Inverse-Affine Dependence of Recovery-Time Sensitivities on Critical Disturbance Parameters: A Nonlinear Dynamics Explanation

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***Abstract*— We study the sensitivity properties of non-linear dynamical systems subject to disturbances, when disturbance parameters are near critical values. In particular, in seeking to explain recent experimental and simulation results, we give an explanation for why the inverses of response-time sensitivities to disturbance parameters often display an affine dependence on the parameters near critical points. Application of the results in electrical power system analysis and virus-spreading control examples are presented.**

I. Introduction

In infrastructure systems such as electric power systems and air traffic networks, various disturbance parameters are known to have critical threshold values that delineate different operating regimes and dynamical responses, see for example [1]–[4]. In many cases, when the value of a parameter is on one side of the threshold, the system gracefully returns to its nominal pre-disturbance operating characteristics; meanwhile, parameter values beyond the threshold yield entirely different transient responses and steady-state operation. For instance, when a fault-clearing time in an electric power system is below a threshold, the system will return gracefully to the pre-fault operating condition. In contrast, if the fault-clearing time exceeds the threshold, the trajectory may leave the region of attraction of the operating point, resulting in vastly different dynamic behavior. Similarly, in air traffic systems, it is well known that weather constriction of one severity or duration may have little impact on the traffic flows, while a slightly stronger or longer constriction has network-wide impact.

In studying these threshold behaviors in infrastructure systems, it is particularly important to characterize the dynamics and their sensitivities when the disturbance parameters are near the threshold, for several reasons:

1) Responses in this case may incur significant cost, for example because of slow recovery,
2) Response characteristics will tend to be highly sensitive to disturbance parameter changes, and
3) Such characterizations can be helpful in predicting the threshold.

The purpose of this article is to give a preliminary discussion on an interesting dependence of response-characteristic *sensitivities* to disturbance parameters near a critical point. In particular, some studies of electric power systems as well as other infrastructure networks have observed the following: the inverse of the sensitivity of a temporal response characteristic to a disturbance parameter (henceforth called an *inverse-sensitivity*) often turns out to be an affine function of the parameter near its threshold (critical point), with the inverse-sensitivity equalling zero exactly at the critical point [2]. Such a linear dependence of the inverse-sensitivities, if it is a common phenomenon, can be valuable for easily estimating the critical parameter value and for *a priori* predicting temporal response characeristics/costs near the critical value. Of course, large infrastructure systems are incredibly complex and multi-faceted entities, and this inverse-sensitivity relationship may not always hold nor admit a common explanation even when it occurs. However, it is worthwhile to explore whether simple explanations can be obtained for typical threshold behaviors or criticalities that are observed in nonlinear dynamics.

We consider the particular case that the criticality is a consequence of the system's trajectory exiting the domain of attraction of the post-disturbance equilibrium point, and entering a different equilibrium's domain of attraction. For this case, we make the argument that the affine inverse-sensitivity relationship is a typical phenomenon that can result from the increasing duration spent by the trajectory near an unstable equilibrium (or its stable manifold) as the disturbance parameter approaches its critical value.

Trajectories of non-linear systems have been extensively characterized, see for example the texts [5], [6], and the sensitivity of trajectory characteristics due to small changes in initial conditions and/or parameters is well-understood [7]. Of particular interest to us, parameterized models of transient disturbances and post-disturbance system dynamical responses have been studied in several contexts. In a few of these studies, and more broadly in a few studies of bifurcation phenomenon, the possibility that the trajectory remains near an unstable equilibrium point for an extended duration has been noted and analyzed [8]–[11]. To the best of our knowledge, however, sensitivities for such trajectories that remain near unstable equilibria have not been characterized systematically. Here, we initiate such a study, and motivate its use in analyzing and designing infrastructure systems.

The remainder of this article is organized as follows. In Section II, we demonstrate linear inverse-sensitivity properties of canonical nonlinear systems, including scalar and certain planar systems. In the process, we introduce simple examples from virus-spread control and electrical power system analysis. In Section III, we briefly discuss the possible broader applications of the inverse-sensitivity results. Section IV presents conclusions and suggests directions for future work.

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II. Inverse-Sensitivity Results for Canonical Systems

Our goal in this section is to explain the observed affine dependence of inverse sensitivities (of dynamical response/recovery characteristics to disturbance parameters) near critical parameter values, in the context of certain canonical nonlinear differential systems. Specifically, we consider disturbances which result in states near an unstable equilibrium or its stable manifold in the post-disturbance system (for near-critical disturbance parameter values). We argue that it is the extended time spent near this unstable equilibrium or manifold that largely governs the time required by the system to return to an equilibrium state, and hence primarily determines recovery characteristics and their sensitivities. We will initially pursue this analysis for a class of scalar systems (Subsection II-A), and then progress to a broader class that includes many planar systems of interest (Subsection II-C). In each case, we conclude the analytical development with a canonical infrastructure-system example.

A. The scalar case

Let us consider an autonomous scalar time-invariant differential equation $\dot{x} = f(x)$, where $f()$ is a continuous, differentiable function with bounded values and first-derivative, that describes a system's post-fault dynamics. We assume that the system has $n > 1$ distinct equilibrium points $x_1, \dots, x_n$, at least one of which (say x_1) is asymptotically stable in the sense of Lyapunov. We notice that the region of attraction of the stable equilibrium is an open set, that is bounded on at least one side by another equilibrium point, say x_2. Let us consider the case that x_2 is an unstable equilibrium, and in fact the slope of $f()$ at x_2 is non-vanishing. Let us consider the circumstance that the system's state is nominally at the equilibrium x_1, and that the state is modified during a disturbance to a different location, according to a disturbance parameter value w. Specifically, we assume that the disturbance impacts the system until a *recovery start time* t_s, and places the state at a value $x(t_s)$ that is a function of w, specifically at $x(t_s) = g(w)$. We assume that for a critical parameter value $w = w_c$, the disturbance places the state precisely at x_2 (i.e., $g(w_c) = x_2$). Without loss of generality, we assume that $x_1 < x_2$ and $x_1 < g(w) < x_2$ (respectively, $g(w) > x_2$) for $w < w_c$ (respectively, $w > w_c$). Thus, the state remains in the domain of attraction of the nominal equilibrium point for a sufficiently small disturbance parameter w, and leaves the domain for larger w. Note that, based on our assumption regarding the equilibrium x_2, we have that $\frac{df}{dx}|_{x=x_2} > 0$

Our aim is to characterize the dynamical response of the scalar differential equation $\dot{x} = f(x)$, $x(t_s) = g(w)$, for w near but less than the critical point w_c and $t \geq t_s$. In particular, let us focus on measures of *recovery time*, for instance the time required for the state to return to a specified interval around the nominal equilibrium point. We note that, as the disturbance parameter value is brought closer and closer to its critical value, such recovery times are dominated by the time spent by the trajectory near the unstable equilibrium point, since the rate of change of the state is vanishingly small there. Thus, we can approximately characterize such return times and their sensitivities to disturbance parameter changes by studying the *escape time* of the trajectory from the unstable equilibrium. The following simple theorem makes explicit the sensitivity of the escape time from the unstable equilibrium to changes in the initial state $x(t_s)$.

Theorem 1: Consider an initial state that is in the domain of attraction of x_1 and is a distance b from the unstable equilibrium point, i.e. $x_2 - x(t_s) = b > 0$. Let T be the time until the state escapes to a distance $c > b$ from x_2, i.e. the time T such that $x_2 - x(t_s + T) = c$. As b and c are made small, the inverse sensitivity of the time T with respect to b, or $1/\frac{dT}{db}$, is approximated arbitrarily well by the linear function $-ab$, where $a = \frac{df}{dx}|_{x=x_2} > 0$. Thus, we see that the inverse sensitivity of the escape time with respect to the initial condition is indeed linear in the initial condition.

Proof: As b and c are made small, the trajectory until escape becomes arbitrarily close to the trajectory when the linear approximation of $f(x)$ around the unstable equilibrium is applied, and hence the escape time can be approximated arbitrarily well from the linear approximation. Using the linearization, we find that $x_2 - x(t_s + t) = e^{at}b$. Setting this response equal to the escape value c, we find that the escape time is given by $T = \frac{1}{a}ln(\frac{c}{b})$. The inverse sensitivity result then follows with just a little algebra. ■

Remark: It is worth noting that, in the case that $\frac{df}{dx}|_{x=x_2} = 0$ (i.e., the slope of the non-linear function is vanishingly small at the equilibrium point, the dependence of the inverse sensitivity of the escape time with respect to the initial condition can be determined from polynomial-approximations of the function $f()$ near x_2; these cases in general do not yield a fully affine dependence.

We have thus far characterized the inverse sensitivity of the escape time with respect to the distance of the initial state (state at time t_s) from the unstable equilibrium. In our formulation, we view the initial state as being governed by a disturbance parameter, for instance a fault clearing time in electric power system applications. Let us characterize the inverse sensitivity of the escape time with respect to the disturbance parameter. To do so, we notice that $b = x_2 - x(t_s) = x_2 - g(w)$. Then, using the chain rule, we immediately obtain

$$\frac{1}{\frac{dT}{dw}} = \frac{a(x_2 - g(w))}{\frac{dg}{dw}}. \quad (1)$$

In general, the dependence of the inverse sensitivity on the parameter w may or may not be linear. However, in the case where $x_2 - g(w)$ takes the form $k(w_c - w)^\alpha$ for some $\alpha > 0$, then

$$\frac{1}{\frac{dT}{dw}} = \frac{a}{\alpha}(w_c - w) \quad (2)$$

or in other words the inverse sensitivity is affine in w. We argue that this sort of polynomial dependence, and hence affineness of the inverse sensitivities, is quite common. In fact, considering a Taylor series approximation of $g(w)$, we can argue under broad continuity conditions that an affine inverse sensitivity is approximately valid for some small range around the critical parameter value.

Let us continue the discussion of the scalar case with an illustrative example.

B. Example: Saturating control of virus spread

Classical non-linear population-dynamics models are commonly used to represent the spread of viruses, see for example [12]. A control action, for example quarantine of suspected infectives, can be modeled as removing infectives at a rate that is proportional to the number of infectives, subject to a saturation effect [13]. The following is a specific canonical example of a controlled virus-spread model:

$$\dot{x} = x(1-x) - 0.09\sigma(\frac{4x}{0.09}), \tag{3}$$

where $x \in [0,1]$ is the fraction of a population that is infected by a virus, and $\sigma(.)$ is the standard saturation function. It is easy to check that this system has stable equilibria at $x = 0$ and $x = 0.9$, and an unstable equilibrium at $x = 0.1$. Consider the recovery time to the stable equilibrium point $x = 0$ as the initial condition is made close to the unstable equilibrium point $x = 0.1$. Linearizing the model at that unstable equilibrium point gives $a = \frac{df}{dx}|_{x=0.1} = 0.8$. This suggests that the inverse sensitivity (2), with $\alpha = 1$, will have a slope of -0.8. The inverse sensitivity of the time until the infected fraction recovers to near $x = 0$ (specifically, to $x = 0.02$) is shown as a function of the initial condition in Figure 1. We see that the inverse sensitivity indeed satisfies an affine dependence for initial conditions near $x = 0.1$, with slope equal to -0.8 and critical point (or infinite-sensitivity point) at $x = 0.1$.

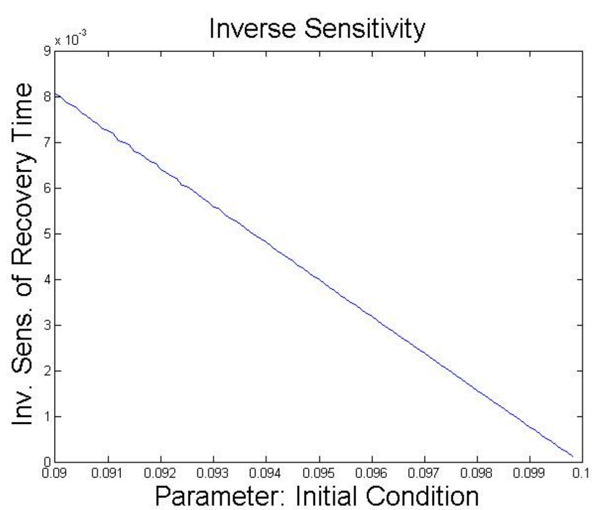

Fig. 1. A nonlinear model for virus-spread control is considered. The inverse sensitivity of the recovery time to a virus-free state with respect to the initial infection fraction is shown. For initial conditions near the critical point (unstable equilibrium), the inverse-sensitivity is affine with respect to the initial condition with slope 0.8, as predicted.

C. A more general case

The phenomenon verified above for scalar systems, namely that inverse sensitivities of temporal response characteristics with respect to disturbance parameters are affine near critical points, is in fact common to a broader family of systems. This is because the linear inverse sensitivity property fundamentally results from the increasingly long time spent by the trajectory near an unstable equilibrium when the initial condition is brought close to that unstable equilibrium; similar response characteristics will be in force for multivariate nonlinear systems whose trajectories are near an unstable equilibrium point or its stable manifold upon disturbance.

Let us give a more explicit description of a class of multivariate models that dispay linear inverse sensitivities. First, recalling that the time spent by a trajectory near an unstable equilibrium can be studied through a linearization, we formalize the escape time from an unstable linear system's equilibrium (or from its stable manifold). Using this result, we argue at a conceptual level that certain nonlinear systems will also exhibit linear inverse sensitivities, and give a formal statement thereof for a common class of planar systems.

Let us first study the escape time from an unstable linear system's equilibrium (or stable manifold). In the case where the system has a single right-half-plane (RHP) eigenvalue, the escape-time characterization is analogous to that for the scalar case. In particular, we see that the component of the response in the eigenvector direction associated with the RHP eigenvalue escapes from the stable manifold according to the scalar dynamics described above, and so a linear inverse-sensitivity relationship is obtained. The result can be formalized as follows:

Lemma 1: Consider the real autonomous linear system $\dot{\mathbf{x}} = A\mathbf{x}$ of dimension n, and assume that $n-1$ eigenvalues of A are in the closed left half plane, with only simple eigenvalues on the $j\omega$-axis. Assume that the remaining eigenvalue has value $a > 0$, with corresponding right and left eigenvectors $\mathbf{v}$ and $\mathbf{w}^\top$. It follows that the equilibrium $\mathbf{x} = \mathbf{0}$ has a stable manifold of dimension $n-1$. Consider the case where the component of the initial condition away from the stable manifold, $\mathbf{w}^\top\mathbf{x}(0)$, equals $b > 0$, and consider the time T such that $\mathbf{w}^\top\mathbf{x}(T) = c > b$. The inverse of the sensitivity of the time T to the initial condition b is given by $-ab$.

Since the analysis of the escape time is similar to that for the scalar case, we omit the proof. We may conclude that the sensitivity of the escape time from a stable manifold with respect to the initial distance to the manifold is inversely linear.

When an autonomous LTI system has multiple RHP eigenvalues, escape times in each modal direction display a linear inverse sensitivity relationship with the initial condition in that modal direction[1]. This is a simple generalization of the single-unstable-equilibrium case discussed above, so we omit

[1]When the unstable eigenvalues are complex, the escape-time measure must be defined rather carefully, as the time required for the trajectory to escape from a circle of particular radius. Upon definition in this way, the result follows immediately.

the details.

The above characterization of unstable LTI systems can yield inverse-sensitivity characterizations for nonlinear dynamical systems with unstable or saddle-point equilibria, in a local sense. Specifically, through linearization arguments, it follows (under some smoothness criteria) that the inverses of escape time sensitivities are linear with respect to the distance of the initial condition from the stable manifold, as long as the initial condition and escape surface are sufficiently close to the unstable/saddle equilibrium. In turn, we can also argue that recovery time sensitivities to disturbance parameters are inversely linear. We note that this argument is applicable when the return of the trajectory to a nominal (stable) equilibrium is disproportionately governed by capture of the trajectory near an unstable equilibrium; such a circumstance may result, for instance, when the unstable/saddle equilibrium is on the boundary of the domain of attraction of the nominal stable equilibrium, and further the disturbance sets the trajectory close to the unstable equilibrium. The structure of nonlinear systems is in general very complicated, and we will not attempt to characterize the class of systems/disturbances that display linear inverse sensitivity properties in any general way. However, we will formalize the result for one class of planar systems.

Theorem 2: Consider a planar system with a saddle equilibrium (i.e., an equilibrium such that the linearization has one strictly-positive eigenvalue and one strictly-negative one). Notice that the saddle point will have a one-dimensional stable manifold. Consider the escape time of the system from near the saddle: specifically, from a point that is a distance b from the saddle on the perpendicular to the stable manifold, to a distance c from the equilibrium. For sufficiently small b and c, the sensitivity of escape time to b is inversely linear.

The proof of the theorem is very similar to that for the scalar case, and so the details are omitted. We note that, as in the scalar case, the inverse sensitivity with respect to a disturbance parameter rather than the initial condition is also linear under broad conditions when the critical disturbance parameter value places the state near the unstable equilibrium. As in the scalar case (2), the slope of the inverse sensitivity is scaled by the power in the dependence between the disturbance parameter and the initial condition.

Let us conclude our study of inverse sensitivities for multivariate systems with an example.

D. Example: Simple power system

We consider a classical planar model for the swing dynamics of a single machine infinite bus system that is subject to a three-phase fault [2]. It is well known that the *fault-clearing time*, i.e. the time at which the system is reset from faulted conditions to post-fault operation, determines whether or not the system returns to its nominal equilibrium point. In particular, when the fault-clearing time is less than a critical value, the system recovers to the nominal operating point, though more and more slowly as the clearing time approaches the critical value. This example studies the inverse sensitivity of the recovery time with respect to the fault-clearing time.

Specifically, consider the switched differential equation model,

$$\ddot{\delta} = 0.6, \qquad 0 \le t \le T_c \tag{4}$$

$$\ddot{\delta} = -\sin(\delta) - 0.5\dot{\delta} + 0.6, \quad t \ge T_c \tag{5}$$

where the system is initially at the stable equilibrium $\delta(0) = \arcsin(0.6)$, $\dot{\delta}(0) = 0$, and where T_c is a parameter that represents the fault-clearing time in this power system application. It is easy to check that $T_c \approx 1.94$ is a critical value, with the state returning to the equilibrium $(\arcsin(0.6), 0)$ for smaller fault-clearing times and traveling to other equilibria for larger fault-clearing times. We will consider fault-clearing times below the threshold, and study the time required for the system to return to the stable equilibrium. One can verify that this return time is dominated by the time spent near the (stable manifold of the) unstable equilibrium $(\pi - \arcsin(0.6), 0)$, and thus that the escape time from this manifold predicts the return time.

Linearizing the dynamics around the equilibrium $(\pi - \arcsin(0.6), 0)$ gives the unstable eigenvalue $a = 0.68$. Furthermore, it can be shown with a little algebra that the distance of the initial condition (the state after fault-clearing) from the stable manifold roughly scales with $\sqrt{1.94 - T_c}$. Thus, we predict that the inverse sensitivity of the return time to the fault-clearing time should be linear, and the slope of the dependence should be approximately $-\frac{a}{\alpha} = -\frac{0.67}{0.5} = -1.34$.

Simulations indeed reveal an affine dependence, as shown in Figure 2, with the line having a slope of approximately -1.2. The difference between the simulated return time and the predicted one results from several non-idealities, including that 1) the trajectory spends a non-trivial time period away from the equilibrium manifold and 2) the linearization that we have used for prediction is exactly valid only for the equilibrium point and not along the dynamical trajectory followed by the state, even when the trajectory is close to the manifold of interest.

We have also verified that when the dependence of the distance to the unstable equilibrium's stable manifold is linear in the fault-clearing time (due to large damping), then the inverse sensitivity is fairly linear and has a slope of roughly -0.6 (as compared to a theoretical prediction of -0.67).

Based on the intersection of the linear asymptote with the horizontal axis, the critical value of the fault clearing time can be estimated as $T_c = 1.95$. This compares favorably with the value $T_c = 1.94$ obtained by repeated simulation.

III. Applications

Several applications of the affine inverse-sensitivity phenomenon can be envisioned, particularly in the analysis, design, and operation of large-scale infrastructure systems:

1) Determining critical disturbance parameter values is extremely important in analyzing large-scale infrastructure networks. For instance, power engineers require knowledge of maximum allowable fault-clearing times, to permit resolution of fault events without

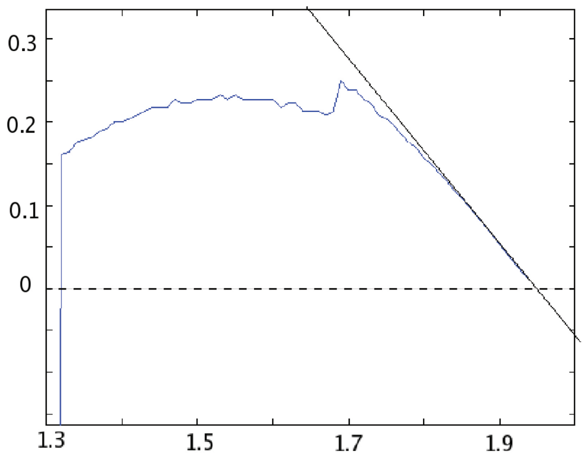

Fig. 2. Inverse sensitivities for a power system example. The x-axis represents the fault-clearing time, while the y-axis is the inverse sensitivity of the response time. The inverse sensitivity has an affine dependence on the disturbance parameter, with slope equal to roughly -1.2 (as indicated by the straight line in the plot).

cascading power failures. However, because the simulation of these large-scale infrastructure systems is computationally expensive, determining critical parameter values through exhaustive search may be difficult, especially in real time. The affineness of inverse sensitivities permits us to easily approximate critical disturbance parameter values from only a few simulations. In particular, we note that the sensitivity is infinite, and so the inverse sensitivity is zero, at the critical parameter value. Thus, we can use sensitivity computations at a few parameter values to obtain an approximate affine inverse-sensitivity relationship, and in turn estimate the zero-inverse-sensitivity (or critical) parameter value.

2) The affineness of inverse sensitivities may in some cases help in characterizing domains of attraction of nominal operating points (equilibria) in large-scale networks. In particular, upon predicting the critical parameter values using the inverse sensitivities, the system response for near-critical parameter values in some cases can be used to characterize the boundary of the nominal equilibrium's domain of attraction.
3) Increasingly, as infrastructure networks become more and more stressed and interconnected, disturbances induce near-critical dynamics. For instance, air traffic disturbances routinely decrease capacity to a point where very slow recovery ensues, while faults in stressed electrical power systems are often cleared just quickly enough. Given this tendency for operation near critical points, the inverse-sensitivity results can help characterize the recovery time of these systems, providing insights that may help prevent cascading disturbances.

IV. Conclusions and Future Work

In this article, we have explored the observation that inverse sensitivities of response characteristics to near-critical disturbance parameter values exhibit an affine dependence. In particular, we have given a non-linear dynamics explanation for the phenomenon, arguing that the dependence stems from the increasing time spent by the trajectory near an unstable equilibrium as the disturbance parameter approaches the critical value. We should acknowledge that the result is an approximate one and is not accurate for all non-linear dynamical systems. Furthermore, even when linear inverse sensitivities are observed, the phenomenon may admit other explanations, for instance queueing-theoretic explanations are appropriate in some air traffic applications [14]. Nevertheless, our preliminary studies suggests that affineness of inverse sensitivities may be a relatively common phenomenon in dynamical systems.

We are pursuing future work in two directions. First, we intend to extend the analysis pursued here to more general classes of nonlinear systems. A particularly promising first step in this direction is to extend the result to the class of nonlinear systems with equilibria on domain-of-attraction boundaries given in [15]. Second, we will pursue application of linearity in inverse sensitivities in larger-scale electric power systems and air traffic management applications.

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