciated by observing Fig. 7. The shape of this cost function highlights the potential difficulties of attempting direct solution, and justifies the use of a continuation process.

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