

Constrained Sequential Resource Allocation and Guessing Games

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Abstract—In this paper, we consider a constrained sequential resource allocation problem where an individual needs to accomplish a task by repeatedly guessing/investing a sufficient level of effort/input. If the investment falls short of a minimum required level that is unknown to the individual, she fails; with each unsuccessful attempt, the individual then increases the input and tries again until she succeeds. The objective is to complete the task with as little resources/cost as possible subject to a delay constraint. The optimal strategy lies in the proper balance between 1) selecting a level (far) below the minimum required and therefore having to try again, thus wasting resources, and 2) selecting a level (far) above the minimum required, and therefore, overshooting and wasting resources. A number of motivating applications arising from communication networks are provided. Assuming that the individual has no knowledge on the distribution of the minimum effort required to complete the task, we adopt a worst-case cost measure and a worst-case delay measure to formulate the above constrained optimization problem. We derive a class of optimal strategies, shown to be randomized, and obtain their performance as a function of the constraint.

Index Terms—Competitive analysis, constrained optimization, data query and search, exponential functions, minimax problems, online algorithms, randomized algorithms, stochastic analysis.

I. INTRODUCTION

IN this paper, we consider a sequential resource allocation problem where an individual needs to accomplish a task by repeatedly guessing/investing a sufficient level of effort/input. If the investment falls short of a minimum required level, which is unknown to the individual and which may be random, she fails; with each unsuccessful attempt, the individual then increases the input and tries again until she succeeds. In general, the more

resources she commits, the more likely she is to succeed. In deciding how much resources to commit in each attempt, the individual needs to balance between consuming minimal resources and increasing her chances of success. This is because if she selects a level (far) below the minimum required, she would have to try again, thus wasting resources; on the other hand, if she selects a level (far) above the minimum required, she overshoots and ends up wasting resources.

The above problem is motivated by a number of applications in communications and networks. The first is that of power control, where in order to find a nearby radio receiver, a transmitter gradually increases its transmission power until the signal is successfully received/decoded by the receiver [1]. The transmitter does not know *a priori* the minimum power needed to reach the receiver. Each failed transmission wastes resources, while transmission using more power than necessary is also a waste. Therefore, it has to choose the sequence of transmission powers carefully to balance these two factors. A second example is that of searching for a node (or a piece of data) in a wireless network using flooding techniques [2], [3]. In this application, the source node looking for the target broadcasts a query packet and prespecifies how many times it will be relayed (which determines how far it travels within the network). If the target is found within that range, then the search is complete; otherwise, the source node times out and has to broadcast a second query specifying a bigger range. This continues until the target is found. Each broadcast incurs a certain amount of transmissions and receptions, thus consuming energy. A third example is that of reliable point-to-point communication using automatic retransmission request (ARQ) [4], where a sender strengthens the encoding in a packet (in the form of increased redundancy or number of bits sent, e.g., using parity check code [1]) with each failed reception. More motivating scenarios may be found in other areas besides communications. For instance, in performing a certain computation task, one may not know ahead of time the correct step size to use to achieve a desired accuracy. Setting the step size too large may cause the task to fail while setting it too small may cause the computation to run for an excessively long time.

Note that in most, if not all, of the examples mentioned above, the resources are committed *in advance* regardless of the outcome of the attempt. Also note that in many cases, resources expended in a failed attempt are completely wasted as the next attempt has to start from scratch. In other cases, we may be able to resume/continue from the point of failure and commit *additional* resources for the next attempt rather than starting anew. A final note is that for most of these applications, an increase in resources committed is accompanied by an increase in the time it takes to process the task. For example, this is reflected in

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the increased size of the network to be searched in the flooding application, and in the increased packet size in the ARQ application.

In this paper, we formulate the above problem as a constrained sequential resource allocation problem. The successive amount of resources committed by the individual form a *strategy*, which determines the total cost paid and time expended in accomplishing the task. The primary goal of this paper is to derive strategies that minimize a cost measure subject to a delay constraint. Specifically, assuming that the individual has no knowledge of the minimum resource required to complete the task (either its distribution or its realization), we adopt a worst-case cost measure and a worst-case delay measure in the constrained optimization problem. These worst-case measures are in the form of competitive ratios against an offline adversary. Thus, this sequential resource allocation problem also takes on the form of a guessing game against the adversary. The solution to this problem results in strategies that minimize the worst-case cost and successfully complete the task within a specified worst-case delay constraint. We will show that the dual problem of minimizing a delay measure subject to a cost constraint is also solved by this framework.

Imposing a delay (or cost) constraint allows us to study the tradeoff between cost and delay. This tradeoff can be seen by considering the strategy of selecting the maximum amount of resource possible. Such a strategy would most likely result in a short delay, as the task is likely to be completed during the first attempt. On the other hand, this strategy is likely not the most cost effective [2], [3]. The unconstrained version becomes a special case of this formulation.

We analyze the above problem for two cases. In the first case, problem P1, one must begin the task from scratch after every failed attempt. In the second case, problem P2, one is allowed to continue from previous failed attempt. We first analyze P1 and then use a similar approach to derive optimal strategies for P2.

It is worth noting the difference between the competitive analysis-based [5] approach used in this paper and the commonly used stochastic optimization-based [6] approach. Under the competitive analysis approach, no *a priori* knowledge is assumed on the random processes underlying the system, and the objective is to obtain the best worst-case performance over all possible distributions. This is, in general, a pessimistic approach; however, using this method, one can provide a performance guarantee with respect to the best strategy (the adversary). Consequently, the resulting solution is generally more robust to imperfect (or perturbation in) knowledge of the probability distribution describing the system model. By contrast, the stochastic optimization-based approach assumes *a priori* knowledge on the random distributions underlying the system model. In this case, the objective is to determine the strategy that minimizes the average cost over all strategies given that assumption. An important thing to note is that strategies obtained using this approach are optimal only with respect to the assumed *a priori* stochastic description driving the system model, and they are typically highly sensitive to changes in such assumptions. In this sense, while these strategies may be optimal, they are, in general, not very robust. Thus, one can view the two approaches as

complementary to each other; the preferred method to analyze a problem may depend on the specific problem scenario and how much information is known regarding the systems distribution.

The works most closely related to our mathematical abstraction in this paper are [7] and [8]. While these two papers also use a competitive-analysis-based approach, there are some key differences. In particular, both [7] and [8] consider a variant of the sequential resource allocation problem of this paper where the user can choose from among multiple tasks to attempt, but the costs are *not* paid in advance for each attempt. Thus, the cost is determined by whether there is success, whereas in our model the cost depends on the resource level committed before the attempt. Thus, the analysis of [7] and [8] does not apply to the problem we consider here. Additionally, both these papers consider a specific cost function, i.e., the cost of an input level (IL) is equal to the IL. Our work, on the hand, is derived for a more general class of cost and delay functions to be precisely defined in Section II. Furthermore, [7] and [8] consider an unconstrained problem where only a single performance measure is adopted, i.e., minimizing the cost. By contrast, in this paper, we seek to derive optimal strategies for a constrained optimization problem where one must balance between two competing performance measures, cost and delay. To the best of our knowledge, our work represents one of the first studies that use competitive analysis for a *constrained* optimization problem. Thus, the proof techniques we use in this paper could potentially serve as a framework for the analysis of other constrained worst-case optimization problems. Due to the above differences, the optimal strategies of [7] and [8] are not optimal for our problem. As will be seen, we derive a class of optimal strategies, which can be tuned with respect to the delay (cost) constraint.

The main contributions of this paper are summarized as follows.

- 1) We provide an analytical framework within which the delay and cost of strategies can be studied for P1 and P2. An unconstrained version (simply minimizing a cost measure) becomes a special case under this framework when the delay constraint is not binding.
- 2) When a worst-case delay constraint is imposed for P1 and P2, we derive a class of optimal strategies that minimize a worst-case cost measure among all strategies that satisfy the delay constraint. We show that randomized strategies outperform deterministic strategies when a worst-case delay constraint is imposed for both problems. Similarly, when a worst-case cost constraint is imposed, we derive the class of optimal strategies that minimize worst-case delay measure among all strategies satisfying the cost constraint.
- 3) For both problems P1 and P2, we establish an understanding of the tradeoff between delay constraints and corresponding optimal achievable cost and show specifically how the two conflicting objectives can affect each other.

The remainder of this paper is organized as follows. The first part of the paper, Sections II–IV, focuses on problem P1. In Section II, we describe the model and associated assumptions. The formulation and main results are presented in Section III. Optimal strategies satisfying a delay constraint for this problem are derived in Section IV. We then formulate and solve problem

P2 in Section V. Results on both problems are discussed in Section VI. Section VII concludes this paper.

II. MODEL AND ASSUMPTIONS FOR PROBLEM P1

In this section, we outline the model and assumptions we will use. Motivated by the applications described in the previous section, we make the following assumptions.

- 1) There is a minimum level of resources/effort X required to accomplish the task. X is a random variable (r.v.) whose distribution may be unknown to the individual. Its realization is not known in advance.
- 2) The individual may choose from a range of resources/effort levels she is willing to put in the task. If the individual chooses a level $v \geq X(\omega)$, where $X(\omega)$ denotes a realization of X , then she succeeds, in which case the process terminates. Otherwise, if $v < X(\omega)$, the task fails.
- 3) Whenever the task fails, she must attempt the task from scratch by using a higher resources/effort level to increase her success probability. Thus, the process occurs in *rounds* until the task is completed successfully.
- 4) When a level v is chosen, the individual commits in advance to paying a cost $C(v)$, where $C(\cdot)$ is a function whose properties will be described in more detail later. Note that this cost only depends on v and not on whether she succeeds.
- 5) At the same time, with a level v , the task takes a certain amount of time to process (either with a success or a failure outcome). This *delay* is given by $D_t(v)$ when there is a failure, and $D_s(X)$ when there is a success. Both $D_t(\cdot)$ and $D_s(\cdot)$ are functions to be described later.

We will use the term IL to describe the amount of effort/resources that the individual applies to the task. Depending on the particular problem of interest, the IL can have different meanings. For example, it can denote how many times the query packet should be forwarded in the search problem, while in the power control problem, it may denote one of a finite number of power levels the transmitter can use. The IL has two main effects on the problem: it determines the probability of successful completion and the amount of resources (e.g., cost or delay) incurred.

We let L denote the maximum permissible IL. We will assume that by applying an IL of L , the user can complete the task with probability 1. We will use the term *task level*, denoted by random variable X , to indicate the minimum IL required to successfully complete the task. The tail distribution of the random variable X is denoted by $\bar{F}_X(u) = P(X > u)$.

In some scenarios, it is natural to only consider integer-valued (*discrete*) policies. This corresponds to the scenario where only a countable number of ILs are available to the user. In this case, a strategy \mathbf{u} is a sequence of IL values of certain length N , $\mathbf{u} = [u_1, u_2, \dots, u_N]$. It can be either fixed/deterministic where $u_i, i = 1, \dots, N$, are deterministic values, or random where u_i are drawn from probability distributions. For a fixed strategy, we assume that \mathbf{u} is an increasing sequence. For randomized strategies, we assume all realizations are increasing sequences. The requirement for the sequence to be increasing is a natural one under the assumption that using IL u will always complete the task if $X(\omega) \leq u$. Note that in a specific experiment we may

not need to use the entire sequence; the process stops whenever the task is completed. When considering discrete strategies, IL values are integers and the task level X is assumed to be a positive integer taking values between 1 and L .

We will also consider the case where effort levels can take an uncountable number of values and are described using real numbers. It will be seen that considering real-valued sequences also proves to be helpful in deriving optimal integer-valued strategies. We refer to strategies for this case as *continuous* (real-valued) strategies, denoted by \mathbf{v} , where $\mathbf{v} = [v_1, v_2, \dots, v_N]$, and v_i is either a fixed or continuous r.v. that takes real values. In analyzing continuous strategies, X is assumed to be a real number in the interval $[1, L]$.

A strategy is *admissible* if it completes the task with probability 1. For a fixed strategy, this implies $u_N = L$. For a random strategy, this implies $Pr(u_i = L \text{ for some } 1 \leq i \leq N) = 1$. In the asymptotic case as $L \rightarrow \infty$, a strategy \mathbf{u} is admissible if $\forall x \geq 1, Pr(u_n \geq x \text{ for some } n \in \mathbb{Z}^+) = 1$. This implies that in the asymptotic case, \mathbf{u} is an infinite-length vector. We let V and U denote the set of all real-valued and integer-valued admissible strategies, respectively. These sets include all random or fixed strategies.

We associate a cost $C(u)$ with using IL value u , i.e., the individual's cost in a single round is only dependent on the amount of effort she applies and not whether her attempt is successful. Meanwhile, we associate a delay $D_t(u)$ with using IL value u if the user fails, and a delay of $D_s(X)$ if she is successful. This assumption is made for cases where, for example, the processing ends more quickly in a round if it is successfully completed. We will assume that these two functions are proportional to each other, i.e., $D_t(\cdot) = \eta D_s(\cdot)$ for some constant $\eta > 0$. We will show in Section III-A that there is no loss of generality in assuming that $\eta = 1$.

For real-valued sequences, we require that the functions $C(v)$, $D_t(v)$, and $D_s(v)$ be defined for all $v \in [1, \infty)$, while for integer-valued sequences, we only require that the cost and delay functions be defined for positive integers. When the cost function is invertible, we write $C^{-1}(\cdot)$ to denote its inverse. We will adopt the assumption that if $v_1 > v_2$, then $C(v_1) > C(v_2)$, $D_t(v_1) > D_t(v_2)$, and $D_s(v_1) > D_s(v_2)$. This assumption implies that a higher IL correlates with more resource consumption and may require more processing time.

We define the following class of cost functions for real-valued sequences.

Definition 1: The function $C : [1, \infty) \rightarrow [C(1), \infty)$ belongs to the class \mathbb{C} if $0 < C(1) < \infty$, $C(v)$ is strictly increasing and differentiable (hence continuous), and $\lim_{v \rightarrow \infty} C(v) = \infty$. Note that for every $y \in [C(1), \infty)$, there exists exactly one $v \in [1, \infty)$ such that $C(v) = y$.

In Section III, we provide an explanation for only considering strictly increasing cost functions.

When considering discrete strategies, we will restrict our results to the following subclass of \mathbb{C} .

Definition 2: A function $C(\cdot) \in \mathbb{C}$ belongs to the class \mathbb{C}_q for some $q \geq 1$ if: 1) $\lim_{x \rightarrow \infty} \frac{C(x+1)}{C(x)} = q$ and 2) $\frac{C(x+1)}{C(x)} \geq q$ for all $x \in [1, \infty)$.

Note that because $C(\cdot)$ is strictly increasing, when $q = 1$ condition 2) is automatically satisfied. The case $q = 1$ also contains all polynomial cost functions. The case $q > 1$ includes, for example, exponential cost functions of the form q^x .

III. PROBLEM FORMULATION AND MAIN RESULTS ON P1

A. Problem Formulation

We will consider the performance in the asymptotic regime as $L \rightarrow \infty$. This is because it is difficult, if at all possible, to obtain a general strategy that is optimal for all problems with finite L , as the optimal IL sequence often depends on the specific value of L . In this sense, an asymptotically optimal strategy may provide much more insight into the intrinsic structure of the problem. It will become evident that asymptotically optimal IL sequences also perform very well for problems of arbitrary finite L .

Let $J_X^{\mathbf{u}}$ denote the expected cost of using strategy \mathbf{u} when the task level is X . This quantity can be calculated as follows:

$$\begin{aligned} J_X^{\mathbf{u}} &= E_{\mathbf{u}} E_X \left[\sum_{k=1}^{\infty} I_{\{X > u_{k-1}\}} C(u_k) \right] \\ &= E_{\mathbf{u}} \left[\sum_{k=1}^{\infty} \bar{F}_X(u_{k-1}) C(u_k) \right] \end{aligned} \quad (1)$$

where $u_0 = 0$, $C(0) = 0$, $E_{\mathbf{u}}$, and E_X denote expectations with respect to \mathbf{u} and X , respectively, and $I_{\{\cdot\}}$ denotes the indicator function. The expectation and summation can be interchanged due to the monotone convergence theorem [9]. We will drop the variable from the subscript when it is clear which variable the expectation is taken with respect to.

Similarly, let $D_X^{\mathbf{u}}$ denote the expected delay induced by strategy \mathbf{u} for X . This quantity can be calculated as follows:

$$\begin{aligned} D_X^{\mathbf{u}} &= E_{\mathbf{u}} E_X \left[\sum_{k=1}^{\infty} \{ I_{\{X > u_k\}} D_t(u_k) + I_{\{u_k \geq X > u_{k-1}\}} D_s(X) \} \right] \\ &= E_{\mathbf{u}} \left[\sum_{k=1}^{\infty} \bar{F}_X(u_k) D_t(u_k) \right] + E_X [D_s(X)]. \end{aligned} \quad (2)$$

When the distribution of X is known in advance, a natural objective is to determine strategies that minimize $J_X^{\mathbf{u}}$ subject to some constraint on $D_X^{\mathbf{u}}$. In general, such computations are numerical and the optimal solutions can be determined by standard constrained optimization techniques [10], [11]. In Section IV-B, we will derive the optimal strategy for a particular distribution of X and delay constraint under which the optimal strategy has a very interesting structure.

On the other hand, when the distribution of X is not known, as is often the case, a different approach is required. In this study, we adopt a worst-case performance measure. Consider an omniscient observer (or genie) who knows the task level in advance and thus uses an IL of X , incurring an expected cost of $E[C(X)]$. We can then measure the performance of a strategy \mathbf{u} by the following:

$$\rho^{\mathbf{u}} = \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \quad (3)$$

where $\{p_X(x)\}$ denotes the set of all probability distributions of X such that $E[C(X)] < \infty$. The term $\rho^{\mathbf{u}}$ is an upper bound, or worst-case measure, on the ratio between the cost of strategy \mathbf{u} and the omniscient observer, over all X . We will refer to $\rho^{\mathbf{u}}$ as the competitive ratio, or worst-case cost ratio, of \mathbf{u} . This type of worst-case measure is commonly used in many online decision and computation problems [5]. It was introduced in [3] as a method of analyzing strategies for the controlled flooding scheme described in the Introduction, and generalized in [2] to study randomized strategies.

We apply a similar worst-case analysis to delay. The minimum expected delay is $E[D_s(X)]$, obtainable by either an omniscient observer or a strategy that uses the highest IL ($\mathbf{u} = [L]$ as $L \rightarrow \infty$). Hence, the worst-case delay ratio is defined as

$$\tau^{\mathbf{u}} = \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D_s(X)]} \quad (4)$$

where we note in this case $\{p_X(x)\}$ is the set of all distributions such that $E[D_s(X)] < \infty$. Note that the worst-case cost and delay ratios are always strictly greater than 1 for any admissible strategy as it is impossible to equal or do better than the omniscient observer.

We define the following set:

$$U_d = \left\{ \mathbf{u} \in U : \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D_s(X)]} \leq d \right\} \quad (5)$$

for some constant $d > 1$. This is the set of all strategies whose delay is always within a factor d of the delay of the omniscient observer, regardless of X . We will call d the *delay constraint*. Note that as $d \rightarrow \infty$, the delay constraint becomes less restrictive and the set U_d approaches U .

We seek a strategy that satisfies this delay constraint d and has the smallest worst-case cost ratio, i.e., achieves the minimum worst-case cost ratio among all $\mathbf{u} \in U_d$

$$\rho_d^* = \inf_{\mathbf{u} \in U_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]}. \quad (6)$$

This essentially constitutes our constrained optimization problem P1, given as follows:

$$\inf_{\mathbf{u}} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \text{ s.t. } \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D_s(X)]} \leq d. \quad (7)$$

Note that the two suprema in P1, one in the objective and the other in the constraint, are in general *not* achieved under the same distribution $p_X(x)$. The intention for adopting such a worst-case formulation is to place an upper bound on both the delay and the cost over all possible distributions.

The above definitions also hold analogously for continuous strategies, by simply replacing U with V , and replacing the set $\{p_X(x)\}$ with $\{f_X(x)\}$, which is the set of density functions such that $E[C(X)] < \infty$, or $E[D_s(X)] < \infty$ depending on whether we consider worst-case cost or delay. We will thus denote by $\rho^{\mathbf{v}}$, $\tau^{\mathbf{v}}$, and V_d the continuous versions of (3)–(5), respectively. We will use the same notation ρ_d^* to denote the minimum worst-case cost ratio achieved by continuous strategies

satisfying a delay constraint d ; the distinction should be clear from the context. V_d is defined as follows for any $d > 1$:

$$V_d = \left\{ \mathbf{v} \in V : \sup_{\{f_X(x)\}} \frac{D_X^{\mathbf{v}}}{E[D_s(X)]} \leq d \right\}. \quad (8)$$

From the equations for $J_X^{\mathbf{v}}$ and $D_X^{\mathbf{v}}$, we can now explain the reason for examining strictly increasing, rather than possibly nondecreasing cost or delay functions in Definition 1. Suppose the cost and delay functions are constant over some interval $[a, b]$. Then, one can see from the equations for $J_X^{\mathbf{v}}$ and $D_X^{\mathbf{v}}$ that any optimal strategy would not use any IL v where $a \leq v < b$; it cannot do worse by instead using level b . Thus, one can “remove” the intervals over which cost and delay are constant to produce strictly increasing cost and delay functions that belong to \mathbb{C} . Furthermore, if the cost function is strictly increasing while the delay function is nonincreasing, then it can easily be shown that the delay constraint in (7) becomes non-binding and P1 reduces to an unconstrained problem of simply minimizing a worst-case cost measure. The optimal strategy for this unconstrained problem is given in Theorem 1.

As stated earlier, we assume that $D_t(\cdot) = \eta D_s(\cdot)$ for some $\eta > 0$. We now show that there is no loss of generality in assuming that $\eta = 1$. Let $\tilde{D}_X^{\mathbf{u}}$ denote expected delay of strategy \mathbf{u} for task level X when these two functions are equal. Then, note the following:

$$\begin{aligned} \frac{D_X^{\mathbf{u}}}{E[D_s(X)]} &= \frac{E[D_t(X)]}{E[D_s(X)]} \frac{E_{\mathbf{u}} \left[\frac{\sum_{k=1}^{\infty} \tilde{F}_X(u_k) D_t(u_k)}{E[D_t(X)]} \right] + 1}{E[D_t(X)]} \\ &= \frac{E[D_t(X)]}{E[D_s(X)]} \left(\frac{\tilde{D}_X^{\mathbf{u}}}{E[D_t(X)]} - 1 \right) + 1 \\ &= \eta \left(\frac{\tilde{D}_X^{\mathbf{u}}}{E[D_t(X)]} - 1 \right) + 1. \end{aligned} \quad (9)$$

Hence, the delay ratio when $D_t(\cdot) = \eta D_s(\cdot)$ is simply a rescaling of the ratio when $\eta = 1$. Specifically, a strategy \mathbf{u} satisfies $D_X^{\mathbf{u}}/E[D_s(X)] \leq d$ if and only if

$$\frac{\tilde{D}_X^{\mathbf{u}}}{E[D_t(X)]} \leq \frac{E[D_s(X)]}{E[D_t(X)]} (d-1) + 1 = \frac{d-1}{\eta} + 1. \quad (10)$$

Therefore, the set U_d that we defined for the case of $D_t(\cdot) = \eta D_s(\cdot)$ can easily be redefined for $\eta = 1$, by simply rescaling the delay constraint d . Note that this result holds in both the discrete and continuous cases. Therefore, for the rest of the analysis, we will assume these two functions are equal while noting that the results apply to the unequal case by scaling the constant d . We let $D(u) = D_t(u) = D_s(u)$ for all u . It follows that using an IL u will incur a delay of $D(\min\{X, u\})$.

We will also consider the dual problem of P1, i.e., minimizing delay subject to a constraint on cost. As in the previous problem, we define the following set:

$$\hat{U}_c = \left\{ \mathbf{u} \in U : \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq c \right\}. \quad (11)$$

That is, \hat{U}_c is the set of strategies satisfying a worst-case cost constraint. Then, the corresponding objective is to achieve the following minimum:

$$\hat{\rho}_d^* = \inf_{\mathbf{u} \in \hat{U}_c} \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D(X)]}. \quad (12)$$

Thus, the constrained optimization problem Q1, the dual of P1, is given by

$$\inf_{\mathbf{u}} \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D(X)]} \text{ s.t. } \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq c. \quad (13)$$

The analogous term \hat{V}_c is defined similarly to (11) by replacing U and $\{p_X(x)\}$ with V and $\{f_X(x)\}$, respectively.

B. Main Results for P1

Next we present our main results to be proven and discussed in Section III-B1. We begin by examining optimal continuous strategies for P1, i.e., finding the strategy in V_d that achieves minimum worst-case cost ratio. We define the following class of continuous strategies.

Definition 3: Assume that the cost function $C(\cdot) \in \mathbb{C}$. Let $\mathbf{v}[r, F_{v_1}(\cdot)]$ denote a jointly defined sequence $\mathbf{v} = [v_1, v_2, \dots]$ with a configurable parameter r , generated as follows.

- J.1) The first IL v_1 is a continuous random variable taking values in the interval $[1, C^{-1}(r \cdot C(1))]$, with its cumulative distribution function (cdf) given by some nondecreasing, right-continuous function $F_{v_1}(x) = Pr(v_1 \leq x)$. Note that this means $F_{v_1}(1) = 0$ and $F_{v_1}(C^{-1}(r \cdot C(1))) = 1$.
- J.2) The k th IL v_k is defined by $v_k = C^{-1}(r^{k-1} C(v_1))$ for all positive integers k .

From J.1) and J.2), it can be seen that r and $F_{v_1}(\cdot)$ uniquely define the IL strategy, and that given the selection of v_1 , the cost of successive IL values essentially forms a geometric sequence of base r , i.e., $C(v_k) = r^{k-1} C(v_1)$. More discussion on this structure is given in Section VI.

Our main theorem regarding the class of continuous strategies V is as follows.

Theorem 1: When $C(\cdot) \in \mathbb{C}$ and $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$, we have the following.

- 1) For any fixed $1 < d < m + 1$

$$\inf_{\mathbf{v} \in V_d} \sup_{\{f_X(x)\}} \frac{J_X^{\mathbf{v}}}{E[C(X)]} = \frac{(d-1)}{m} e^{\frac{m}{d-1}}. \quad (14)$$

Moreover, this minimum worst-case ratio is achieved by using the strategy $\mathbf{v} \left[r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)} \right]$ with $r = e^{\frac{m}{d-1}}$.

- 2) For $d \geq m + 1$, we have

$$\inf_{\mathbf{v} \in V_d} \sup_{\{f_X(x)\}} \frac{J_X^{\mathbf{v}}}{E[C(X)]} = e. \quad (15)$$

Moreover, this minimum worst-case ratio is achieved by using the strategy $\mathbf{v} \left[r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)} \right]$ with $r = e$.

Note that the optimal strategy of Theorem 1 can be adjusted for different delay constraints by varying the parameter r . For discrete strategies, we have the following.

Theorem 2: When $C(\cdot) \in \mathbb{C}$ and $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$, we have the following.

- 1) For $1 < d < m + 1$

$$\inf_{\mathbf{u} \in \mathcal{U}_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq \frac{(d-1)}{m} e^{\frac{m}{d-1}}. \quad (16)$$

- 2) For $d \geq m + 1$

$$\inf_{\mathbf{u} \in \mathcal{U}_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq e. \quad (17)$$

Whether the upper bounds in Theorem 2 become equalities appears to depend on the specific cost function $C(\cdot)$. By restricting our attention to cost functions $C(\cdot) \in \mathbb{C}_1$, we have the following result.

Theorem 3: Consider $C(\cdot) \in \mathbb{C}_1$ and $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$.

- 1) For $1 < d < m + 1$

$$\inf_{\mathbf{u} \in \mathcal{U}_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} = \frac{(d-1)}{m} e^{\frac{m}{d-1}} \quad (18)$$

where this minimum worst-case ratio can be achieved by the discrete strategy \mathbf{u}^* constructed as follows. Take the strategy $\mathbf{v}^* \left[e^{\frac{m}{d-1}}, \frac{d-1}{m} \ln \frac{C(\cdot)}{C(1)} \right]$ given by Definition 3, and set $u_k^* = \lfloor v_k^* \rfloor$ for all k to obtain the discrete strategy $\mathbf{u}^* = [u_1^*, u_2^*, \dots] = \lfloor \mathbf{v}^* \rfloor$.

- 2) For $d \geq m + 1$, we have

$$\inf_{\mathbf{u} \in \mathcal{U}_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} = e. \quad (19)$$

Moreover, this minimum worst-case cost ratio is achieved by the strategy $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$, where \mathbf{v}^* denotes the strategy $\mathbf{v}^* \left[e, \ln \frac{C(\cdot)}{C(1)} \right]$.

This result shows that we can take the floor of the optimal continuous strategy to obtain a discrete strategy, which is optimal when the cost is a subclass of \mathbb{C} .

These theorems for P1 lead to the following corresponding results for Q1, the dual of P1.

Theorem 4: Suppose $C(\cdot) \in \mathbb{C}$ and $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$. For any $c \geq e$, we have

$$\inf_{\mathbf{v} \in \hat{\mathcal{V}}_c} \sup_{\{f_X(x)\}} \frac{D_X^{\mathbf{v}}}{E[D(X)]} = ma_c + 1 \quad (20)$$

where a_c is the unique number in $[1, \infty)$ satisfying $a_c e^{a_c} = c$. Moreover, this minimum worst-case ratio is achieved by using the strategy $\mathbf{v}^* \left[e^{\frac{1}{a_c}}, a_c \ln \frac{C(\cdot)}{C(1)} \right]$. In addition, using the same definition of a_c , we have the following if the cost function belongs to \mathbb{C}_1 :

$$\inf_{\mathbf{u} \in \hat{\mathcal{U}}_c} \sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D(X)]} = ma_c + 1 \quad (21)$$

where this minimum worst-case cost ratio is achieved by the strategy $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$.

C. Discussion of Main Results

The worst-case performance measures used to derive the main results imply that for any task level, the optimal (for $1 < d < m + 1$) strategy \mathbf{v} of Theorem 1 has an expected cost within $\frac{d-1}{m} e^{\frac{m}{d-1}}$ times the expected cost of the omniscient observer. Similarly, its expected delay is always within factor d of the delay incurred by an omniscient observer.

The differentiation between the two cases, $1 < d < m + 1$ versus $d \geq m + 1$, in the first three theorems, is due to the fact that P1 has an active/binding constraint in the former, and an inactive/nonbinding constraint in the latter, as we show in Section IV.

The main results rely on the relationship $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$, where the factor m essentially describes the relative rate at which the cost and delay functions grow with respect to IL. Note that the constant positive factor β simply cancels out in the cost or delay ratio calculated in (3) and (4). Hence, we can assume that $\beta = 1$ without loss of generality. The relationship $C(\cdot) = D(\cdot)^m$ is justified in many application scenarios such as those mentioned earlier. For example, for the flooding search problem, this relationship is very representative of searching in a two-dimensional network where the search cost is proportional to the number of transmissions incurred. In this case, $C(v)$ is well approximated by a quadratic function (see, e.g., [2] and [3]) and $D(v)$ can be chosen to be a linear function of v (implying $m = 2$), or quadratic (implying $m = 1$).

IV. P1: OPTIMAL STRATEGIES WITH DELAY CONSTRAINTS

In this section, we prove the results shown in the previous Section III, i.e., the solution to problem P1. The solution to Q1, the dual of P1, follows from these results as described in part G of the Appendix.

The solution approach we take is outlined as follows. We first (in Section IV-B) consider the continuous version of P1 and derive a tight lower bound to the minimum worst-case cost under the delay constraint. This is accomplished by interchanging the \inf and \sup in (6), and introducing a constrained optimization problem whose objective is to minimize the average cost subject to a delay constraint. Then, in Section IV-C, we derive a class of randomized continuous strategies whose worst-case cost ratio matches this lower bound for all d , proving that they are optimal. These continuous strategies are then used in Section IV-D to derive good discrete strategies whose performance is at least as good in the worst-case. We will also prove that they are optimal for the subclass \mathbb{C}_1 .

Unless otherwise stated, all proofs can be found in the Appendix.

A. Preliminaries

The following lemmas are critical in our subsequent analysis. We will let $J_x^{\mathbf{u}}$ and $D_x^{\mathbf{u}}$ denote the expected cost and delay, respectively, of using strategy \mathbf{u} when $P(X = x) = 1$.

Lemma 1: For any strategy $\mathbf{v} \in V$ and $\mathbf{u} \in U$, we have

$$\sup_{\{f_X(x)\}} \frac{J_X^{\mathbf{v}}}{E[C(X)]} = \sup_{x \in [1, \infty)} \frac{J_x^{\mathbf{v}}}{C(x)} \quad (22)$$

$$\sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} = \sup_{x \in \mathbb{Z}^+} \frac{J_x^{\mathbf{u}}}{C(x)} \quad (23)$$

where \mathbb{Z}^+ denotes the set of natural numbers.

Proof of this lemma can be found in [12]. In other words, this lemma states that the cost ratio is maximized when the task level is a single point. We also have an analogous lemma for delay.

Lemma 2: For any strategy $\mathbf{v} \in V$ and $\mathbf{u} \in U$

$$\sup_{\{f_X(x)\}} \frac{D_X^{\mathbf{v}}}{E[D(X)]} = \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)}, \quad (24)$$

$$\sup_{\{p_X(x)\}} \frac{D_X^{\mathbf{u}}}{E[D(X)]} = \sup_{x \in \mathbb{Z}^+} \frac{D_x^{\mathbf{u}}}{D(x)}. \quad (25)$$

Proof: We begin by noting that for every $x \in [1, \infty)$, there corresponds a singleton probability density $f_X(x') = \delta(x' - x)$, such that $E[D(X)] = D(x)$ and $D_X^{\mathbf{v}} = D_x^{\mathbf{v}}$. We thus have the following inequality:

$$\sup_{\{f_X(x)\}} \frac{D_X^{\mathbf{v}}}{E[D(X)]} \geq \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)} \quad (26)$$

because the left-hand side is a supremum over a larger set.

On the other hand, setting $A = \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)}$, we have $\frac{D_x^{\mathbf{v}}}{D(x)} \leq A$ for all $x \in [1, \infty)$. Thus, $D_x^{\mathbf{v}} \leq AD(x)$. Then, for any random variable X , we can use this inequality along with the independence between \mathbf{v} and X to obtain

$$\begin{aligned} \frac{D_X^{\mathbf{v}}}{E[D(X)]} &= \frac{\int_{[1, \infty)} D_x^{\mathbf{v}} f_X(x) dx}{\int_{[1, \infty)} C(x) f_X(x) dx} \\ &\leq \frac{\int_{[1, \infty)} AD(x) f_X(x) dx}{\int_{[1, \infty)} D(x) f_X(x) dx} = A. \end{aligned} \quad (27)$$

Equation (27) implies that $\frac{D_X^{\mathbf{v}}}{E[D(X)]} \leq A = \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)}$. Because this inequality holds for all possible random variables X , we have

$$\sup_{\{f_X(x)\}} \frac{D_X^{\mathbf{v}}}{E[D(X)]} \leq \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)}. \quad (28)$$

Inequalities (26) and (28) collectively imply the equality in (24). Equation (25) can be proven using similar steps. Lemma 2 then follows. \square

These two lemmas reduce the space over which the worst-case cost or delay can occur, and thus are very useful in subsequent analysis.

B. A Tight Lower Bound

Consider any $d > 1$. To establish a tight lower bound to the minimum worst-case cost ratio, we interchange infimum and supremum [10] to obtain the following:

$$\sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in V_d} \frac{J_X^{\mathbf{v}}}{E[C(X)]} \leq \inf_{\mathbf{v} \in V_d} \sup_{\{f_X(x)\}} \frac{J_X^{\mathbf{v}}}{E[C(X)]}. \quad (29)$$

Any lower bound of the left-hand side of (29) can be found by fixing some distribution f_X and finding the strategy within V_d that minimizes the expected cost. Note that the strategy in V_d that minimizes the cost may be randomized, which makes the minimization very difficult.

Therefore, we further lower bound the left-hand side by considering a larger set of strategies than V_d . In particular, let $V_d(Y)$ denote the following set of strategies for some Y such that $E[D(Y)] < \infty$

$$V_d(Y) = \left\{ \mathbf{v} \in V : \frac{D_Y^{\mathbf{v}}}{E[D(Y)]} \leq d \right\}. \quad (30)$$

Clearly, $V_d(Y) \supseteq V_d$ for any Y because any strategy $\mathbf{v} \in V_d$ has a delay ratio upper bounded by d for all task levels. Therefore

$$\sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in V_d(Y)} \frac{J_X^{\mathbf{v}}}{E[C(X)]} \leq \sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in V_d} \frac{J_X^{\mathbf{v}}}{E[C(X)]} \quad (31)$$

because for any task level X , the infimum on the right-hand side is over a smaller set.

A valid lower bound of the left-hand side of (31) can be obtained by choosing particular distributions for X and Y , and finding the strategy within $V_d(Y)$ that minimizes the expected cost. To obtain a tight lower bound, we need to find a combination of X and Y such that the optimal average cost strategy under X satisfying the delay constraint induced by Y has a high expected cost ratio. It is important to note that it is *not* necessary that Y and X have the same distribution.

We consider the following problem, whose solution not only provides a tight lower bound to (29) but also serves as an example for deriving optimal average cost strategies subject to a delay constraint.

Problem 1: Suppose $C(\cdot) = \beta D(\cdot)^m$. For some $\alpha > 1$, let $\bar{F}_X(x) = P(X > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha}$, and $\bar{F}_Y(x) = P(Y > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha+1-\frac{1}{m}}$, for all $x \geq 1$. Consider the following constrained optimization problem:

$$\inf_{\mathbf{v}} \frac{J_X^{\mathbf{v}}}{E[C(X)]} \text{ s.t. } \frac{D_Y^{\mathbf{v}}}{E[D(Y)]} \leq d. \quad (32)$$

We solve the above problem for the following choice of α .

1) If $1 < d < m + 1$, choose α to be such that

$$1 < \alpha < 1 + \frac{m+1-d}{m(d-1)}. \quad (33)$$

2) If $d \geq m + 1$, choose any $\alpha > 1$.

The distinction between the two cases is that problem 1 under the former ($1 < d < m + 1$) has a binding constraint, and reduces to an unconstrained problem under the latter ($d \geq m + 1$).

Solution: The optimal strategy \mathbf{v} for this problem satisfies $C(v_j)/C(1) = \gamma^j$ for all j . The value of γ depends on d as follows (details can be found in the Appendix).

If $1 < d < m + 1$, then γ is

$$\gamma = \left(1 + \frac{(\alpha-1)m}{[(\alpha-1)m+1](d-1)} \right)^{\frac{1}{\alpha-1}}. \quad (34)$$

	Continuous	Discrete
$1 < d < m + 1$	Thm 1 (1) $\rho_d^* = \frac{d-1}{m} e^{\frac{m}{d-1}}$ optimal strategy: $\mathbf{v}^* [e^{\frac{m}{d-1}}, \frac{d-1}{m} \ln \frac{C(\cdot)}{C(1)}]$	Thm 2 (1) $\rho_d^* \leq \frac{d-1}{m} e^{\frac{m}{d-1}}$ Thm 3 (1), $C(\cdot) \in \mathbb{C}_1$ $\rho_d^* = \frac{d-1}{m} e^{\frac{m}{d-1}}$ optimal: $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$
$d \geq m + 1$	Thm 1 (2) $\rho_d^* = e$ optimal strategy: $\mathbf{v}^* [e, \ln \frac{C(\cdot)}{C(1)}]$	Thm 2 (2) $\rho_d^* \leq e$ Thm 3 (2), $C(\cdot) \in \mathbb{C}_1$ $\rho_d^* = e$ optimal: $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$

Fig. 1. Summary of the results on optimal worst-case strategies under P1.

The optimal cost ratio for this case is given by

$$\frac{J_X^\gamma}{E[C(X)]} = \left[\frac{d-1}{m\alpha} + d \left(\frac{\alpha-1}{\alpha} \right) \right] \gamma. \quad (35)$$

If $d \geq m + 1$, then $\gamma = \alpha^{\frac{1}{\alpha-1}}$ and the optimal cost ratio is $\alpha^{\frac{1}{\alpha-1}}$.

Using this solution, we see that as α approaches 1 from above, the optimal cost ratio for the case $1 < d < m + 1$ has the following limit:

$$\lim_{\alpha \rightarrow 1^+} \frac{J_X^\gamma}{E[C(X)]} = \frac{(d-1)}{m} e^{\frac{m}{d-1}} \quad (36)$$

where the limit is reached from below. When $d \geq m + 1$, the optimal cost ratio satisfies $\lim_{\alpha \rightarrow 1^+} \alpha^{\frac{1}{\alpha-1}} = e$. Hence, the minimum cost ratio is lower bounded as follows.

Theorem 5: When $C(\cdot) = \beta D(\cdot)^m$ for $\beta, m > 0$, then for any $1 < d < m + 1$, the best worst-case cost ratio is lower bounded by the following:

$$\inf_{\mathbf{v} \in V_d} \sup_{\{f_X(x)\}} \frac{J_X^\gamma}{E[C(X)]} \geq \frac{(d-1)}{m} e^{\frac{m}{d-1}}.$$

Therefore, any strategy in V_d that achieves a worst-case cost ratio of $\frac{(d-1)}{m} e^{\frac{m}{d-1}}$ must be optimal.

Similarly, when $d \geq m + 1$, we have

$$\inf_{\mathbf{v} \in V_d} \sup_{\{f_X(x)\}} \frac{J_X^\gamma}{E[C(X)]} \geq e. \quad (37)$$

Therefore, any strategy in V_d that achieves a worst-case cost ratio of e must be optimal.

C. Optimal Delay-Constrained Strategies

We proceed to find strategies that match the lower bounds established in Section IV-B. For convenience, we summarize the main results Fig. 1. In this and the next subsection, we will prove these results. To do so, we will consider strategies of the form $\mathbf{v}[r, F_{v_1}(x)]$ given by Definition 3.

Lemma 3: Assume $C(\cdot) \in \mathbb{C}$ and $C(\cdot) = \beta D(\cdot)^m$ for some $\beta, m > 0$. Then, for any strategy $\mathbf{v}[r, F_{v_1}(\cdot)]$, its worst-case delay ratio $\sup_{x \in [1, \infty)} \frac{D_v^*(x)}{D(x)}$ is given by

$$\sup_{1 \leq z < r^{\frac{1}{m}}} \left\{ \frac{1}{r^{\frac{1}{m}} - 1} \frac{g(r^{\frac{1}{m}}) + (r^{\frac{1}{m}} - 1)g(z)}{zD(1)} - \frac{g'(z)}{D(1)} \right\} + 1$$

where $g'(z)$ denotes the derivative of g with respect to z , and $g(z)$ is defined as follows for $1 \leq z < r^{\frac{1}{m}}$:

$$g(z) = D(1) + \int_{D(1)}^{z \cdot D(1)} [1 - F_{v_1}(D^{-1}(y))] dy. \quad (38)$$

We also have an analogous result for the worst-case cost ratio of these strategies.

Lemma 4: Suppose $C(\cdot) \in \mathbb{C}$. For any strategy $\mathbf{v}[r, F_{v_1}(x)]$, the worst-case cost ratio is given by

$$\sup_{x \in [1, \infty)} \frac{J_x^\gamma}{C(x)} = \sup_{1 \leq z < r} \left\{ \frac{r}{r-1} \frac{h(r) + (r-1)h(z)}{zC(1)} - r \frac{h'(z)}{C(1)} \right\}$$

where $h(z)$, for all $1 \leq z < r$, is defined similarly to $g(z)$ by replacing $D(\cdot)$ with $C(\cdot)$ in (38).

Considering the family of strategies of the form $\mathbf{v}[r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)}]$, we have

$$g(z) = D(1) \left[z - \frac{mz \ln z}{\ln r} + \frac{zm}{\ln r} - \frac{m}{\ln r} \right] \quad (39)$$

and $g'(z) = D(1)(1 - m \frac{\ln z}{\ln r})$ for all z . Similarly, we have

$$h(z) = C(1) \left[z - \frac{z(\ln z - 1)}{\ln r} \right] \quad (40)$$

and $h'(z) = C(1)(1 - \frac{\ln z}{\ln r})$ for all z . Thus, we have the following results regarding this family of strategies.

- 1) The worst-case delay ratio of these strategies is $\frac{m}{\ln r} + 1$. This is easily verified by using Lemma 3.
- 2) The worst-case cost ratio of these strategies is $\frac{r}{\ln r}$. This is also easily verified by using Lemma 4.

We consider two special cases of this family of strategies. The first case is when $r = e^{\frac{m}{d-1}}$ for some $1 < d < m + 1$. With the above results, the worst-case delay ratio of this strategy is exactly d . Hence, this specific strategy belongs to V_d . On the other hand, its worst-case cost ratio is $\frac{(d-1)}{m} e^{\frac{m}{d-1}}$ (plugging $r = e^{\frac{m}{d-1}}$ into $\frac{r}{\ln r}$), achieving the lower bound established in Theorem 5.

The second case is when $r = e$. In this case, we achieve a worst-case delay ratio of $m + 1$, and the worst-case cost ratio of exactly e . Hence, when $d \geq m + 1$, this strategy belongs to V_d and is optimal because it matches the lower bound established in Theorem 5. If $d > m + 1$, then the delay constraint becomes inactive/nonbinding under this strategy. Thus, for $d > m + 1$, this is also the solution to the unconstrained problem. This result was proven separately in [12] within the context of an unconstrained optimization problem, which we have now shown to be a special case of the more general result in this paper.

Combining these two cases, we obtain Theorem 1. Therefore, we have obtained the optimal worst-case continuous strategies for any delay constraint $d > 1$.

D. Optimal Discrete Strategies

We now return to deriving robust integer-valued strategies, i.e., finding $\mathbf{u} \in U_d$ achieving the minimum worst-case cost ratio. For notation, we will let $\lfloor \mathbf{v} \rfloor$ be the strategy $[\lfloor v_1 \rfloor, \lfloor v_2 \rfloor, \dots]$. We begin with the following lemma.

Lemma 5: For all $x \in \mathbb{Z}^+$, we have $D_x^{\lfloor \mathbf{v} \rfloor} \leq D_x^{\mathbf{v}}$ and $J_x^{\lfloor \mathbf{v} \rfloor} \leq J_x^{\mathbf{v}}$. That is, we can take the floor of any continuous strategy to find a discrete strategy that performs just as well if the task level is restricted to integers.

Using this result, we can prove Theorem 2. The proof is given in part E of the Appendix. This theorem gives an upper bound on the best worst-case discrete strategy, for all $C(\cdot) \in \mathbb{C}$. It appears that the actual value of the minimum worst-case cost will depend on the specific function $C(\cdot)$. A general result is currently not available, but if we restrict ourselves to cost functions of the simple polynomial form $C(\cdot) \in \mathbb{C}_1$, then we can obtain Theorem 3 presented earlier. This proof is provided in the Appendix.

Below we provide a counter example to illustrate the reason for limiting cost functions to the class \mathbb{C}_1 . Suppose $C(x) = r^{x-1}$ for all integers x and some constant $r > 1$. Furthermore, suppose d is very large so that the delay constraint for P1 becomes nonbinding. Note from Definition 2 that $C(\cdot) \in \mathbb{C}_r$. Consider the strategy $\mathbf{v} = [1, 2, 3, \dots]$, i.e., a strategy that increases the IL by 1 after every unsuccessful attempt. It can easily be seen that the cost ratio of such a strategy for any positive integer x is

$$\frac{J_x^{\mathbf{v}}}{C(x)} = \frac{\sum_{j=1}^x r^{j-1}}{r^{x-1}} = \frac{r^x - 1}{(r-1)r^{x-1}} = \frac{r(1 - r^{-x})}{r-1} \quad (41)$$

which is increasing in x . Taking the limit as $x \rightarrow \infty$, we see that the worst-case cost ratio of \mathbf{v} is $\frac{r}{r-1}$. Note that for very large r , this worst-case ratio approaches 1, which is the best possible worst-case ratio. This example illustrates that if the cost function increases very fast (very large r in this case), then the best worst-case cost ratio may be very small (close to 1). Consequently, it becomes very difficult if at all possible to lower bound this ratio and derive the optimal strategy. This is the reason why we have limited our results to the class \mathbb{C}_1 .

V. PROBLEM P2

Problem P1 requires that the user starts the task from scratch during each round and pays the full cost and delay of using an IL. As mentioned in the Introduction, there are motivating scenarios where the user may be able to resume the task from a previous attempt and thus pay only incremental cost and delays associated with the increase in IL. One example illustrating this difference is the ARQ application described in the introduction. P1 corresponds to the scenario where the receiver discards all previous reception failures (i.e., previously received copies that could not be decoded correctly) and the transmitter completely encodes the packet (with longer code) each time. P2, on the

other hand, corresponds to the scenario where the transmitter encodes the packet with different parity bits each time instead of increasing the number of parity bits. At the same time, the receiver saves all received copies, and although it may not be able to decode the packet correctly in a single reception, it may be able to decode successfully given a sufficient number of receptions. In this case, the transmitter does not have to increase the packet length each time. In this sense, each transmission provides the receiver with additional information about the packet, and thus, the transmitter's effort may be viewed as expended *incrementally*.

Motivated by this, in this section, we study problem P2 that differs from P1 in this regard. It will be seen that the class of optimal strategies for P2 are derived similarly and share many similar properties to the optimal strategies for P1.

A. Problem Formulation

Consider the description of P1 given in Section II (points 1–5). P2 differs from P1 in the last two points, which we elaborate on below.

- 4) When the level is increased from v_{k-1} to v_k , the individual commits to paying a cost $C(v_k) - C(v_{k-1})$, with the initial level $v_0 = 0$, and $C_0 = 0$ for notation. Whenever a level v_k successfully completes the task, there is an extra cost incurred denoted by $C_s(v_k)$.
- 5) With a level v_k on the k th round, the task takes a certain amount of time $D_1(\cdot, \cdot)$ to process plus an extra time depending on whether the outcome is success or failure, e.g., time for verification or resumption. Specifically, when the task completes successfully, this delay is given by $D_1(v_{k-1}, X) + \bar{D}_s(X)$; if it fails, then the delay is $D_1(v_{k-1}, v_k) + \bar{D}_t(v_k)$.

Thus, the main differences between P2 and P1 are in the cost and the delay. In particular, because the individual is allowed to resume her previous attempt during each round, she does not need to pay the full cost of using IL v_k . Instead, she only needs to pay the incremental cost $C(v_k) - C(v_{k-1})$ incurred by increasing her IL from v_{k-1} to v_k . We have also included an extra cost $C_s(\cdot)$, which models the extra cost needed for verification purposes (e.g., the cost of sending a query reply in controlled flooding or an ACK in ARQ). We will assume that these two cost functions are proportional, i.e., $C_s(\cdot) = \zeta C(\cdot)$ for some $\zeta \geq 0$. Thus, we can set $\zeta = 0$ when there is no extra cost associated with completing the task.

For the delay, we assume that $D_1(\cdot, \cdot)$ is piecewise additive so that $D_1(a, b) + D_1(b, d) = D_1(a, d)$ for any constants $a < b < d$. We will also assume proportionality between the delay functions, so that $\bar{D}_t(\cdot) = \theta D_1(0, \cdot)$ and $\bar{D}_s(\cdot) = w D_1(0, \cdot)$ for some constants $\theta, w \geq 0$. We have assumed that the verification delay is a function of X rather than the last increment ($v_k - v_{k-1}$) in the process. This is because even though the task is completed in incremental steps, the verification may require that the user start from the beginning. For instance, in the controlled flooding example, this verification means sending a suppression message from the source node to all nodes within the range that includes the target, and it is thus a function of the location of the target. For some applications, this verification delay may be more appropriately modeled as a function of the last task level

increase step (or as a constant). Such a change in the model makes the problem quite different from the one examined here and is thus out of the scope of this paper.

We let H_X^u denote the expected cost of using strategy u for a given task level X . This can be calculated as follows:

$$\begin{aligned} H_X^u &= E_{\mathbf{u}} E_X \left[\sum_{k=1}^{\infty} I_{\{u_k \geq X > u_{k-1}\}} (C(u_k) + C_s(u_k)) \right] \\ &= (1 + \zeta) \cdot E_{\mathbf{u}} \left[\sum_{k=1}^{\infty} (\bar{F}_X(u_{k-1}) - \bar{F}_X(u_k)) C(u_k) \right]. \end{aligned}$$

Consider an omniscient observer that knows X in advance, and does not need to pay the verification cost $C_s(\cdot)$. The expected cost of this genie is $E[C(X)]$. Thus, we can take the ratio between H_X^u and $E[C(X)]$ to form a similar worst-case performance measure analogous to the one used for P1. Notice that even if the genie was required to pay $C_s(X)$ upon completing the task, this would simply change his expected cost to $(1 + \zeta)E[C(X)]$. Thus, the worst-case performance measure in this case would simply be a rescaling of the results when not paying a verification cost.

Similarly, the expected delay T_X^u is given by $T_X^u = E_{\mathbf{u}} E_X [\sum_{k=1}^{\infty} I_{\{X > u_k\}} (D_1(u_{k-1}, u_k) + \bar{D}_t(u_k)) + \sum_{k=1}^{\infty} I_{\{u_k \geq X > u_{k-1}\}} (D_1(u_{k-1}, X) + \bar{D}_s(X))]$, which can be simplified as follows:

$$\begin{aligned} T_X^u &= E_{\mathbf{u}} \left[\sum_{k=1}^{\infty} \bar{F}_X(u_k) (\bar{D}_t(u_k)) \right] \\ &\quad + E_X [D_1(0, X) + \bar{D}_s(X)] \\ &= \theta \cdot E_{\mathbf{u}} \left[\sum_{k=1}^{\infty} \bar{F}_X(u_k) (D_1(0, u_k)) \right] \\ &\quad + (1 + w) \cdot E_X [D_1(0, X)]. \end{aligned} \quad (42)$$

Meanwhile, the expected delay of the omniscient observer is $(1 + w)E_X[D_1(0, X)]$. We can thus define \bar{U}_d and \bar{V}_d analogously to U_d and V_d of (5) and (8) by replacing D_X^u and $D(\cdot)$ with T_X^u and $D_1(0, \cdot)$, respectively. That is, \bar{U}_d is the set of all admissible discrete strategies satisfying $T_X^u \leq d \cdot (1 + w)E[D_1(0, X)]$. Similar to the steps taken in (9) and (10) for P1, we can show that there is no loss of generality in choosing particular values of w and θ . For convenience, we will assume $1 + w = \theta$. The results for this case can be generalized to the case of $1 + w \neq \theta$ by proper rescaling.

B. Optimal Strategies for P2

Using the formulation described in the previous section, we have the following results.

Theorem 6: When $C(\cdot) \in \mathcal{C}$, $C(\cdot) = \beta D_1(0, \cdot)^m$ for some $m, \beta > 0$, and $C_s(\cdot) = \zeta C(\cdot)$ for some $\zeta \geq 0$, we have for any fixed $d > 1$

$$\inf_{\mathbf{v} \in \bar{V}_d} \sup_{\{f_X(x)\}} \frac{H_X^{\mathbf{v}}}{E[C(X)]} = (1 + \zeta) \frac{(d-1)}{m} [e^{\frac{m}{d-1}} - 1]. \quad (43)$$

This minimum worst-case ratio is achieved by using the strategy $\mathbf{v}^* [r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)}]$ with $r = e^{\frac{m}{d-1}}$.

Theorem 7: Consider $C(\cdot) \in \mathcal{C}_1$, $C(\cdot) = \beta D(\cdot)^m$ for some $m, \beta > 0$, and $C_s(\cdot) = \zeta C(\cdot)$ for some $\zeta \geq 0$. Then, we have for any fixed $d > 1$

$$\inf_{\mathbf{u} \in \bar{U}_d} \sup_{\{p_X(x)\}} \frac{H_X^{\mathbf{u}}}{E[C(X)]} = (1 + \zeta) \frac{(d-1)}{m} [e^{\frac{m}{d-1}} - 1] \quad (44)$$

where this minimum worst-case ratio can be achieved by the discrete strategy $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$, where \mathbf{v}^* denotes strategy $\mathbf{v}^* [e^{\frac{m}{d-1}}, \frac{d-1}{m} \ln \frac{C(\cdot)}{C(1)}]$.

We can prove Theorem 6 similarly to the way we proved Theorem 1 in Section IV. For brevity, the complete proof is omitted; a sketch of the proof is provided as follows. First, it can be shown that the following inequality holds for any random variable Y :

$$\inf_{\mathbf{v} \in \bar{V}_d} \sup_{\{f_X(x)\}} \frac{H_X^{\mathbf{v}}}{E[C(X)]} \geq \sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in \bar{V}_d(Y)} \frac{H_X^{\mathbf{v}}}{E[C(X)]} \quad (45)$$

where $\bar{V}_d(Y)$ is the set of all admissible \mathbf{v} such that $T_Y^{\mathbf{v}} \leq d \cdot E[D_1(0, Y)]$.

To obtain a tight upper bound to the right-hand side, we consider the following problem.

Problem 2: Suppose $C(\cdot) = \beta D(\cdot)^m$. Let $\bar{F}_X(x) = P(X > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha}$, and $\bar{F}_Y(x) = P(Y > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha+1-\frac{1}{m}}$, for some $\alpha > 1$ and for all $x \geq 1$. Consider the following constrained optimization problem:

$$\inf_{\mathbf{v}} \frac{H_X^{\mathbf{v}}}{E[C(X)]} \text{ s.t. } \frac{T_Y^{\mathbf{v}}}{E[D_1(0, Y)]} \leq d. \quad (46)$$

Solution: The optimal strategy \mathbf{v} for this problem satisfies $C(v_j)/C(1) = \gamma^j$ for all j , where γ is

$$\gamma = \left(1 + \frac{(\alpha-1)m}{[(\alpha-1)m+1](d-1)}\right)^{\frac{1}{\alpha-1}}. \quad (47)$$

The optimal cost ratio $\frac{H_X^{\mathbf{v}}}{E[C(X)]}$ is given by

$$(1 + \zeta) \left[\frac{d-1}{m\alpha} + d \left(\frac{\alpha-1}{\alpha} \right) \right] (\gamma-1) + \frac{\alpha-1}{\alpha} (1 + \zeta).$$

The solution for problem 2 is proven similarly to problem 1. Using this solution, we see that as α approaches 1 from above, the optimal cost ratio has the following limit:

$$\lim_{\alpha \rightarrow 1^+} \frac{H_X^{\mathbf{v}}}{E[C(X)]} = (1 + \zeta) \frac{(d-1)}{m} [e^{\frac{m}{d-1}} - 1]. \quad (48)$$

Thus, we have the following.

Lemma 6: When $C(\cdot) = \beta D(\cdot)^m$ for $\beta, m > 0$, for any $1 < d < m+1$, the best worst-case cost ratio is lower bounded by the following:

$$\inf_{\mathbf{v} \in \bar{V}_d} \sup_{\{f_X(x)\}} \frac{H_X^{\mathbf{v}}}{E[C(X)]} \geq (1 + \zeta) \frac{(d-1)}{m} [e^{\frac{m}{d-1}} - 1].$$

Therefore, any strategy in \bar{V}_d that achieves a worst-case cost ratio of $(1+\zeta)\frac{(d-1)}{m}[e^{\frac{m}{d-1}}-1]$ must be optimal. It can be shown that strategy $\mathbf{v}^*\left[e^{\frac{m}{d-1}}, \frac{d-1}{m}\ln\frac{C(\cdot)}{C(1)}\right]$ achieves this worst-case cost ratio by using the following equation that relates $H_x^{\mathbf{v}^*}$ and $J_x^{\mathbf{v}^*}$ for any $x \in [1, \infty)$:

$$\begin{aligned} H_x^{\mathbf{v}^*} &= (1+\zeta) \cdot E_{\mathbf{v}^*} \left[\sum_{k=1}^{\infty} (I_{\{x>v_{k-1}\}} - I_{\{x>v_k\}}) C(v_k) \right] \\ &= (1+\zeta) \left(J_x^{\mathbf{v}^*} - E_{\mathbf{v}^*} \left[\sum_{k=1}^{\infty} I_{\{x>v_k\}} C(v_k) \right] \right). \end{aligned}$$

Thus, by using similar steps to the proofs of Lemmas 3 and 4, we have the following:

$$\frac{H_x^{\mathbf{v}^*}}{C(x)} = (1+\zeta) \left[\frac{h(r) + (r-1)h(z)}{zC(1)} - (r-1)\frac{h'(z)}{C(1)} \right]$$

where $z = \frac{C(x)}{r^{m-1}C(1)}$ and $h(\cdot)$ is the function defined in Lemma 4. Plugging (40) into this equation, we can thus show that the worst-case ratio of \mathbf{v}^* is given by $(1+\zeta)\frac{(d-1)}{m}[e^{m/(d-1)}-1]$.

Finally, to show that \mathbf{v}^* belongs to \bar{V}_d , we note that if $D(x) = D_1(0, x)$ for all x , then $T_x^{\mathbf{v}^*} = D_x^{\mathbf{v}^*}$ by comparing (2) and (42) under the assumptions $\theta = 1$, $\omega = 0$, and $D_s(\cdot) = D_t(\cdot)$. Thus, from Lemma 3 and (39), the worst-case delay ratio of \mathbf{v}^* is d . Thus, we have shown \mathbf{v}^* is in \bar{V}_d , therefore completing the proof of Theorem 6.

The proof of Theorem 7 is also very similar to the proof for Theorem 3, and therefore, only a sketch of the proof is provided in part H of the Appendix.

VI. APPLICATIONS, EXAMPLES AND DISCUSSION

A. Cost-Delay Tradeoff for P1

Having derived optimal strategies for any delay constraint, it is worth examining how the delay constraint affects the minimum achievable worst-case cost ratio. Fig. 2 depicts the tradeoff between optimal worst-case cost ratio as given by Theorem 1 and the delay constraint d when $C(\cdot) = \beta D(\cdot)^m$. The dotted portion of each curve indicates when the delay constraint is not binding, i.e., for $d \geq m+1 = 1.5, 2, 3$, respectively. In these cases, the optimal unconstrained strategy (using $r = e$) has a minimum worst-case cost ratio of e . Note that the plot is logarithmic. As d approaches 1 from above, the best worst-case cost ratio approaches ∞ for all m . Hence, as the constraint on delay becomes tighter, the minimum worst-case cost increases unboundedly.

For any fixed d , as m increases, the minimum worst-case cost also increases. This can be understood by fixing some delay function $D(\cdot)$. As m increases, the cost function $C(\cdot) = \beta D(\cdot)^m$ increases faster. For any given delay constraint, it then becomes more difficult to achieve a low cost ratio.

B. Examples of P1

We present an example scenario where the delay function grows linearly in the IL value used, while the cost function grows quadratically. Specifically, consider $D(x) = \beta x$ for all x and $C(x) = \xi x^2$ so $m = 2$. As mentioned earlier, for the

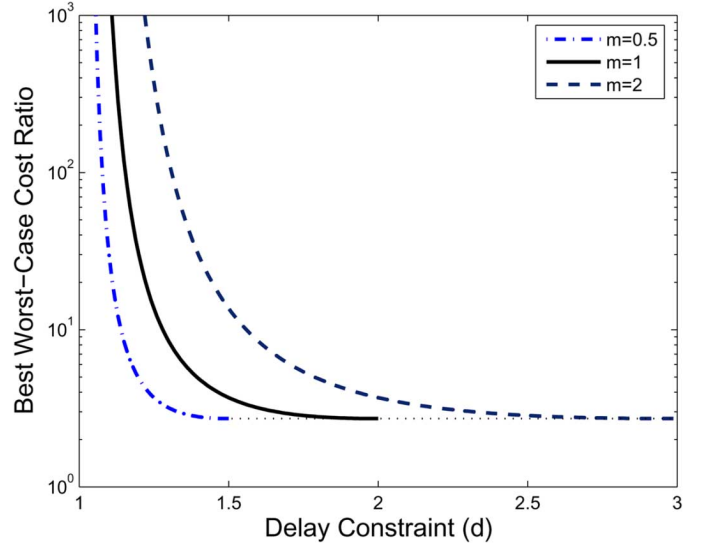


Fig. 2. Logarithmic plot of the minimum worst-case cost ratio as a function of the delay constraint d , when $C(\cdot) = \beta D(\cdot)^m$. Dotted portions indicate when the delay constraint is not binding and hence the unconstrained strategy of Theorem 1, part 2) is optimal. For $d \geq 3$, the best worst-case cost ratio is e for all three curves.

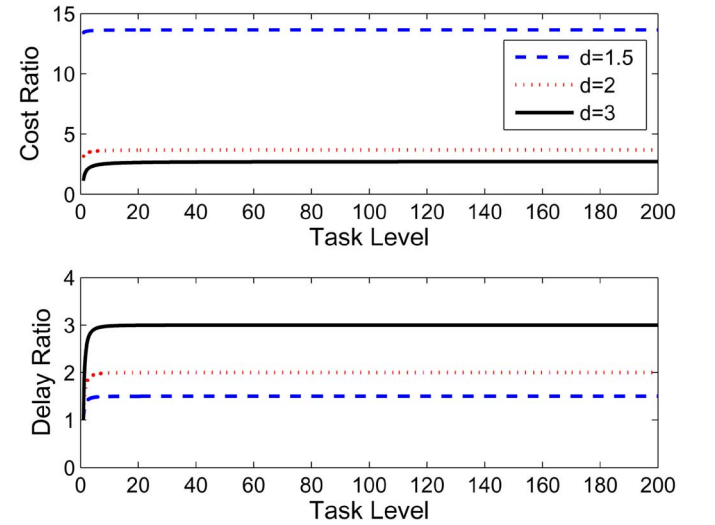


Fig. 3. Plot of cost and delay ratios of optimal strategies under different delay constraints, when cost is quadratic and delay is linear, i.e., $C(\cdot) = D(\cdot)^m$ for $m = 2$. Note that the delay ratio and cost ratio curves approach their maximum values very rapidly.

flooding search application, this could be a good representation of a two-dimensional network, where transmissions are on the order of x^2 , and the delay is proportional to number of hops.

From Theorem 1, the optimal strategy is $\mathbf{v}[e^{\frac{m}{d-1}}, (d-1)\ln x]$ whenever $1 < d < 3$. When $d \geq 3$, the optimal strategy is $\mathbf{v}[e, 2 \cdot \ln x]$. Fig. 3 depicts the cost and delay ratio curves, with respect to task level, of the corresponding optimal strategies when $d = 1.5, 2$, and 3 .

Note that both the delay and cost ratio curves approach their maximum values very rapidly. Hence, the worst-case value of the cost and the delay under asymptotic maximum permissible IL (as $L \rightarrow \infty$) can approximate the performance when L is finite. At the same time, the worst-case is approached asymptotically. Hence, the cost (delay) ratio at any finite task level is less than the worst-case cost (delay) ratio. Also note that the cost

and delay ratio curves are smooth and nearly flat with respect to task level. Thus, the actual task level does not significantly change the performance of these strategies. One can view this as a built-in robustness for both the cost and delay criteria.

Similar results hold for other values of d and m , and other functional forms of $C(\cdot)$ and $D(\cdot)$. They are not repeated here.

C. Comparison of P1 Strategies

It was shown in [2] and [12] that when adopting a worst-case cost measure, randomized strategies outperform deterministic ones. The results of the previous sections show that randomized strategies also perform better when delay constraints are added. Here we illustrate this in more detail.

Note that both the optimal deterministic strategy for Problem 1 and the optimal randomized strategies of Section IV-C share the property that the costs of the IL values grow geometrically. That is, for any realization, $C(v_k) = r^{k-1}C(v_1)$ for all k . It was shown in [3] that the unconstrained optimal deterministic strategy under linear cost $C(\cdot)$ is also a geometric sequence: $u_k = 2^{k-1}$ for all k .

Below we compare deterministic and randomized geometric strategies to examine the effect of randomization. For deterministic geometric strategies with parameter r , $C(v_k) = r^{k-1}C(v_1)$ for all $k \geq 1$. Consider when both the cost and the delay are linear, so $D(v) = C(v) = v$ for all v . Then, for any $k \geq 0$ and $x = r^k + \epsilon$, where $0 < \epsilon \leq r^{k+1} - r^k$, we have

$$\frac{D_x^{\mathbf{v}}}{D(x)} = \frac{\sum_{j=0}^k r^j + r^k + \epsilon}{r^k + \epsilon} = \frac{r^{k+1} - 1}{(r-1)(r^k + \epsilon)} + 1.$$

For each k , this ratio is maximized by taking the limit as ϵ approaches 0 from above. The maximum value of this ratio over all k is derived by letting $k \rightarrow \infty$, giving

$$\sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)} = \lim_{k \rightarrow \infty} \frac{r^{k+1} - 1}{(r-1)r^k} + 1 = \frac{2r-1}{r-1} \quad (49)$$

which is strictly greater than 2 for all values of r . At the same time, similar calculations show that the worst-case cost ratio for such strategies is $\frac{r^2}{r-1}$.

Now consider the randomized strategies $\mathbf{v} \left[r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)} \right]$, shown to be optimal in Theorem 1. Every realization of \mathbf{v} is a geometric deterministic strategy with growth rate r . For any $r > 1$ and $m = 1$, it was shown that the worst-case cost ratio of \mathbf{v} is $\frac{r}{\ln r}$ and the worst-case delay ratio is $\frac{1}{\ln r} + 1$. In addition, there is an interesting interpretation of these strategies. Specifically, if Z denotes a random variable uniformly distributed in the interval $[0, 1)$, then for any r , the strategy has costs satisfying $C(v_k) = C(1)r^{k-1}r^Z$ for all $k \geq 1$. Thus, the costs of ILs grow geometrically.

In Fig. 4, we plot the worst-case cost and delay ratios, as functions of r , for the aforementioned geometric deterministic and randomized strategies. Note that for any r , the randomized strategy achieves a lower worst-case cost and a lower worst-case delay than its deterministic counterpart. Hence, randomization has the effect of decreasing the worst-case cost and delay at the same time.

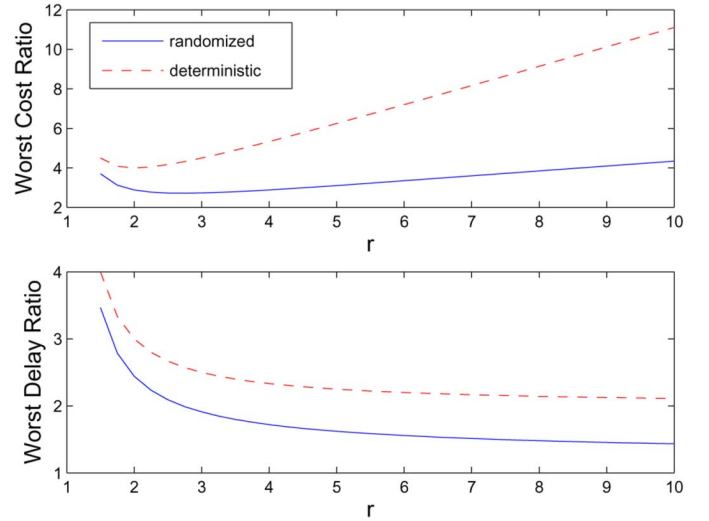


Fig. 4. Comparison of deterministic and randomized strategies as a function of r , for the strategies discussed in Section VI-C. Note that for any r , the randomization achieves lower worst-case cost ratio and delay ratio.

In addition, note that the worst-case delay ratio of the randomized strategies approaches 1 as $r \rightarrow \infty$, but for the fixed strategies, this limit is 2. In fact, for randomized geometric strategies using $r \geq e$, the worst-case delay ratio is always below 2. The class of optimal randomized strategies in Theorem 1 used $r \geq e$ for all values of d . Therefore, even by arbitrarily increasing the value of r for deterministic geometric strategies, it is not possible to match the worst-case delay ratio of the optimal randomized geometric strategies that we have derived in this study.

By varying the cost/delay functions and m , the curves in Fig. 4 may change but the general relationship between randomized and deterministic strategies will still hold.

D. Comparison Between P1 and P2

As mentioned in Section V-A, problems P1 and P2 differ in how the cost and the delay are applied. For P1, the user starts the task from scratch during each round and thus pays the full cost and delay of using an IL. By contrast, in P2, the user is allowed to resume her previous attempts during each round and thus pays the incremental cost and delay associated with increasing the IL.

In some applications, the user may have the option of choosing between strategies that fit the description of P1 (which we call a *P1 strategy*) or those that fit the description of P2 (*P2 strategy*). Thus, in this section, we provide a comparison between the optimal P1 and P2 strategies.

We make the following assumptions for comparing a P1 strategy and a P2 strategy. First, we assume that the same cost function $C(\cdot)$ describes the cost for both strategies. We will assume that $D_1(0, \cdot) = D(\cdot)$ so that the time it takes for the user to start from scratch (for both strategies) is the same. Finally, recall that for P2 we assumed an extra cost $C_s(\cdot) = \zeta C(\cdot)$ for the P2 strategy.

Note that the results we will obtain under the above assumptions can be generalized to the case when the costs and delay functions for the P1 and P2 strategies are proportional, because we showed earlier that relaxing this assumption simply scales the delay constraints in P1 and P2. Thus, with these assumptions, we can use Theorems 1 and 6 to analyze the performance

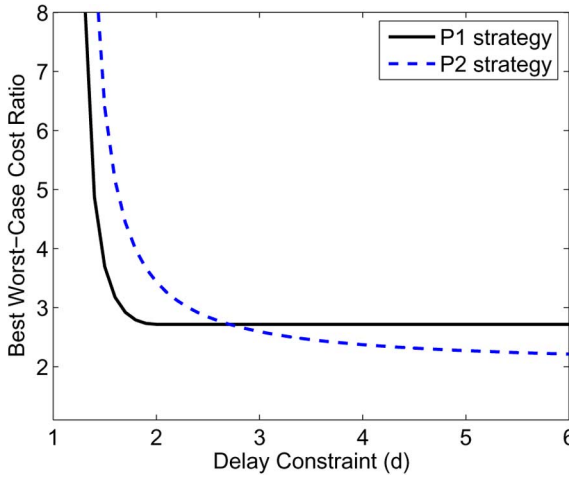


Fig. 5. Plot of the minimum worst-case cost ratio as a function of the delay constraint d for both P1 and P2 strategies, when $C(\cdot) = \beta D(\cdot)$ and $\zeta = 1$.

of each of these schemes. Fig. 5 depicts the minimum cost ratio for the P1 and P2 schemes when $C_s(\cdot) = C(\cdot) = \beta D(\cdot)$ (so $\zeta = m = 1$).

Note that for large d , the P2 scheme performs better. This is because if the delay is not a factor, then the minimum cost strategy is to increase IL by the smallest possible increment after every round. Such a strategy incurs high delay, but only a minimal cost is committed for every round. Thus, for larger delay tolerance, the P2 strategy performs better because more low-cost strategies are admissible. On the other hand, when d is small, then the opposite is true. As ζ is varied between 0 and 1, the curve for the P2 strategy essentially shifts up and down, but the general relationship described earlier between P1 and P2 still holds.

VII. CONCLUSION

In this paper, we studied a class of sequential constrained resource allocation problems where an individual must decide how much effort (IL) to apply for completing a task while consuming minimal resources. We presented a constrained optimization framework in order to derive strategies that minimize a worst-case cost measure subject to a worst-case delay constraint. Optimal strategies were obtained in the continuous as well as discrete cases for ILs and their performance was studied. These results were used to discuss the tradeoff between the cost and the delay using this type of method.

APPENDIX

A. Solution to Problem 1

To begin, we calculate the mean of object location cost and delay, noting that X takes values on $[1, \infty)$

$$\begin{aligned} E[C(X)] &= \int_0^\infty Pr(C(X) > x) dx \\ &= C(1) + \int_{C(1)}^\infty \left[\frac{C(C^{-1}(x))}{C(1)} \right]^{-\alpha} dx \\ &= \frac{\alpha}{\alpha - 1} C(1). \end{aligned} \quad (50)$$

$$\begin{aligned} E[D(Y)] &= E \left[C(Y)^{1/m} \right] \\ &= \int_0^\infty Pr \left(C(Y)^{1/m} > x \right) dx \\ &= \frac{1 + (\alpha - 1)m}{(\alpha - 1)m} C(1)^{1/m}. \end{aligned} \quad (51)$$

Using (2) to evaluate the delay ratio for (deterministic) \mathbf{v}

$$\frac{D_Y^{\mathbf{v}}}{E[D(Y)]} = \frac{\sum_{k=1}^\infty \bar{F}_Y(v_k) D(v_k)}{E[D(Y)]} + 1. \quad (52)$$

By rearranging (52) and observing that only the numerator of the cost ratio depends on \mathbf{v} , Problem 1 is equivalent to

$$\inf_{\mathbf{v}} J_X^{\mathbf{v}} \text{ s. t. } \sum_{k=1}^\infty \bar{F}_Y(v_k) D(v_k) \leq (d-1)E[D(Y)].$$

Therefore, we define the Lagrangian for $\lambda \geq 0$

$$\begin{aligned} G(\mathbf{v}, \lambda) &= \sum_{k=0}^\infty \bar{F}_X(v_k) C(v_{k+1}) \\ &\quad - \lambda \left((d-1)E[D(Y)] - \sum_{k=1}^\infty \bar{F}_Y(v_k) D(v_k) \right) \\ &= \sum_{k=0}^\infty \left(\frac{C(v_k)}{C(1)} \right)^{-\alpha} C(v_{k+1}) - \lambda (d-1)E[D(Y)] \\ &\quad + \lambda \sum_{k=1}^\infty \left(\frac{C(v_k)}{C(1)} \right)^{-\alpha+1-1/m} C(v_k)^{1/m}. \end{aligned}$$

A necessary condition [10] for optimality of \mathbf{v} is that the partial derivative of G with respect to v_j is 0, for all j . In other words

$$\begin{aligned} \frac{\partial G}{\partial v_j} &= \frac{\partial C(v_j)}{\partial v_j} C(1)^\alpha \left[C(v_{j-1})^{-\alpha} - \alpha C(v_{j+1}) C(v_j)^{-\alpha-1} \right. \\ &\quad \left. + \lambda(1-\alpha) C(v_j)^{-\alpha} C(1)^{\frac{1}{m}-1} \right] \\ &= 0. \end{aligned} \quad (53)$$

Because the derivative of the cost function is strictly positive and $C(1) > 0$, then (53) is satisfied if and only if the term inside the brackets is equal to 0. Setting this term equal to 0, letting $\bar{\lambda} = \lambda C(1)^{\frac{1}{m}-1}$ for notational convenience, and rearranging yields the following recursion for $j \geq 1$:

$$C(v_{j+1}) = \frac{C(v_j)}{\alpha} \left[\left(\frac{C(v_j)}{C(v_{j-1})} \right)^\alpha + \bar{\lambda}(1-\alpha) \right]. \quad (54)$$

Hence, any optimal strategy must satisfy the recursion given by (54). Let $\gamma_j = \frac{C(v_j)}{C(v_{j-1})}$ for all $j \geq 1$. This quantity indicates the relative amount of cost increase after every unsuccessful attempt. Then, (54) reduces to

$$\gamma_{j+1} = \frac{1}{\alpha} (\gamma_j^\alpha - \bar{\lambda}(\alpha-1)). \quad (55)$$

Note that the value of γ_1 uniquely defines the remaining values γ_j for all $j \geq 2$. At the same time, the entire sequence $\{\gamma_j\}$

uniquely defines the strategy. Hence, it remains to determine values of γ_1 that define optimal strategies.

Lemma 7: Fix $\bar{\lambda} \geq 0$. A necessary condition for optimality is that for all $j \geq 1$, $\gamma_j = \gamma$ where γ is the unique solution to the following equation:

$$\frac{1}{\alpha}\gamma^\alpha - \gamma = \bar{\lambda} \frac{(\alpha-1)}{\alpha}. \quad (56)$$

Hence, any optimal strategy for Problem 1 will have costs increasing geometrically by factor γ .

Proof: It should be noted for completeness that (56) has a unique solution because the function $\frac{1}{\alpha}x^\alpha - x$ is strictly increasing in x (this can be seen by differentiating with respect to x), is equal to 0 when $x = \alpha^{\frac{1}{\alpha-1}}$, is continuous, and increases to ∞ as $x \rightarrow \infty$. At the same time, $\bar{\lambda} \frac{(\alpha-1)}{\alpha}$ is a nonnegative finite quantity.

Note that if $\gamma_1 = \gamma$, then $\gamma_j = \gamma$ for all $j \geq 2$. Hence, it suffices to prove that $\gamma_1 = \gamma$ is necessary for optimality. We proceed by contradiction.

Case 1: $\gamma_1 > \gamma$.

Note that if $\gamma_j > \gamma$ for some j , then we have the following:

$$\begin{aligned} \gamma_{j+1} &= \frac{1}{\alpha}(\gamma_j^\alpha - \bar{\lambda}(\alpha-1)) = \frac{1}{\alpha}\gamma_j^\alpha - \gamma_j + \gamma_j - \frac{\bar{\lambda}(\alpha-1)}{\alpha} \\ &> \frac{1}{\alpha}\gamma^\alpha - \gamma - \frac{\bar{\lambda}(\alpha-1)}{\alpha} + \gamma_j \\ &= \frac{\bar{\lambda}(\alpha-1)}{\alpha} - \frac{\bar{\lambda}(\alpha-1)}{\alpha} + \gamma_j = \gamma_j \end{aligned}$$

where the last inequality follows from the fact that $\frac{1}{\alpha}x^\alpha - x$ is strictly increasing in x , as noted earlier. Hence, we have the following: if $\gamma_j > \gamma$ for some j , then $\gamma_{j+1} > \gamma_j$. This means that because $\gamma_1 > \gamma$, then $\gamma_2 > \gamma_1 > \gamma$, and so on. Hence, by induction, the $\{\gamma_j\}$ form a strictly increasing sequence, where $\gamma_j > \gamma$ for each j ; so for each $j \geq 1$ by rearranging the recursion (55)

$$\frac{\gamma_{j+1}}{\gamma_j^\alpha} = \frac{1}{\alpha} - \bar{\lambda} \frac{(\alpha-1)}{\gamma_j^\alpha} \geq \frac{1}{\alpha} - \bar{\lambda} \frac{(\alpha-1)}{\gamma^\alpha} = \gamma^{1-\alpha} \quad (57)$$

where the inequality holds because $\gamma_j^\alpha > \gamma^\alpha$, and the last equality holds from the definition of γ . The inequality becomes strict when $\bar{\lambda} > 0$.

Note that for any $j \geq 1$, we have by the definition of γ_j that $C(v_j) = C(1) \prod_{k=1}^j \gamma_k$. Hence, the expected cost of any such strategy is given by

$$\begin{aligned} J_X^{\mathbf{v}} &= \sum_{j=0}^{\infty} \bar{F}_X(v_j) C(v_{j+1}) = \sum_{j=0}^{\infty} \left(\frac{C(v_j)}{C(1)} \right)^{-\alpha} C(v_{j+1}) \\ &= \sum_{j=0}^{\infty} C(1) \gamma_1 \prod_{k=1}^j \frac{\gamma_{k+1}}{\gamma_k^\alpha} \end{aligned} \quad (58)$$

where the product is defined to be equal to 1 if $j = 0$.

We have shown that if $\gamma_1 > \gamma$, then $\frac{\gamma_{k+1}}{\gamma_k^\alpha} > \gamma^{1-\alpha}$ for all k . Hence, for any such strategy where $\gamma_1 > \gamma$, the expected cost is lower bounded by

$$\sum_{j=0}^{\infty} C(1) \gamma_1 \prod_{k=1}^j \frac{\gamma_{k+1}}{\gamma_k^\alpha} > \sum_{j=0}^{\infty} C(1) \gamma \prod_{k=1}^j \gamma^{1-\alpha}. \quad (59)$$

However, note that the right-hand side of (59) is simply the expected cost for a strategy such that $\gamma_j = \gamma$ for all j [plug $\gamma_j = \gamma$ into (58)]. Hence, from (59), any strategy where $\gamma_j > \gamma$ for all j has an expected cost strictly greater than using $\gamma_j = \gamma$, and these strategies cannot be optimal.

Case 2: $\gamma_1 < \gamma$.

Note that for any optimal strategy, $\gamma_j > 1$ for all j , because only strictly increasing IL sequences can be optimal. Hence, the sequence $\{\gamma_j\}$ is always lower bounded by 1. Note that if $\gamma_j < \gamma$ for some j , then we have the following:

$$\begin{aligned} \gamma_{j+1} &= \frac{1}{\alpha}(\gamma_j^\alpha - \bar{\lambda}(\alpha-1)) \\ &= \frac{1}{\alpha}\gamma_j^\alpha - \gamma_j + \gamma_j - \frac{\bar{\lambda}(\alpha-1)}{\alpha} \\ &< \frac{1}{\alpha}\gamma^\alpha - \gamma - \frac{\bar{\lambda}(\alpha-1)}{\alpha} + \gamma_j \\ &= \frac{\bar{\lambda}(\alpha-1)}{\alpha} - \frac{\bar{\lambda}(\alpha-1)}{\alpha} + \gamma_j = \gamma_j. \end{aligned}$$

Hence, if $\gamma_1 < \gamma$, then $\gamma_2 < \gamma_1 < \gamma$, and so on.

Because the $\{\gamma_j\}$ are bounded, then the sequence converges (since all monotonic bounded sequences converge). Let $\gamma_\infty = \lim_{j \rightarrow \infty} \gamma_j$. Because the γ_j are strictly less than γ and form a decreasing sequence, then $\gamma_\infty < \gamma$. On the other hand, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \gamma_{j+1} &= \lim_{j \rightarrow \infty} \left\{ \frac{1}{\alpha} [\gamma_j^\alpha - \bar{\lambda}(\alpha-1)] \right\} \\ \implies \gamma_\infty &= \frac{1}{\alpha} [\gamma_\infty^\alpha - \bar{\lambda}(\alpha-1)] \\ \implies \frac{1}{\alpha} \gamma_\infty^\alpha - \gamma_\infty &= \bar{\lambda} \frac{(\alpha-1)}{\alpha}. \end{aligned}$$

We defined γ as the unique number satisfying (56). Because we have just shown that γ_∞ is bounded and also satisfies the same equation, we have that $\gamma_\infty = \gamma$. However, this contradicts the fact that $\gamma_\infty < \gamma$, which we showed earlier. Hence, it is not possible to have $\gamma_1 < \gamma$ if the $\{\gamma_j\}$ are bounded.

Therefore, combining Case 1 and Case 2 proves that $\gamma_j = \gamma$ for all j is the only possibly optimal strategy for fixed $\bar{\lambda}$. \square

For any strategy \mathbf{v} where $\gamma_j = \gamma$, we have the following: $C(v_j) = C(1) \prod_{k=1}^j \gamma_k = C(1) \gamma^j$. Therefore, we have the following geometric sum, which converges because $\gamma > 1$ (necessary for increasing sequence) implies $0 < \gamma^{1-\alpha} < 1$

$$\begin{aligned} &\sum_{k=1}^{\infty} \bar{F}_Y(v_k) D(v_k) \\ &= C(1)^{1/m} \sum_{k=1}^{\infty} \left(\frac{C(v_k)}{C(1)} \right)^{1-\alpha} \\ &= C(1)^{1/m} \sum_{k=1}^{\infty} (\gamma^{1-\alpha})^k = \frac{C(1)^{1/m}}{\gamma^{\alpha-1} - 1}. \end{aligned} \quad (60)$$

Using (60) into (52), we can see that it is possible to achieve a delay ratio arbitrarily close to 1 by choosing a sufficiently high enough value of γ . Therefore, for every $d > 1$, there exists a strategy achieving delay ratio below d . Hence, the optimal strategy for Problem 1 must satisfy the Kuhn–Tucker condition (see [10] and [11])

$$\bar{\lambda} \left(\sum_{k=1}^{\infty} \bar{F}_Y(v_k) D(v_k) - (d-1)E[D(Y)] \right) = 0. \quad (61)$$

Therefore, either $\bar{\lambda} = 0$ or the delay constraint is satisfied with equality. We use this to prove solutions for two cases, depending on whether $1 < d < m+1$ or $d \geq m+1$.

Case 1: $1 < d < m+1$.

When $\bar{\lambda} = 0$, then $\lambda = 0$ and we have an unconstrained optimization problem. In this case, $\gamma = \alpha^{\frac{1}{\alpha-1}}$ from Lemma 7. The summation in (60) is then equal to $C(1)^{1/m}/(\alpha-1)$ for this value of γ . From (52), we know that dividing this summation by (51) and then adding 1 gives the delay ratio

$$\frac{D_Y^v}{E[D(Y)]} = \frac{\sum_{k=1}^{\infty} \bar{F}_Y(v_k) D(v_k)}{E[D(Y)]} + 1 = \frac{m}{1 + (\alpha-1)m} + 1.$$

From inequality (33) on α , this delay ratio is thus strictly greater than d . Hence, this strategy does not meet the delay inequality requirement.

Therefore, we seek solutions for which the delay constraint is met with equality, i.e., the term inside the brackets of (61) is equal to 0. In this case, (60) needs to be equal to $(d-1)C(1)^{\frac{1}{m}} \frac{(\alpha-1)m+1}{(\alpha-1)m}$, and solving for γ gives

$$\gamma = \left(1 + \frac{(\alpha-1)m}{[(\alpha-1)m+1](d-1)} \right)^{\frac{1}{\alpha-1}}. \quad (62)$$

From the earlier (56) relating γ and $\bar{\lambda}$, we have that $\bar{\lambda}$ can be calculated as

$$\bar{\lambda} = \gamma \left(\frac{\gamma^{\alpha-1} - \alpha}{\alpha-1} \right) = \gamma \left(\frac{m}{[(\alpha-1)m+1](d-1)} - 1 \right).$$

The cost ratio can be calculated by multiplying both sides of (54) by $\bar{F}_X(v_j)$ and then summing over $j \geq 1$ to give

$$\begin{aligned} & \sum_{j=1}^{\infty} \bar{F}_X(v_j) C(v_{j+1}) \\ &= \frac{1}{\alpha} \sum_{j=0}^{\infty} \bar{F}_X(v_j) C(v_{j+1}) - \bar{\lambda} \frac{\alpha-1}{\alpha} \sum_{j=1}^{\infty} C(v_j) \bar{F}_X(v_j). \end{aligned}$$

The left-hand side sum is simply $J_X^v - C(v_1)$, so rearranging and solving for J_X^v gives

$$\begin{aligned} J_X^v \frac{\alpha-1}{C(1)\alpha} &= \frac{C(v_1)}{C(1)} - \bar{\lambda} \frac{\alpha-1}{C(1)\alpha} \sum_{j=1}^{\infty} C(v_j) \bar{F}_X(v_j) \\ \implies \frac{J_X^v}{E[C(X)]} &= \frac{C(v_1)}{C(1)} - \bar{\lambda} \frac{\sum_{j=1}^{\infty} C(v_j) \bar{F}(v_j)}{E[C(X)]}. \quad (63) \end{aligned}$$

To evaluate this ratio, note that

$$\begin{aligned} \sum_{j=1}^{\infty} C(v_j) \bar{F}(v_j) &= C(1) \sum_{k=1}^{\infty} \left(\frac{C(v_k)}{C(1)} \right)^{1-\alpha} = \frac{C(1)}{\gamma^{\alpha-1} - 1} \\ &= C(1)(d-1) \left(1 + \frac{1}{(\alpha-1)m} \right). \quad (64) \end{aligned}$$

Dividing by $E[C(X)]$ from (50) gives

$$\frac{\sum_{j=1}^{\infty} C(v_j) \bar{F}(v_j)}{E[C(X)]} = (d-1) \left(\frac{(\alpha-1)m+1}{m\alpha} \right). \quad (65)$$

Using (62), the corresponding $\bar{\lambda}$, and (65) into (63) gives

$$\begin{aligned} \frac{J_X^v}{E[C(X)]} &= \frac{C(v_1)}{C(1)} - \bar{\lambda}(d-1) \left(\frac{(\alpha-1)m+1}{m\alpha} \right) \\ &= \gamma - \gamma \left(\frac{m}{[(\alpha-1)m+1](d-1)} - 1 \right) \\ &\quad \times (d-1) \left(\frac{(\alpha-1)m+1}{m\alpha} \right) \\ &= \left[\frac{d-1}{m\alpha} + d \left(\frac{\alpha-1}{\alpha} \right) \right] \gamma. \quad (66) \end{aligned}$$

Hence, (66) gives the optimal cost ratio when $1 < d < m+1$.

Case 2: $d \geq m+1$.

As explained in Case 1, it follows from Lemma 7 that when $\bar{\lambda} = 0$ (the unconstrained case), then using $\gamma = \alpha^{\frac{1}{\alpha-1}}$ is optimal. Because this strategy is the optimal unconstrained strategy, it achieves minimum average cost when it satisfies the delay constraint. It was shown that the delay ratio for this strategy is $1 + \frac{m}{1+(\alpha-1)m}$, which is always strictly less than $m+1$ for all $\alpha > 1$. Hence, for $d \geq m+1$, the delay constraint is not binding and the optimal strategy uses $\gamma = \alpha^{\frac{1}{\alpha-1}}$. From (63), the optimal cost ratio is $\alpha^{\frac{1}{\alpha-1}}$ because $\bar{\lambda} = 0$.

B. Proof of Lemma 3

Because $C(v_k) = r^{k-1}C(v_1)$ for all positive integers k , then $D(v_k) = r^{\frac{k-1}{m}}D(v_1)$. Let $S_k = D(v_1) + D(v_2) + \dots + D(v_k)$. Then, for any k

$$\begin{aligned} E[S_k] &= E \left[\sum_{j=1}^k D(v_j) \right] = E \left[\sum_{j=1}^k r^{\frac{j-1}{m}} D(v_1) \right] \\ &= \sum_{j=0}^{k-1} r^{\frac{j}{m}} E[D(v_1)] = E[D(v_1)] \frac{r^{\frac{k}{m}} - 1}{r^{\frac{1}{m}} - 1}. \quad (67) \end{aligned}$$

For any $1 \leq l < C^{-1}(rC(1))$

$$\begin{aligned} & E[D(v_1)I_{\{v_1 \leq l\}}] \\ &= P(v_1 \leq l)E[D(v_1)|v_1 \leq l] \\ &= P(v_1 \leq l) \int_0^{\infty} Pr(D(v_1) > y | v_1 \leq l) dy \\ &= F_{v_1}(l)D(1) + \int_{D(1)}^{\infty} Pr(D^{-1}(y) < v_1 \leq l) dy \\ &= F_{v_1}(l)D(1) + \int_{D(1)}^{D(l)} [\bar{F}_{v_1}(D^{-1}(y)) - \bar{F}_{v_1}(l)] dy \\ &= D(1) + \int_{D(1)}^{D(l)} \bar{F}_{v_1}(D^{-1}(y)) dy - D(l) \cdot \bar{F}_{v_1}(l). \quad (68) \end{aligned}$$

We use the following notation. Let $D_x^{\mathbf{v}}|_{v_n > x}$ denote the expected delay of using strategy \mathbf{v} when $v_n > x$. Similarly, $D_x^{\mathbf{v}}|_{v_n \leq x}$ is the expected delay given that $v_n \leq x$. Now consider any real number $x \geq 1$. There must exist a positive integer n such that $r^{n-1}C(1) \leq C(x) < r^n C(1)$. Equivalently, $r^{\frac{n-1}{m}} D(1) \leq D(x) < r^{\frac{n}{m}} D(1)$. Then, we have

$$\begin{aligned} D_x^{\mathbf{v}} &= D_x^{\mathbf{v}}|_{v_n > x} Pr(v_n > x) + D_x^{\mathbf{v}}|_{v_n \leq x} Pr(v_n \leq x) \\ &= E[S_{n-1} + D(x)|v_n > x] Pr(v_n > x) \\ &\quad + E[S_n + D(x)|v_n \leq x] Pr(v_n \leq x) \\ &= E[S_{n-1}] + D(x) + E[D(v_n)I_{\{v_n \leq x\}}] \\ &= \frac{r^{\frac{n-1}{m}} - 1}{r^{\frac{1}{m}} - 1} E[D(v_1)] + D(x) \\ &\quad + r^{\frac{n-1}{m}} E \left[D(v_1) I_{\left\{ v_1 \leq D^{-1} \left(\frac{D(x)}{r^{\frac{n-1}{m}}} \right) \right\}} \right]. \end{aligned} \quad (69)$$

Using (68), we obtain the following:

$$\begin{aligned} D_x^{\mathbf{v}} &= \frac{r^{\frac{n}{m}} - 1}{r^{\frac{1}{m}} - 1} E[D(v_1)] + D(x) + r^{\frac{n-1}{m}} \cdot D(1) \\ &\quad + r^{\frac{n-1}{m}} \int_{D(1)}^{\frac{D(x)}{r^{\frac{n-1}{m}}}} \bar{F}_{v_1}(D^{-1}(y)) dy \\ &\quad - D(x) \bar{F}_{v_1} \left(D^{-1} \left(\frac{D(x)}{r^{\frac{n-1}{m}}} \right) \right) \\ &= \frac{r^{\frac{n-1}{m}}}{r^{\frac{1}{m}} - 1} E[D(v_1)] + r^{\frac{n-1}{m}} D(1) \\ &\quad + r^{\frac{n-1}{m}} \int_{D(1)}^{\frac{D(x)}{r^{\frac{n-1}{m}}}} \bar{F}_{v_1}(D^{-1}(y)) dy + D(x) \\ &\quad - D(x) \bar{F}_{v_1} \left(D^{-1} \left(\frac{D(x)}{r^{\frac{n-1}{m}}} \right) \right) - \frac{E[D(v_1)]}{r^{\frac{1}{m}} - 1}. \end{aligned} \quad (70)$$

Letting $z = \frac{D(x)}{r^{\frac{n-1}{m}} D(1)}$, we obtain the following:

$$\begin{aligned} \frac{D_x^{\mathbf{v}}}{D(x)} &= \frac{1}{r^{\frac{1}{m}} - 1} \left[\frac{E[D(v_1)]}{zD(1)} \right] + 1 - \bar{F}_{v_1}(D^{-1}(zD(1))) \\ &\quad + \frac{D(1) + \int_{D(1)}^{zD(1)} \bar{F}_{v_1}(D^{-1}(y)) dy}{zD(1)} \\ &\quad - \frac{E[D(v_1)]}{(r^{\frac{1}{m}} - 1)zr^{\frac{n-1}{m}} D(1)} \\ &= \frac{1}{r^{\frac{1}{m}} - 1} \left[\frac{g(r^{\frac{1}{m}}) + (r^{\frac{1}{m}} - 1)g(z)}{zD(1)} \right] \\ &\quad + 1 - \frac{g'(z)}{D(1)} - \frac{g(r^{\frac{1}{m}})}{(r^{\frac{1}{m}} - 1)zr^{\frac{n-1}{m}} D(1)} \end{aligned} \quad (71)$$

where we have used the fact that $g(r^{\frac{1}{m}}) = E[D(v_1)]$ (by the relationship between expectation and tail distribution), and $g'(z) = \bar{F}_{v_1}(D^{-1}(z \cdot D(1))) \cdot D(1)$ by the fundamental theorem of calculus. Note that the following is true for any x and y , where $y = D^{-1}(r^{\frac{1}{m}} D(x))$: $\frac{D_x^{\mathbf{v}}}{D(x)} < \frac{D_y^{\mathbf{v}}}{D(y)}$. This statement holds because all terms except for the last one in (71) are the same for x and y , and the last term increases with increasing x . In addition, when x ranges from $D^{-1}(r^{\frac{n-1}{m}} D(1))$ to $D^{-1}(r^{\frac{n}{m}} D(1))$, then z takes values between 1 and $r^{\frac{1}{m}}$.

Hence, the following is true, where $x_n = D^{-1}(r^{\frac{n}{m}} D(1))$ for notation:

$$\begin{aligned} &\sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{v}}}{D(x)} \\ &= \sup_{n \in \mathbb{Z}^+} \left\{ \sup_{x_{n-1} \leq x < x_n} \frac{D_x^{\mathbf{v}}}{D(x)} \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \sup_{x_{n-1} \leq x < x_n} \frac{D_x^{\mathbf{v}}}{D(x)} \right\} \\ &= \sup_{1 \leq z < r^{\frac{1}{m}}} \left\{ \frac{1}{r^{\frac{1}{m}} - 1} \frac{g(r^{\frac{1}{m}}) + (r^{\frac{1}{m}} - 1)g(z)}{zD(1)} - \frac{g'(z)}{D(1)} \right\} + 1 \end{aligned}$$

because only the last (rightmost) term in (71) depends on n , and as x goes to ∞ , then this term goes to 0. \square

C. Proof of Lemma 4

First note from Definition 3 that $C(v_k) = r^{k-1}C(v_1)$ for all positive integers k . Let $S_k = C(v_1) + C(v_2) + \dots + C(v_k)$ for any $k \geq 1$. Note that $E[S_k]$ can be calculated from (67) when $C(\cdot) = D(\cdot)$, i.e., $m = 1$. Thus, $E[S_k] = E[C(v_1)] \frac{r^{\frac{k}{m}} - 1}{r^{\frac{1}{m}} - 1}$. Similarly, we can calculate $E[C(v_1)I_{\{v_1 \leq l\}}]$ from (68) using $m = 1$.

We now use the following notation. Let $J_x^{\mathbf{v}}|_{v_n > x}$ denote the expected cost of using strategy \mathbf{v} when $v_n > x$. Similarly, $J_x^{\mathbf{v}}|_{v_n \leq x}$ is the expected cost given that $v_n \leq x$.

Now consider any real number $x \geq 1$. There must exist a positive integer n such that $r^{n-1}C(1) \leq C(x) < r^n C(1)$, or in other words, $C^{-1}(r^{n-1})C(1) \leq x < C^{-1}(r^n C(1))$. Then, the expected cost can be calculated using the earlier equation for $E[S_k]$

$$\begin{aligned} J_x^{\mathbf{v}} &= J_x^{\mathbf{v}}|_{v_n > x} Pr(v_n > x) + J_x^{\mathbf{v}}|_{v_n \leq x} Pr(v_n \leq x) \\ &= E[S_n|v_n > x] Pr(v_n > x) \\ &\quad + E[S_n|v_n \leq x] Pr(v_n \leq x) \\ &\quad + E[C(v_{n+1})|v_n \leq x] Pr(v_n \leq x) \\ &= E[S_n] + E[C(v_{n+1})|v_n \leq x] Pr(v_n \leq x) \\ &= \frac{r^n - 1}{r - 1} E[C(v_1)] + r^n E[C(v_1)I_{\{v_1 \leq C^{-1}(C(x)/r^{n-1})\}}]. \end{aligned}$$

Comparing this to (69) for the case $m = 1$, which implies $C(\cdot) = D(\cdot)$, we have the following relationship between $J_x^{\mathbf{v}}$ and $D_x^{\mathbf{v}}$, with $z = \frac{C(x)}{r^{n-1}C(1)}$:

$$J_x^{\mathbf{v}} = r(D_x^{\mathbf{v}} - D(x)) + E[C(v_1)]. \quad (72)$$

Thus, combining this relationship with (71) for $m = 1$, we obtain the following:

$$\begin{aligned} \frac{J_x^{\mathbf{v}}}{C(x)} &= \frac{r}{r-1} \frac{h(r) + (r-1)h(z)}{zC(1)} \\ &\quad - r \frac{h'(z)}{C(1)} - \frac{h(r)}{(r-1)zr^{n-1}C(1)}. \end{aligned} \quad (73)$$

Note the following is true for any x and $y = C^{-1}(rC(x))$: $\frac{J_x^{\mathbf{v}}}{C(x)} < \frac{J_y^{\mathbf{v}}}{C(y)}$. This statement holds because the first two terms for the cost ratio above are the same for x and y , and the last term

increases with increasing x . In addition, when x ranges from $C^{-1}(r^{n-1}C(1))$ to $C^{-1}(r^n C(1))$, then z takes values between 1 and r . Hence, the following is true, where $x_n = C^{-1}(r^n C(1))$ for notation:

$$\begin{aligned} & \sup_{x \in [1, \infty)} \frac{J_x^v}{C(x)} \\ &= \sup_{n \in \mathbb{Z}^+} \left\{ \sup_{x_{n-1} \leq x < x_n} \frac{J_x^v}{C(x)} \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \sup_{x_{n-1} \leq x < x_n} \frac{J_x^v}{C(x)} \right\} \\ &= \sup_{1 \leq z < r^{\frac{1}{m}}} \left\{ \frac{r}{r-1} \frac{h(r) + (r-1)h(z)}{zC(1)} - r \frac{h'(z)}{C(1)} \right\} \end{aligned}$$

which completes the proof of the lemma.

D. Proof of Lemma 5

Fix any $x \in \mathbb{Z}^+$. Note that $x > \lfloor v_k \rfloor$ if and only if $x > v_k$, because x is an integer. Therefore, for all k , $I_{\{x > \lfloor v_k \rfloor\}} = I_{\{x > v_k\}}$ w.p.1 (with probability 1). In addition, $D(\lfloor v_k \rfloor) \leq D(v_k)$ w.p.1, because the delay function is increasing. This gives us

$$\begin{aligned} D_x^{\lfloor v \rfloor} &= D(x) + E \left[\sum_{k=1}^{\infty} I(x > \lfloor v_k \rfloor) D(\lfloor v_k \rfloor) \right] \\ &\leq D(x) + E \left[\sum_{k=1}^{\infty} I(x > v_k) D(v_k) \right] = D_x^v. \end{aligned} \quad (74)$$

This proves the delay part of the lemma. Similarly, $C(\cdot)$ being increasing implies $C(\lfloor v_k \rfloor) \leq C(v_k)$ w.p.1. Therefore

$$\begin{aligned} J_x^{\lfloor v \rfloor} &= E \left[\sum_{k=1}^{\infty} I(x > \lfloor v_{k-1} \rfloor) C(\lfloor v_k \rfloor) \right] \\ &\leq E \left[\sum_{k=1}^{\infty} I(x > v_{k-1}) C(v_k) \right] = J_x^v \end{aligned} \quad (75)$$

which establishes the inequality on the expected cost. \square

E. Proof of Theorem 2

For $1 < d < m + 1$, consider the strategy $\mathbf{v}^* \left[e^{\frac{m}{d-1}}, \frac{d-1}{m} \ln \frac{C(\cdot)}{C(1)} \right]$ of Theorem 1. Let $\mathbf{u}^* = \lfloor \mathbf{v}^* \rfloor$. From Lemma 5, we have that $\mathbf{v}^* \in V_d$ implies $\mathbf{u}^* \in U_d$. Also from Lemma 5, $J_x^{\mathbf{u}^*} \leq J_x^{\mathbf{v}^*}$ for all integers x . From Theorem 1, the worst-case ratio of \mathbf{v}^* is $\frac{d-1}{m} e^{\frac{m}{d-1}}$. Hence, the worst-case cost ratio of \mathbf{u}^* over all integers is less than or equal to $\frac{d-1}{m} e^{\frac{m}{d-1}}$, which establishes the theorem for $1 < d < m + 1$.

For $d \geq m + 1$, similar steps can be applied to the floor of $\mathbf{v}^* \left[e, \ln \frac{C(\cdot)}{C(1)} \right]$ to establish the theorem. \square

F. Proof of Theorem 3

Because $\mathbb{C}_1 \subset \mathbb{C}$, then from Theorem 2, the best worst-case cost ratio is upper bounded as follows.

1) For $1 < d < m + 1$

$$\inf_{\mathbf{u} \in U_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq \frac{(d-1)}{m} e^{\frac{m}{d-1}}. \quad (76)$$

2) For $d \geq m + 1$

$$\inf_{\mathbf{u} \in U_d} \sup_{\{p_X(x)\}} \frac{J_X^{\mathbf{u}}}{E[C(X)]} \leq e. \quad (77)$$

Hence, to prove the theorem, it suffices to prove that the minimum worst cost ratio is upper bounded by the above quantities.

Consider any $\mathbf{u} \in U$. For any integer $x \geq 2$

$$\frac{J_x^{\mathbf{u}}}{C(x)} = \lim_{\epsilon \rightarrow 0} \frac{J_{x-1+\epsilon}^{\mathbf{u}}}{C(x+\epsilon)} = \sup_{y \in [x-1, x)} \frac{J_y^{\mathbf{u}}}{C(y+1)} \quad (78)$$

because $J_{x-1+\epsilon}^{\mathbf{u}} = J_x^{\mathbf{u}}$ for all $0 < \epsilon \leq 1$, and $C(\cdot)$ is strictly increasing and continuous. Hence, we have

$$\begin{aligned} \sup_{x \in \mathbb{Z}^+} \frac{J_x^{\mathbf{u}}}{C(x)} &= \sup \left\{ \frac{J_1^{\mathbf{u}}}{C(1)}, \sup_{x \in \mathbb{Z}^+, x \geq 2} \sup_{y \in [x-1, x)} \frac{J_y^{\mathbf{u}}}{C(y+1)} \right\} \\ &= \sup \left\{ \frac{J_1^{\mathbf{u}}}{C(1)}, \sup_{x \in [1, \infty)} \frac{J_x^{\mathbf{u}}}{C(x+1)} \right\}. \end{aligned} \quad (79)$$

We can repeat the above steps for the delay to obtain for any $\mathbf{u} \in U$

$$\begin{aligned} \sup_{x \in \mathbb{Z}^+} \frac{D_x^{\mathbf{u}}}{D(x)} &= \sup \left\{ \frac{D_1^{\mathbf{u}}}{D(1)}, \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{u}}}{D(x+1)} \right\} \\ &= \sup_{x \in [1, \infty)} \frac{D_x^{\mathbf{u}}}{D(x+1)} \end{aligned} \quad (80)$$

because $\frac{D_1^{\mathbf{u}}}{D(1)} = 1$.

Next, it can be shown similarly to Lemmas 1 and 2

$$\sup_{\{f_X(x)\}} \frac{J_X^v}{E[C(X+1)]} = \sup_{x \in [1, \infty)} \frac{J_x^v}{C(x+1)}, \quad (81)$$

$$\sup_{\{f_X(x)\}} \frac{D_X^v}{E[D(X+1)]} = \sup_{x \in [1, \infty)} \frac{D_x^v}{D(x+1)}. \quad (82)$$

Define \tilde{V}_d similarly to V_d in (8), but replacing $D(X)$ with $D(X+1)$. Also, define $\tilde{V}_d(Y)$ similarly to $V_d(Y)$ in (30) but replacing $D(X)$ with $D(X+1)$. In other words

$$\tilde{V}_d = \left\{ \mathbf{v} \in V : \sup_{x \in [1, \infty)} \frac{D_x^v}{D(x+1)} \leq d \right\} \quad (83)$$

$$\tilde{V}_d(Y) = \left\{ \mathbf{v} \in V : \frac{D_Y^v}{E[D(Y+1)]} \leq d \right\}. \quad (84)$$

Note that from (80) and $U \subseteq V$, we have that $U_d \subseteq \tilde{V}_d$. Therefore, if we can prove a lower bound on the best cost ratio in \tilde{V}_d , the same bound will also apply to U_d . To proceed, we use the following:

$$\begin{aligned} & \sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in \tilde{V}_d(Y)} \frac{J_X^v}{E[C(X+1)]} \\ & \leq \sup_{\{f_X(x)\}} \inf_{\mathbf{v} \in \tilde{V}_d} \frac{J_X^v}{E[C(X+1)]} \\ & \leq \inf_{\mathbf{v} \in \tilde{V}_d} \sup_{x \in [1, \infty)} \frac{J_x^v}{E[C(x+1)]}. \end{aligned} \quad (85)$$

We will use this inequality to obtain a tight upper bound. In order to do this, we use the following.

Lemma 8: Let X_α be the random variable with tail distribution $P(X_\alpha > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha}$, and let Y_α be the random variable with $P(Y_\alpha > x) = \left(\frac{C(x)}{C(1)}\right)^{-\alpha+1-\frac{1}{m}}$, for some $\alpha > 1$ and for all $x \geq 1$. Then, we have for $C(\cdot) \in \mathbb{C}_1$

$$\lim_{\alpha \rightarrow 1^+} \frac{E[C(X_\alpha + 1)]}{E[C(X_\alpha)]} = q \tag{86}$$

$$\lim_{\alpha \rightarrow 1^+} \frac{E[D(Y_\alpha + 1)]}{E[D(Y_\alpha)]} = q^{\frac{1}{m}}. \tag{87}$$

Proof: Because $\frac{C(x+1)}{C(x)} \geq q$ for all x , we have that $\frac{E[C(X_\alpha+1)]}{E[C(X_\alpha)]} \geq q$ for all α . Hence, to complete the proof, we need to show that for any $\epsilon > 0$, there exists $\bar{\alpha}$ such that $\frac{E[C(X_\alpha+1)]}{E[C(X_\alpha)]} < q + \epsilon$ for all $1 < \alpha < \bar{\alpha}$.

Fix $\epsilon > 0$. Because $\lim_{x \rightarrow \infty} \frac{C(x+1)}{C(x)} = q$, there exists a x^* such that $\frac{C(x+1)}{C(x)} < q + \frac{\epsilon}{2}$ for all $x > x^*$. Let $1(\cdot)$ denote the indicator function, so $1(A) = 1$ if A is true; otherwise, it equals 0. Thus, we have

$$\begin{aligned} E[C(X_\alpha + 1)I_{\{X_\alpha > x^*\}}] &< \left(q + \frac{\epsilon}{2}\right) E[C(X_\alpha)I_{\{X_\alpha > x^*\}}] \\ &\leq \left(q + \frac{\epsilon}{2}\right) E[C(X_\alpha)]. \end{aligned} \tag{88}$$

At the same time, we have

$$\lim_{\alpha \rightarrow 1^+} \frac{E[C(X_\alpha + 1)1_{\{X_\alpha \leq x^*\}}]}{E[C(X_\alpha)]} \leq \lim_{\alpha \rightarrow 1^+} \frac{C(x^* + 1)}{E[C(X_\alpha)]} = 0$$

because $C(x^* + 1) < \infty$ and $E[C(X_\alpha)] = \frac{\alpha}{\alpha-1}$, which approaches ∞ as α goes to 1. Hence, there exists an $\bar{\alpha}$ such that for all $1 < \alpha < \bar{\alpha}$

$$\frac{E[C(X_\alpha + 1)I_{\{X_\alpha \leq x^*\}}]}{E[C(X_\alpha)]} < \frac{\epsilon}{2}. \tag{89}$$

Therefore, combining (88) and (89) gives for all $1 < \alpha < \bar{\alpha}$

$$\begin{aligned} \frac{E[C(X_\alpha + 1)]}{E[C(X_\alpha)]} &= \frac{E[C(X_\alpha + 1)I_{\{X_\alpha > x^*\}}]}{E[C(X_\alpha)]} \\ &\quad + \frac{E[C(X_\alpha + 1)I_{\{X_\alpha \leq x^*\}}]}{E[C(X_\alpha)]} \\ &< q + \frac{\epsilon}{2} + \frac{\epsilon}{2} = q + \epsilon \end{aligned} \tag{90}$$

which completes the proof for the cost function.

Note that because $C(\cdot) \in \mathbb{C}_q$ and $D(\cdot) = C(\cdot)^{1/m}$, then $D(\cdot) \in \mathbb{C}_{q^{1/m}}$. Hence, we can repeat the above steps, replacing X_α with Y_α and $C(\cdot)$ with $D(\cdot)$ to prove the delay portion of the lemma. \square

By modifying Problem 1 and using the above lemma, we obtain the following tight upper-bound.

Problem 3: Define X_α and Y_α as in Lemma 8

$$\inf_{\mathbf{v}} \frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha + 1)]} \text{ s.t. } \frac{D_{Y_\alpha}^{\mathbf{v}}}{E[D(Y_\alpha + 1)]} \leq d.$$

Solution: Similar to Problem 1, the optimal cost ratio for this problem satisfies the following.

1) For $1 < d < m + 1$

$$\lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha + 1)]} = \frac{(d-1)}{m} e^{\frac{m}{d-1}} \tag{91}$$

where the limit is reached from below.

2) For $d \geq m + 1$, then

$$\lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha + 1)]} = e. \tag{92}$$

Proof: Because only the numerator of the cost ratio depends on \mathbf{v} , then Problem 3 is equivalent to the following:

$$\begin{aligned} \inf_{\mathbf{v}} J_{X_\alpha}^{\mathbf{v}} \text{ s.t. } \sum_{k=1}^{\infty} \bar{F}_{Y_\alpha}(v_k) D(v_k) \\ \leq \left(d \frac{E[D(Y_\alpha + 1)]}{E[D(Y_\alpha)]} - 1\right) E[D(Y_\alpha)]. \end{aligned}$$

We note that for α sufficiently close to 1, the above problem can be solved by using the solution to Problem 1. To see this, define the following:

$$\tilde{d} = d \frac{E[D(Y_\alpha + 1)]}{E[D(Y_\alpha)]}. \tag{93}$$

Note that $\lim_{\alpha \rightarrow 1^+} \tilde{d} = d$ from Lemma 8.

If $1 < d < m + 1$, then this implies

$$\lim_{\alpha \rightarrow 1^+} (\tilde{d} - 1)[1 + (\alpha - 1)m] = d - 1 < m. \tag{94}$$

Hence, there must exist an $\bar{\alpha}$ such that for all $1 < \alpha < \bar{\alpha}$, we have $(\tilde{d} - 1)[1 + (\alpha - 1)m] < m$. Rearranging this gives for all $1 < \alpha < \bar{\alpha}$

$$1 < \alpha < 1 + \frac{m + 1 - \tilde{d}}{m(\tilde{d} - 1)} \tag{95}$$

which is equivalent to (33), with \tilde{d} replacing d . Note also that $(\tilde{d} - 1)[1 + (\alpha - 1)m] < m$ implies $\tilde{d} < m + 1$. Hence, the above problem is equivalent to part 1) of Problem 1, with \tilde{d} replacing d , so using (66), we have the following for the optimal strategy:

$$\frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha)]} = \left[\frac{\tilde{d} - 1}{m\alpha} + \tilde{d} \left(\frac{\alpha - 1}{\alpha} \right) \right] \gamma. \tag{96}$$

Therefore, we have the following limit:

$$\begin{aligned} \lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha + 1)]} \\ = \lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^{\mathbf{v}}}{E[C(X_\alpha)]} \frac{E[C(X_\alpha)]}{E[C(X_\alpha + 1)]} \\ = \lim_{\alpha \rightarrow 1^+} \left[\frac{\tilde{d} - 1}{m\alpha} + \tilde{d} \left(\frac{\alpha - 1}{\alpha} \right) \right] \gamma \cdot \lim_{\alpha \rightarrow 1^+} \frac{E[C(X_\alpha)]}{E[C(X_\alpha + 1)]} \\ = \frac{d-1}{m} e^{\frac{m}{d-1}}. \end{aligned}$$

This establishes (91).

If $d \geq m + 1$, then $\tilde{d} > d > m + 1$. Hence, we can use the solution to part (2) of Problem 1, to obtain

$$\begin{aligned} & \lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^v}{E[C(X_\alpha + 1)]} \\ &= \lim_{\alpha \rightarrow 1^+} \frac{J_{X_\alpha}^v}{E[C(X_\alpha)]} \frac{E[C(X_\alpha)]}{E[C(X_\alpha + 1)]} \\ &= \lim_{\alpha \rightarrow 1^+} \alpha^{\frac{1}{\alpha-1}} \cdot \lim_{\alpha \rightarrow 1^+} \frac{E[C(X_\alpha)]}{E[C(X_\alpha + 1)]} = e \end{aligned}$$

which establishes (92). \square

Plugging this result into (85) gives that for $1 < d < m + 1$

$$\inf_{v \in \tilde{V}_d} \sup_{x \in [1, \infty)} \frac{J_x^v}{C(x+1)} \geq \frac{d-1}{m} e^{\frac{m}{d-1}}. \quad (97)$$

Finally, using the fact that $U_d \subseteq \tilde{V}_d$ gives

$$\inf_{u \in U_d} \sup_{x \in \mathbb{Z}^+} \frac{J_x^v}{C(x)} = \inf_{u \in U_d} \sup_{x \in [1, \infty)} \frac{J_x^v}{C(x+1)} \geq \frac{d-1}{m} e^{\frac{m}{d-1}}.$$

Combining this with Lemma 1 and Theorem 2 proves the theorem for $1 < d < m + 1$.

Similar steps can be applied for the case $d \geq m + 1$.

G. Proof of Theorem 4

We note that Theorem 4 follows from Theorem 1 as follows. Consider any $c \geq e$, and let a_c be the unique number in $[1, \infty)$ satisfying $a_c e^{a_c} = c$. Recall that \hat{V}_c is the set of strategies whose worst-case cost is less than c . From Theorem 1, we can see that the worst-case delay of any strategy in \hat{V}_c must be greater than $ma_c + 1$ (otherwise, there is a contradiction). Thus, we have

$$\inf_{v \in \hat{V}_c} \sup_{\{f_X(x)\}} \frac{D_X^v}{E[D(X)]} \geq ma_c + 1. \quad (98)$$

Now consider strategy $v^* \left[r, \frac{1}{\ln r} \ln \frac{C(\cdot)}{C(1)} \right]$ with $r = e^{\frac{1}{a_c}}$. We have shown earlier that this strategy has worst-case cost ratio of $a_c e^{a_c} = c$ and worst-case delay ratio of $ma_c + 1$. Therefore, this strategy is in \hat{V}_c and achieves the lower bound established in (98), therefore proving the first half of Theorem 4.

The second half of Theorem 4 can be proven using similar steps as the proof of Theorem 2.

H. Sketch of the Proof of Theorem 7

First, using the same steps for proving Lemma 5, we can show that for any strategy v , and any $x \in \mathbb{Z}^+$, we have $T_x^{[v]} \leq T_x^v$ and $H_x^{[v]} \leq H_x^v$. Thus, it follows from Theorem 6 (by applying the same reasoning used to prove Theorem 2) that we have the following bound on the optimal discrete strategy:

$$\inf_{u \in \tilde{U}_d} \sup_{\{p_X(x)\}} \frac{H_X^u}{E[C(X)]} \leq (1 + \zeta) \frac{(d-1)}{m} \left[e^{\frac{m}{d-1}} - 1 \right] \quad (99)$$

where strategy u^* described by Theorem 7 has worst-case cost matching the above bound. Thus, to complete the proof, we need to show that the minimum worst-case ratio is upper bounded by the above quantity.

Similar to (78) and (79), we have the following:

$$\begin{aligned} \sup_{x \in \mathbb{Z}^+} \frac{H_x^u}{C(x)} &= \sup \left\{ \frac{H_1^u}{C(1)}, \sup_{x \in \mathbb{Z}^+, x \geq 2} \sup_{y \in [x-1, x)} \frac{J_y^u}{C(y+1)} \right\} \\ &= \sup \left\{ \frac{H_1^u}{C(1)}, \sup_{x \in [1, \infty)} \frac{H_x^u}{C(x+1)} \right\}. \quad (100) \end{aligned}$$

Furthermore, (80) also holds for any u because $T_x^y = D_x^y$ for all x .

Define \tilde{V}_d^+ similarly to \tilde{V}_d in (83), and $\tilde{V}_d^+(Y)$ similarly to $\tilde{V}_d(Y)$ in (84), but replacing D_x^y with T_x^y in both definitions. Note that from (80) and $U \subseteq V$, we have that $\tilde{U}_d \subseteq \tilde{V}_d^+$. Therefore, if we can prove a lower bound on the best cost ratio in \tilde{V}_d^+ , the same bound will also apply to \tilde{U}_d . To proceed, we use the following:

$$\begin{aligned} & \sup_{\{f_X(x)\}} \inf_{v \in \tilde{V}_d^+(Y)} \frac{H_X^v}{E[C(X+1)]} \\ & \leq \sup_{\{f_X(x)\}} \inf_{v \in \tilde{V}_d^+} \frac{H_X^v}{E[C(X+1)]} \\ & \leq \inf_{v \in \tilde{V}_d^+} \sup_{x \in [1, \infty)} \frac{H_x^v}{E[C(x+1)]}. \quad (101) \end{aligned}$$

To use this inequality to obtain a tight upper bound, we consider the following problem.

Problem 4: Define X_α and Y_α as in Lemma 8

$$\inf_v \frac{H_{X_\alpha}^v}{E[C(X_\alpha + 1)]} \text{ s.t. } \frac{T_{Y_\alpha}^v}{E[D_1(0, Y_\alpha + 1)]} \leq d.$$

Solution: Similar to Problem 2, the optimal cost ratio for this problem satisfies the following:

$$\frac{H_X^v}{E[C(X)]} = (1 + \zeta) \left\{ \left[\frac{d-1}{m\alpha} + d \left(\frac{\alpha-1}{\alpha} \right) \right] (\gamma-1) + \frac{\alpha-1}{\alpha} \right\}.$$

The derivation for this solution follows the same steps as the solution to Problem 3, and is therefore not repeated.

Plugging this result into (101) gives

$$\inf_{v \in \tilde{V}_d^+} \sup_{x \in [1, \infty)} \frac{H_x^v}{C(x+1)} \geq \frac{d-1}{m} e^{\frac{m}{d-1}}. \quad (102)$$

Finally, using the fact that $\tilde{U}_d \subseteq \tilde{V}_d$ gives

$$\inf_{u \in \tilde{U}_d} \sup_{x \in \mathbb{Z}^+} \frac{H_x^v}{C(x)} = \inf_{u \in \tilde{U}_d} \sup_{x \in [1, \infty)} \frac{H_x^v}{C(x+1)} \geq \frac{d-1}{m} e^{\frac{m}{d-1}}.$$

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