Comparative Analysis and Qualitative Integral Representations*

Charles Chiu†
Benjamin Kuipers‡

December, 1991

Abstract

Comparative analysis is applied to a qualitative behavior of an incompletely known mechanism, to determine the effect of a given perturbation on the behavior as a whole. This class of inference is useful in diagnosis, design, planning, and generally for understanding the relations among a set of alternate qualitative behaviors.

Comparative analysis depends on information which is implicit, and relatively difficult to extract, from qualitative differential equations. By introducing the definite integral as a descriptive term linking qualitative variables and their landmarks, we show that the qualitative *integral* representation (QIR) makes the required information easily accessible.

Inspired by observations of expert physicists, we have adopted an approach to inference that allows global algebraic manipulation of the QIR. Within this approach, comparative analysis can be decomposed into a search and algebraic manipulation problems. Several detailed examples are presented to clarify our method.

---

*This work has taken place in the Qualitative Reasoning Group at the Artificial Intelligence Laboratory, The University of Texas at Austin. Research of the Qualitative Reasoning Group is supported in part by NSF grants IRI-8602665, IRI-8905494, and IRI-8904454, by NASA grants NAG 2-507 and NAG 9-200, and by the Texas Advanced Research Program under grant no. 003658175. An earlier version of this paper was presented at the Third Qualitative Physics Workshop, Stanford, California, August 1989. A slightly shorter version was published in Boi Faltings and Peter Struss (Eds.), *Recent Advances in Qualitative Physics*, MIT Press, 1992.

†Physics Department and AI Laboratory, University of Texas, Austin, TX 78712.

‡Computer Sciences Department, University of Texas, Austin TX 78712.

1 Introduction and Overview

The goal of comparative analysis [Forbus, 1984; Weld, 1987, 1988] is to determine how a perturbation to one aspect of a system affects the behavior of other aspects of the system, particularly when the system is incompletely known and described by a qualitative differential equation (QDE) model.

In terms of the QSIM representation for qualitative structure and behavior [Kuipers, 1986], a predicted behavior is a sequence of qualitatively distinct sets of values for the variables in the QDE. The behavior implies a set of relationships among the *landmark values* of the variables. The goal of comparative analysis is to analyze these relations to determine the direction of effect of a perturbation to the value associated with a landmark p on the value associated with the landmark q:

$$C(p,q) = \text{sign}\left(\frac{\partial p}{\partial q}\right).$$

1.1 Why Do Comparative Analysis?

Pure qualitative simulation predicts the set of possible behaviors consistent with a given state of knowledge of a mechanism and initial state.

$$QSIM \vdash QDE, State(t_0) \rightarrow or(Beh_1, Beh_2 \ldots Beh_n)$$

Comparative analysis provides both local information about how changes to one landmark affect another, and global information about how one behavior is related to another as independent parameters vary. This type of inference is valuable in several important problem-solving contexts.

- **Diagnosis**. Derive testable predictions from perturbation experiments on a model. If a model implies that the sign $C(p,q) = +$, then we can test the model by imposing a change $\uparrow p$ and looking for $\uparrow q$.
- **Design**. Determine the best direction for a hill-climbing search. When refining a design, the sign $C(p,q)$ tells how to adjust p to achieve a desired value of q.
- **Planning**. See Design.
- **Transitions**. Determine possible transitions from one qualitative behavior to another as a parameter varies. This exploits relationships *among* sibling behaviors.

Thus, by using comparative analysis, the possible behaviors of an incompletely known mechanism can be promoted from a simple collection of non-deterministic alternatives, to a family of related behaviors with richer qualitative descriptions.

1.2 Experts Use Equations

Our approach here is "Qualitative Physics" (or "Qualitative Mathematics") rather than "Naive Physics." Our approach is motivated primarily by the need to reason effectively with incomplete knowledge. A qualitative differential equation expresses a state of incomplete knowledge about the structure of a system that can be useful even when traditional numerical or analytical methods are inapplicable to a problem, due to incomplete knowledge of monotonic functional relations or the landmark values in quantity spaces.

Some approaches to qualitative reasoning are motivated by the need to reason with limited processing power. This approach suggests the use of propagation as the primary inference step, due to its locality and efficiency.

However, our observations of problem-solving in physics suggest that inference by propagation alone is too restrictive. Physicists tend to formulate models as differential equations. Once the model has been formulated, an extensive collection of algebraic manipulation techniques is applied to transform the equation into alternate forms to make desired inferences easy, even at the cost of eliminating all similarity between the form of the equations and the original physical situation. Experts in physics are very good at obtaining a qualitative result by applying a combination of global algebraic manipulation rules and qualitative simplification methods.

Therefore, once we have a QDE that expresses a state of knowledge about a physical situation, we allow arbitrary (computable) algebraic or analytic reasoning to extract the strongest possible conclusion from the available knowledge. As discussed by Struss [1988] and Kuipers [1988], the set of qualitative conclusions that can be drawn from a given set of qualitative constraints is changed by the application of truth-preserving algebraic manipulations such as the associative and distributive laws. Kuipers & Chiu [1987] took initial steps in this direction in the algebraic derivation of higher-order derivative constraints; Williams [1988], in his MINIMA program, developed a systematic collection of methods combining algebraic and qualitative methods.

Within the qualitative reasoning community, this approach of applying sophisticated mathematics to extract useful conclusions from incomplete knowledge is shared by (among others): Dormoy [1988], Kuipers [1986], Kuipers & Chiu [1987], Lee & Kuipers [1988], Raiman [1986], Sacks [1987, 1988], Struss [1988], Weld [1987, 1988], Williams [1988].

Our approach appears to handle all the comparative analysis examples discussed in the qualitative reasoning literature. It is a superset of Weld's [1987] rigorous method of comparative analysis, which in turn is a superset of Forbus' [1984] DQ analysis. The relationship with Weld's [1988] heuristic method of exaggeration is not clear. Dimensional analysis has also been used for qualitative reasoning [Kokar, 1986] and comparative analysis [Bhaskar and Nigam, 1988]. However, that method relies on a multiplicative form for universal laws, and has problems distinguishing among multiple variables with the same dimensions and qualitatively different roles in the behavior of the mechanism.

1.3 Overview

In this paper, we reduce the problem of comparative analysis to one of algebraic manipulation and simplification. There are several steps to this reduction:

1. Qualitative *integral* equations are required because they make explicit relations among time-points and time-intervals that are implicit in the equivalent qualitative *differential* equations.

2. Given the goal of determining $C(p, q)$ while fixing a subset $\{r, s, t, \ldots\}$ of the parameters, a search process must find an expression $R(p, q; r, s, t, \ldots)$ that follows from the QDE.

   - Where this is impossible, the goal can be broken into a conjunct of simpler subgoals:
   $$C(p, q) = C(p, r) * C(r, q).$$

3. Given an expression $R(p, q; r, s, t, \ldots)$, take the partial derivative with respect to q and solve for $\partial p/\partial q$. There are several approaches to this differentiation, depending on the properties of the expression R.

4. The resulting expression for the partial derivative $\partial p/\partial q$ can appear awesomely complex, but frequently turns out to be relatively easy to simplify since only its sign is of interest, and not its actual value. This simplification step also requires search of a space of truth-preserving transformations.

5. The result is
$$C(p, q) = \text{sign}\left(\frac{\partial p}{\partial q}\right).$$

The research reported in this paper is a theoretical analysis, which reduces the problem of comparative analysis to well-specified search and algebraic manipulation problems. Future work includes implementation of these methods.

2 The Qualitative Integral Representation

Integral equations express the same information as differential equations, but make explicit more of the objects that appear in the behavior. A qualitative differential equation might include the derivative constraint,

$$rate(t) = \frac{d}{dt} amount(t),$$

which the qualitative simulator can use to predict behaviors at time-points t_0, t_1, etc. A corresponding integral equation includes explicit reference to time-points and landmark values of variables:

$$amount(t_1) = amount(t_0) + \int_{t_0}^{t_1} rate(t)\ dt.$$

In fact, this is just a restatement of the Fundamental Theorem of the Calculus. Since we are doing qualitative reasoning, and our knowledge about the function $rate(t)$ may include partially specified monotonic functions, we cannot use the familiar inferential methods of the calculus to evaluate an integral. On the other hand, we can use the definite integral as a descriptive term, and many of the familiar theorems about definite integrals will be useful axioms for our qualitative reasoning method. See section 5 on Simplification and Evaluation.

Note that although the above example involves integration over t, we will have occasion to integrate with respect to other variables in the QDE.

The definite integral provides a new type of constraint, relating time-points and landmark values explicitly with the variables for rate and amount. Thus, quantity spaces (including time) are extended from purely ordinal spaces to ones where it is possible to reason qualitatively about the distances between landmarks.

3 Deriving the Qualitative Integral Representation

To obtain the relation $C(p, q)$, derive an integral equation $p = I(q)$. Note that, from a QDE including n independent variables, there will be $n(n-1)/2$ independent QIRs, differing according to which variables are allowed to vary, while the others are held constant. The following methods allow us to derive a QIR from a QDE.

- For a QDE with following factorizable form:

  $$\frac{\partial p}{\partial q} = f(p)g(q),$$

  the corresponding QIR is given by:

  $$\int_{p_1}^{p_2} \frac{dp}{f(p)} = \int_{q_1}^{q_2} g(q)dq$$

- For a conservative system, where QDEs are:

$$\frac{dx}{dt} = y, \quad and \quad \frac{dy}{dt} = F(x),$$

  following relation must be satisfied:

$$y_2^2 - y_1^2 = 2\int_{x_1}^{x_2} F(x)dx$$

  This implies that for fixed initial values: x_1 and y_1, $y_2(= y)$ is a function of $x_2(= x)$ only. So the QDE: $y = dx/dt$, leads to following QIR:

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{y}$$

- We also use the chain rule, to break a problem into tractable subcases:

$$\frac{\partial p}{\partial q} = \frac{\partial p}{\partial r}\frac{\partial r}{\partial q} \tag{1}$$

So far we have restricted our considerations to the general forms given above.

4 Taking the Partial Derivative

Suppose we have found a way to express p as an integral expression involving q,

$$p = I(q) = \int_a^b G(x, q)dx. \tag{2}$$

There are several ways of exploiting this integral equation to derive an expression for $\partial p/\partial q$.

4.1 The Easier Case

If things go well — i.e. we can integrate $\partial G(x, q)/\partial q$ — we can use the identity:

$$\frac{\partial p}{\partial q} = \frac{\partial b}{\partial q}G(b, q) - \frac{\partial a}{\partial q}G(a, q) + \int_a^b \frac{\partial G(x, q)}{\partial q}dx. \tag{3}$$

4.2 The Not-So-Easy Case

If we are not so fortunate, we can still reason with the more complex equation

$$\frac{\partial p}{\partial q} = \lim_{dq \to 0} \frac{I(q + dq) - I(q)}{dq}. \tag{4}$$

5 Simplification and Evaluation

The partial derivative expressions look awesomely complex, especially when one appreciates that $G(x, q)$ and $I(q)$ may themselves be substantial expressions. However, it turns out that the sign of $\partial p / \partial q$ can (at least sometimes) be evaluated with surprising ease, by applying a set of qualitative integral simplification rules.

5.1 Definitions

Notation:

Suppose $I = \int_a^b F(x)dx$, where a and b are finite.

- $\{F\}$ = "global sign" of $F(x)$, where consistent over (a, b).
- $[I] = sign\ I = \{F\}$

Qualitative integral rules

- If $F(x)$ has a definite sign for $x \in (a, b)$ (except for isolated zeros) then $[I] = \{F\} = -,\ 0,\ or\ +$.
- If $F(x)$ has both $+$ and $-$ signs for $x \in (a, b)$, then $[I] = \{F\} = nil$.

5.2 Qualitative Simplification Rules

1. **Removal of positive multiplicatives**

   For $g > 0$, except for isolated zeros:

   $$[I] = \{fg\} = \{f\}.$$

2. **Elimination of common denominators**

   For f_1 and f_2 arbitrary functions and $\{g_1, g_2\} > 0$,

   $$[I] = \{\frac{f_1}{g_1} - \frac{f_2}{g_2}\} = \{f_1 g_2 - f_2 g_1\}$$

3. **Linearization of differences**

   For $f, g > 0$ and monotonic function h,

   $$[I] = \{h(f) - h(g)\} = \{f - g\}$$

   A useful specific case, when $a > 0$, is:

   $$[I] = \{f^a - g^a\} = \{f - g\}.$$

5.3 Ordinary Algebra With Integrals

1. **Partial derivative rules:**
   - Differential definition rule
   - Integral derivative rule (for regular case)
   - Chain rule

2. **Change limits of integrals:**
   - Displacement rule:

$$\int_a^b f(x)dx = \int_0^{b-a} f(x+a)dx$$

   - Scaling rule:

$$\int_0^c f(x)dx = \int_0^1 f(cx) \cdot c\ dx$$

3. **Combining two integrals:**

$$\int_a^b f dx + \int_a^b g dx = \int_a^b (f+g)dx$$

   To combine two integrals with different limits, use rule 2 to convert to a common range, e.g. 0 to 1.

4. **Combining ranges:** If $t_1 < t_2 < t_3$, then

$$\int_{t_1}^{t_2} f(t)\ dt + \int_{t_2}^{t_3} f(t)\ dt = \int_{t_1}^{t_3} f(t)\ dt.$$

6 Examples

We begin with a simple example of "filling a tank" to illustrate the basic steps involved in equation-based comparative analysis. We then present a sequence of three problems based on the same scenario of one-dimensional motion, demonstrating the complete set of three QIRs derivable from the scenario.

6.1 Problem 1.

"If water pours into the tank faster, it will take less time to fill".

6.1.1 From QDE to QIR

Consider water pouring into a closed tank. The water level is y, and the filling rate is $f(t,c)$, where c is a parameter which controls the rate. For fixed t, $f(t,c) = M^+(c)$; i.e., $\partial f(t,c)/\partial c > 0$. The QDE for this model is:

$$\frac{dy}{dt} = f(t,c).$$

There are two variables, y and t, so one independent QIR. From t_1, y_1 to t_2, y_2,

$$y_2 - y_1 = \int_{t_1}^{t_2} f(t,c)\, dt \tag{5}$$

6.1.2 Define correlation problem

Our goal is to show that $\partial T/\partial c < 0$ for a scenario filling some fixed volume. Instantiate the general QIR to represent the scenario: fill the tank from time $t_1 = 0$ at arbitrary height y_1, to time $t_2 = T$ at the top h.

$$h - y_1 = \int_0^T f(t,c)\, dt \tag{6}$$

6.1.3 Evaluate correlation

Based on (3), take the partial derivative of (6) with respect to c, keeping y_1 and h fixed.

$$0 = \int_0^T \frac{\partial f(t,c)}{\partial c}\, dt + \frac{\partial T}{\partial c} f(T,c)$$

Evaluate signs. Since $\partial f(t,c)/\partial c > 0$,

$$0 = (+) \ + \ \frac{\partial T}{\partial c}(+), \ \ or \ \ \frac{\partial T}{\partial c} < 0$$

.

That is, $C(T,c) = -$, or, the faster the inflow rate, the shorter the time to fill.

6.2 Generic Motion in One Dimension

We provide a complete set of QIRs for a generic problem of motion in one dimension. Let motion be along y. The velocity and acceleration are:

$$v = \frac{dy}{dt} \qquad a = \frac{dv}{dt} \qquad a = -f(y,k)$$

In these equations, k is an exogenous parameter, and a is determined by a parametric function f which can depend on both y and k.

- **ODEs:** f has an explicit algebraic form.
  - $f = 1$. Constant gravity.
  - $f = r^{-2}$. Decreasing gravity.
  - $f = y$. Simple spring.
  - $f = \theta$. Pendulum: small amplitude approximation.
  - $f = \sin\theta$. Pendulum: arbitrary amplitude.
  - $f = s^a$. Rolling on a concave surface, $a > 0$.
- **QDEs:** $f(y, k)$ is a partially specified function, depending on both the variable y and the constant parameter k. We may know facts like the following about f.
  - For fixed k, $f(y, k) = M^+(y)$, increasing monotonic function of y.
  - For fixed k, $f(y, k) = M^-(y)$, decreasing monotonic function of y.
  - For fixed y, $f(y, k) = M^+(k)$, increasing monotonic function of k.
  - For fixed y, $f(y, k) = M^-(k)$, decreasing monotonic function of k.

The variables t, y, and v describe the motion. For a motion from t_1, y_1, v_1 to t_2, y_2, v_2, there are three sets of QIRs.

- The v-y relation. Since $a = dv/dt$, or $v\ dv = a\ v\ dt = a\ dy$, we get

$$v_2^2 - v_1^2 = -2 \int_{y_1}^{y_2} f(y, k)\ dy \tag{7}$$

- The t-y relation. From $v = dy/dt$, we get

$$t_2 - t_1 = \int_{y_1}^{y_2} \frac{dy}{v} \tag{8}$$

  where the dependence of v on y can be obtained from (7).

- The t-v relation. From $a = dv/dt$ and $a = -f(y, k)$, we get

$$t_2 - t_1 = -\int_{v_1}^{v_2} \frac{dv}{f(y, k)} \tag{9}$$

  where the dependence of y (in the argument of f) on v can also be obtained from (7).

There are also three equivalent QIRs which express the inverse relations. For example, the y-v relation expresses y_1 and y_2 in terms of v_1 and v_2. We will not display these relations here.

It can be shown that the QIR (7) amounts to a statement of the conservation of energy. Thus, within the present approach, it is not necessary to express energy conservation as an added constraint in the model.

Below, we present three examples of projectile motion to illustrate our approach. We choose the problem of the rising ball, where gravitational acceleration decreases with height. Here, Weld [1988] encountered the limits of his Differential Qualitative Analysis. Other examples, including a spring system, and more details of the application of the rules, are given in Chiu [1989].

As we step through the search process, note that the branching factor and depth of search are quite small, even though the algebraic manipulation involved in each individual step may appear quite forbidding.

6.3 Problem 2.

"If you throw the ball faster, it will go higher."

For the rising ball, show $\partial h/\partial v_i > 0$, where v_i is the initial velocity, h is the maximum height of the ball, g is the gravitational constant for the current planet, and $f(y, g)$ is the force of gravity, decreasing with height, and increasing with g.

6.3.1 From QDEs to QIRs

Consider the generic QIRs (7), (8), and (9) in section 6.2, replacing the exogenous parameter k by g, and asserting that

$$f(y, g) > 0, \quad \frac{\partial f}{\partial y} < 0 \text{ and } \frac{\partial f}{\partial g} > 0. \tag{10}$$

6.3.2 Define correlation problem

Instantiate the generic equations with information about the scenario in question, searching for a relationship that can demonstrate $\partial h/\partial v_i > 0$. This scenario goes from $t_1 = 0$, $y_1 = 0$, $v_1 = v_i$ to $t_2 = t_{rise}$, $y_2 = h$, $v_2 = 0$.

- Try the t-y relation. Instantiating QIR (8) gives

$$t_{rise} = \int_0^h \frac{dy}{v} \tag{11}$$

Evaluate the dependence of v on y based on (7). From $y_1 = y,\ v_1 = v$ *to* $y_2 = h,\ v_2 = 0$, obtain

$$v^2 = -2\int_y^h f(y,g)dy,\ \textit{i.e.}\ \ v = v(y,\ h;\ g) \tag{12}$$

Substituting (12) into (11) gives the final instantiated form:

$$t_{rise} = \int_0^h \frac{dy}{v(y,\ h;\ g)},\ \textit{i.e.}\ t_{rise} = t_{rise}(h; g). \tag{13}$$

This result does not represent progress toward our current goal.

- Try the v-y relation. Instantiating QIR (7) from $y_1 = 0,\ v_1 = v_i$ to $y_2 = h$, $v_2 = 0$, gives

$$v_i^2 = 2\int_0^h f(y,g)\ dy,\ \textit{i.e.}\ v_i = v_i(h; g) \tag{14}$$

Since our goal is to determine $\partial h/\partial v_i$, for fixed g, equation (14) represents exactly what we have been looking for: a dependency between h and v_i, with no other variables (g, of course, is a constant).

6.3.3 Evaluate correlation

Take the partial derivative of (14) with respect to v_i,

$$2v_i = 2\frac{\partial h}{\partial v_i} f(h,g)$$

Evaluate signs:

$$(+) = \frac{\partial h}{\partial v_i}(+),\quad \textit{so}\quad \frac{\partial h}{\partial v_i} > 0. \tag{15}$$

That is, the faster you throw the ball, the higher it goes.

6.4 Problem 3:

"If you throw the ball faster, it will rise longer."

For the rising ball, where gravity decreases with height y, to find $\partial t_{rise}/\partial v_i$, for fixed g.

6.4.1 From QDEs to QIRs

All equations here are identical to those of problem 2.

6.4.2 Define correlation problem

First, try the previously instantiated QIRs: equations (13) and (14). These do not solve the problem directly, but we can achieve the desired relationship by using the chain rule:

$$\frac{\partial t_{rise}}{\partial v_i} = \frac{\partial t_{rise}}{\partial h}\frac{\partial h}{\partial v_i}.$$

The sign of $\partial h/\partial v_i$ is already known (equation (15)). Proceed to determine $\partial t_{rise}/\partial h$ from equation (13).

6.4.3 Evaluate correlation

- We repeat equation (13) here:

$$t_{rise} = \int_0^h \frac{dy}{v(y,\ h,\ g)}$$

  Using the identity (3), we take the partial of this equation with respect to h, to get

$$\frac{\partial t_{rise}}{\partial h} = \frac{1}{v(h, h, g)} + ... = \frac{1}{0} + ...,$$

  Since $v(h, h; g) = 0$ (see equation (12)), we get a division by zero, so this approach fails.

- Alternative strategy: evaluate

$$\frac{\partial t_{rise}}{\partial h} = \lim_{dh \to 0} \frac{t_{rise}(h + dh) - t_{rise}(h)}{dh}$$

  – Through the set of qualitative integral simplification rules given in section 5, it can be shown [Chiu, 1989]:

$$sign\left(\frac{\partial t_{rise}}{\partial h}\right) = sign\left(\frac{f(y,g)}{y} - \frac{\partial f(y,g)}{\partial y}\right),\ for\ 0 < y < h$$

  .

  – Since $f(y, g)/y > 0$ and $\partial f(y, g)/\partial y < 0$, we know that $\partial t_{rise}/\partial h > 0$.

  – Combining this with $\partial h/\partial v_i > 0$ from equation (15), we get

$$sign\frac{\partial t_{rise}}{\partial v_i} = sign\left(\frac{\partial t_{rise}}{\partial h} \cdot \frac{\partial h}{\partial v_i}\right) = (+)(+) = +.$$

That is, the harder you throw the ball, the longer it continues to rise.

6.5 Problem 4:

"The stronger the gravity, the quicker the ball reaches its peak."

For the rising ball, where gravity decreases with height y, find $\partial t_{rise}/\partial g$, for fixed v_i.

6.5.1 From QDEs to QIRs

All equations here are identical to those of problem 2.

6.5.2 Define correlation problem

The already instantiated t-y relation (13) and v-y relation (14) are not useful. Proceed to obtain the QIR representing the t-v relation by instantiating equation (9).

$$t_{rise} = -\int_{v_i}^{0} \frac{dv}{f(y,g)} \tag{16}$$

where the dependence of y on v is given by equation (7), with the substitutions $v_2 = v$, $y_2 = y$, and $k = g$, i.e.

$$v^2 - v_i^2 = -2\int_0^y f(y',g)\,dy', \text{ or } y = y(v, v_i; g). \tag{17}$$

Substituting (17) into (16) gives

$$t_{rise} = -\int_{v_i}^{0} \frac{dv}{f(y(v,v_i,g),g)}, \;\; \textit{i.e.}\; t_{rise} = t_{rise}(v_i, g) \tag{18}$$

This is exactly the relation we need to determine $\partial t_{rise}/\partial g$ for fixed v_i.

6.5.3 Evaluate correlation

- Take the partial derivative of (18) with respect to g, using the identity (3).

$$\frac{\partial t_{rise}}{\partial g} = \int_0^{v_i} dv\, \frac{\partial}{\partial g}\frac{1}{f} \tag{19}$$

- Apply the chain rule (1)

$$\begin{aligned} \frac{\partial}{\partial g}\frac{1}{f(y(v,v_i,g),g)} &= \left(\frac{\partial}{\partial f}\frac{1}{f}\right)\left[\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial g} + \frac{\partial f}{\partial g}\right] &(20)\\ &= -\frac{1}{f^2}\left[\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial g} + \frac{\partial f}{\partial g}\right] &(21)\end{aligned}$$

Recall that we know that $\partial f/\partial y < 0$ and $\partial f/\partial g > 0$ (equation (10)).

- Determine $\partial y / \partial g$.

  Take the partial derivative of (17) with respect to g, for fixed v and v_i, using the identity (3).

$$\begin{aligned} 0 &= -2\frac{\partial y}{\partial g} f(y,g) - 2\int_0^y \frac{\partial f(y',g)}{\partial g} dy' && (22) \\ &= (-)\frac{\partial y}{\partial g}(+) - (+)\int_0^y (+) dy' && (23) \\ &= (-)\frac{\partial y}{\partial g} - (+) && (24) \\ \text{i.e.} \quad & \frac{\partial y}{\partial g} < 0 && (25) \end{aligned}$$

- Substituting (25) and (10) into (21) gives

$$\frac{\partial}{\partial g}\frac{1}{f} = -(+)[(-)(-) + (+)] < 0 \tag{26}$$

- Finally, substituting (26) into (19),

$$\frac{\partial t_{rise}}{\partial g} = \int_0^{v_i} (-)\ dv < 0 \tag{27}$$

That is, as gravity increases, it takes less time for the ball to reach its peak.

7 Conclusion

We have demonstrated a powerful method for solving comparative analysis problems by recasting them from qualitative differential equations to the qualitative integral representation, where the solution can be found by a relatively straight-forward and well-constrained search process. The individual steps of the search involve substantial algebraic and qualitative manipulation of the integral representation. However, our experience with expert physicists suggests that this is an important component of the "expert common sense" that physicists bring to the task of reasoning qualitatively about physical systems.

8 References

- R. Bhaskar and Anil Nigam. Qualitative physics using dimensional analysis. Manuscript dated 9-25-88.
- Charles Chiu. 1989. Intuitive reasoning in physics — from an expert's point of view. *Journal of the Japanese Society for AI* **4**: 529-536.
- Jean-Luc Dormoy. 1988. Controlling qualitative resolution. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-88)*. Los Altos, CA: Morgan Kaufman.
- K. D. Forbus. Qualitative process theory. *Artificial Intelligence* **24**: 85 - 168, (1984).
- M. M. Kokar. 1987. Critical hypersurfaces and the quantity space. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-87)*. Los Altos, CA: Morgan Kaufman.
- B. J. Kuipers. 1986. Qualitative simulation. *Artificial Intelligence* **29**: 289 - 338.
- B. Kuipers. 1988. The qualitative calculus is sound but incomplete: a reply to Peter Struss. *Int. J. Artificial Intelligence in Engineering* **3**(3): 170-173.
- B. Kuipers and C. Chiu. 1987. Taming intractible branching in qualitative simulation. *Proceedings of the Tenth International Joint Conference on Artificial Intelligence (IJCAI-87)*. Los Altos, CA: Morgan Kaufman.
- W. W. Lee and B. Kuipers. 1988. Non-intersection of trajectories in qualitative phase space: a global constraint for qualitative simulation. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-88)*. Los Altos, CA: Morgan Kaufman.
- Olivier Raiman. 1986. Order of magnitude reasoning. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-86)*. Los Altos, CA: Morgan Kaufman.
- Elisha Sacks. 1987. Hierarchical reasoning about inequalities. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-87)*. Los Altos, CA: William Kaufman.
- Elisha Sacks. 1988. Automatic Qualitative Analysis of Ordinary Differential Equations Using Piecewise Linear Approximations. PhD thesis, Department

of Electrical Engineering and Computer Science, Massachusetts Institute of Technology.

- Peter Struss. 1988. Global filters for qualitative behaviors. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-88).* Los Altos, CA: Morgan Kaufman.

- Peter Struss. 1988. Mathematical aspects of qualitative reasoning *Int. J. Artificial Intelligence in Engineering* **3**(3): 156-169.

- Daniel S. Weld. 1987. Comparative Analysis. *Proceedings of the Tenth International Joint Conference on Artificial Intelligence (IJCAI-87).* Los Altos, CA: Morgan Kaufman.

- D. S. Weld. 1988. Choices for comparative analysis: DQ analysis or exaggeration? *Artificial Intelligence in Engineering* **3**: 174-180.

- B. C. Williams. 1988. MINIMA: A symbolic approach to qualitative algebraic reasoning. In *Proceedings of the National Conference on Artificial Intelligence (AAAI-88).* Los Altos, CA: Morgan Kaufman.