On Jointly Optimal Real-Time Encoding and Decoding Strategies in Multi-Terminal Communication Systems

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***Abstract*—We consider a communication system consisting of two encoders communicating with a single receiver over a noiseless channel. The two encoders make distinct partial observations of a discrete-time Markov source. Each encoder must encode its observations into a sequence of discrete variables. The sequence is transmitted over a noiseless channel to a receiver which attempts to reproduce the output of the Markov source. The system must operate in real-time, that is, the encoding at each encoder and decoding at receiver must be performed without any delay. The goal is to find globally (jointly) optimal real-time encoding and decoding strategies to minimize an expected distortion metric over a finite time horizon. We determine qualitative properties of optimal real-time encoding and decoding strategies. Using these properties, we develop a sequential decomposition of the problem of finding jointly optimal real-time encoding and decoding strategies. Such a sequential decomposition reduces exponentially the complexity of the joint optimization problem.**

***Index Terms*—Dynamic Teams, Information State, Multi-terminal Communication System, Real-time encoding and decoding.**

I. Introduction

A multi-terminal communication system with two encoders communicating with a single receiver over a noiseless channel is considered. The two encoders make distinct partial observations of a discrete-time Markov source. Each encoder must encode its observations into a sequence of discrete variables. This sequence is transmitted over a noiseless channel to a receiver which attempts to reproduce the output of the Markov source. The system must operate in real-time, that is, the encoding at each encoder and decoding at receiver must be performed without any delay. Both the encoders and the receiver have perfect recall, i.e, they remember all of their past observations and actions. The goal is to find globally (jointly) optimal encoding and decoding strategies to minimize an expected distortion metric over a finite time horizon. The problem is motivated by applications such as sensor networks, transportation networks and networked control systems where the communication system is a part of a larger system that requires strict bounds on delays in information transmission.

The key features of the problem are : a) The real-time constraint on information transmission; and b) The presence of multiple encoders with different but correlated information.

The real-time constraint on information-transmission distinguishes our problem from the information-theoretic problem of distributed source coding. Information-theoretic approaches deal with encoding and decoding of long sequences that are asymptotically typical. Encoding long sequences introduces delays and this feature is distinctly different from our real-time constraint. Examples of information-theoretic approaches to distributed source coding appear in [11-13],[14 and references therein].

Point-to-point communication systems under the real-time constraint have been investigated in [1], [2], [10], [6], [5]. The structure of real-time encoders and decoders for the broadcast system under the real-time constraint on information transmission and for a real-time variation of the Wyner-Ziv problem was investigated in [10]. In this paper, we consider a multi-terminal communication system; furthermore, our model is different from the broadcast system and the real-time variation of the Wyner-Ziv problem investigated in [10].

The main feature of a multi-terminal problem that distinguishes it from a point to point communication problem is the presence of coupling between the encoders, (that is, each encoder must take into account what other encoders are doing). This coupling arises because of following reasons : 1) The encoder's observations are correlated with each other. 2) The encoding problems are further coupled because the receiver wants to minimize a non-separable distortion metric. That is, the distortion metric cannot be simplified into separate functions that depend only on one encoder's observations and actions. The nature of optimal strategies strongly depends on the nature and extent of the coupling between the encoders. In this paper, we assume a general distortion metric and a simple model of correlation between the two encoders' observations (described in Section II).

The main contributions of this paper are : 1) The determination of structural properties of optimal real-time encoding and decoding strategies, and 2) A sequential decomposition of the problem of finding jointly optimal encoding and decoding strategies for the model under consideration. Such a decomposition reduces exponentially the complexity of finding jointly optimal real-time encoding and decoding strategies.

The rest of this paper is organized as follows. In Section II, we formulate the problem for a specific source model. In Section III, we present results on the structure of optimal real-time encoders and decoders. In Section IV, we present a

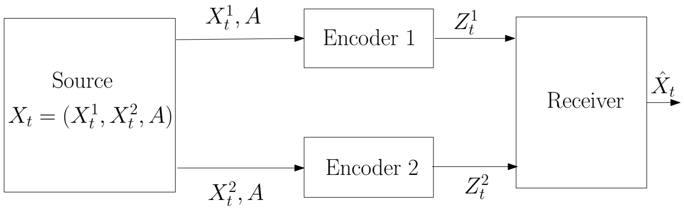

Fig. 1. Multi-Terminal Communication System with two Encoders and one Receiver

method for sequentially determining globally optimal real-time encoding and decoding strategies. We conclude in Section V.
Notation: Throughout this paper, we denote random variables by capital letters. We use superscripts to refer to sequences of random variables. Thus, V^t refers to $V_1, V_2, ...V_t$. In case of 2 superscripts, the first refers to the encoder number for which the variable is being considered and the second refers to the sequence. Thus $X^{1,t}$ indicates the sequence $X_1^1, X_2^1, ...X_t^1$.

II. Problem Formulation

A. The Model

We consider a finite state discrete time Markov chain $X_t \in \mathcal{X}$ with the following structure:
$X_t := (X_t^1, X_t^2, A)$ where $X_t^i \in \mathcal{X}^i$, i=1,2 and $A \in \mathcal{A}$. $\mathcal{X}^i, i = 1, 2$ and $\mathcal{A}$ are finite spaces.
We assume that the parameter A of the source does not change in the evolution of the Markov chain and conditioned on A, X_t^1 and X_t^2 form two independent Markov chains. This can be expressed in terms of the statistics of the initial state and the transition probability as :

$$\begin{aligned} Pr(X_1^1, X_1^2, A) =& Pr(X_1^1, X_1^2/A).Pr(A) \\ =& Pr(X_1^1/A).Pr(X_1^2/A).Pr(A) \end{aligned} \quad (1)$$

$$\begin{aligned} &Pr(X_{t+1}^1, X_{t+1}^2, A^{'}/X_t^1, X_t^2, A) \\ &\quad = Pr(X_{t+1}^1/X_t^1, A).Pr(X_{t+1}^2/X_t^2, A).\delta(A^{'}, A) \end{aligned} \quad (2)$$

At each time t, the first encoder observes X_t^1 and A, and the second encoder observes X_t^2 and A. The two encoders produce Z_t^1 and Z_t^2 that belong to finite alphabets $\mathcal{Z}^1$ and $\mathcal{Z}^2$ respectively. Both encoders must encode in real time hence the encoded symbols at time t are functions of observations available till time t only. Thus,

$$Z_t^i = f_t^i(X^{i,t}, A) \quad (3)$$

for i=1,2 where $X^{i,t} = X_1^i, X_2^i, ..., X_t^i$. The encoders' output at time t, (Z_t^1, Z_t^2) are transmitted to a receiver over a noiseless channel. A perfect memory receiver must produce estimates $\hat{X}_t$ of the state of the source X_t in real time, i.e,

$$\hat{X}_t = g_t(Z^{1,t}, Z^{2,t}) \quad (4)$$

where $Z^{i,t} = Z_1^i, Z_2^i, ..., Z_t^i, i = 1, 2$.
A non-negative distortion function $\rho_t(X_t, \hat{X}_t)$ measures the instantaneous distortion between the source and the estimate at time t. The overall performance of the system is the expected total distortion over a finite time-horizon, T.

B. The Optimization Problem

Given the source statistics, the encoding alphabets, the time horizon T, and the distortion function $\rho_t(X_t, \hat{X}_t)$, the objective is to find globally optimal encoding and decoding functions $f^{1,T}, f^{2,T}, g^T$ so as to minimise

$$J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)], \quad (5)$$

where we use the notation $f^{i,t}$ for $f_1^i, f_2^i, ..., f_t^i$ and g^t for $g^1, ..., g^t$.
Remark: Since the state space of the source, the encoding alphabets and the time -horizon are all finite, the number of possible real-time encoding and decoding strategies $(f^{1,T}, f^{2,T}, g^T)$ is finite. Therefore, an optimal $(\tilde{f}^{1,T}, \tilde{f}^{2,T}, \tilde{g}^T)$ always exists.

C. Features of the Problem

The problem formulated in this paper is a dynamic team problem. Dynamic teams are difficult because they are, in general, non-convex functional optimization problems. We would like to develop a methodology that reduces the complexity of determining an optimal solution to our problem. For that matter, we wish to obtain a sequential decomposition of the optimization problem. The fundamental difficulty in obtaining such a decomposition is the discovery of an information state appropriate for performance evaluation [9]. This difficulty is a fundamental conceptual issue for any decentralized optimization problem. We wish to identify an information state that is not only appropriate for performance evaluation but also has a time-invariant domain, that is, the space in which this state lies does not keep increasing with time. Such an information state would be applicable to both finite as well as infinite time horizon problems.

Identifying structural properties of optimal real-time encoding and decoding strategies could lead to the discovery of information states with a time-invariant domain. Therefore, we proceed as follows. In Section III, we present structural properties of optimal strategies. We use these structural results in Section IV to identify an information state for the joint optimization problem of Section II.B that is appropriate for performance evaluation and has a time-invariant domain. We show how such an information state leads to a sequential decomposition of the joint optimization problem.

III. Structural Properties of an Optimal Design

As shown in Appendix A, for any arbitrary but fixed encoding rules, the decoder can be assumed to have the following structure without any loss of optimality:

$$\hat{X}_t = \tau(\psi_t) \quad (6)$$

where

$$\psi_t = Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}) \quad (7)$$

and

$$\tau(\psi) = \underset{a \in \mathcal{X}}{\arg\min} \sum_{x \in \mathcal{X}} \psi(x) \rho_t(x, a) \quad (8)$$

Note that for a fixed $Z^{1,t}, Z^{2,t}$, the $Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t})$ depends only on the encoding rules used.
We now prove a structural result for the encoders in two steps. In the first step (Theorem 1), we establish a result similar to those proven in Theorem 1 in [1] and Theorem 1 in [2]. A drawback of this result is that the domain of the optimal real-time encoding rule is changing (increasing) with time. We wish to have (if possible) optimal real-time encoding rules whose domain does not change with time. This consideration motivates the structural result we obtain in Theorem 3, where we show the existence of optimal real-time encoding rules whose domain is time-invariant. The result of the second step is similar to that of [2]. However, the presence of two encoders with different but correlated information does not permit us to use the methodology adopted in [2] to achieve the second structural result.

A. First structural Result

Theorem 1 : There is no loss of optimality if one restricts attention to encoders of the form :

$$Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1}) \tag{9}$$

for i=1,2.
Proof: Consider an arbitrary decoding rule $\hat{X}_t = g_t(Z^{1,t}, Z^{2,t})$ and an arbitrary encoding rule for the second encoder $Z_t^2 = f_t^2(X^{2,t}, A)$. We will show that the first encoder can use encoding rules of the form $Z_t^1 = f_t^1(X_t^1, A, Z^{1,t-1})$ without losing optimality.

Define $V_1 := (X_1^1, A)$
and $V_t := (X_t^1, A, Z^{1,t-1})$, for $t = 2, 3..T$.
Then V_t is a conditionally Markov process given the Z_t^1s since

$$\begin{aligned} Pr(V_{t+1}/V^t, Z^{1,t}) =& Pr(X_{t+1}^1, A, Z^{1,t}/X^{1,t}, A, Z^{1,t}) \\ =& Pr(X_{t+1}^1, A, Z^{1,t}/X_t^1, A, Z^{1,t}) \quad (10) \\ =& Pr(X_{t+1}^1, A, Z^{1,t}/V_t, Z_t^1) \\ =& Pr(V_{t+1}/V_t, Z_t^1) \quad (11) \end{aligned}$$

where the equality in (10) is because of the Markovian nature of X_t^1 when conditioned on A.

As seen by the first encoder, the cost function of this system (with second encoder's and decoder's rules fixed) can be written as:

$$\begin{aligned} J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^T \rho_t(X_t, \hat{X}_t)] = \sum_{t=1}^T E[\rho_t(X_t, \hat{X}_t)] \\ = \sum_{t=1}^T E[E[\rho_t(X_t, \hat{X}_t)/X^{1,t}, A, Z^{1,t}]] \end{aligned} \tag{12}$$

$$= \sum_{t=1}^T E[E[\rho_t((X_t^1, X_t^2, A), g_t(Z^{1,t}, Z^{2,t}))/X^{1,t}, A, Z^{1,t}]] \tag{13}$$

where (12) follows from the smoothing property of conditional expectation and (13) by direct substitution. In the inner expectation of (13), the only random quantities are X_t^2 and $Z^{2,t}$ since the rest have been fixed in the conditioning variables. Since the second encoder's rule has been fixed, $Z^{2,t}$ itself is a function of $X^{2,t}$ and A. Thus the only randomness in the inner expectation is due to $X^{2,t}$ which conditioned on A is independent of the first encoders private observations $X^{1,t}$ and actions $Z^{1,t}$. Therefore, the above expectation can be written as:

$$= \sum_{t=1}^T E[E[\rho_t((X_t^1, X_t^2, A), g_t(Z^{1,t}, Z^{2,t}))/X_t^1, A, Z^{1,t}]] \tag{14}$$

$$= \sum_{t=1}^T E[\hat{\rho}_t(X_t^1, A, Z^{1,t})]] \tag{15}$$

$$= \sum_{t=1}^T E[\hat{\rho}_t(V_t, Z_t^1)]] \tag{16}$$

In (15), we have expressed the inner conditional expectation as a function of the conditioning random variables.
Hence, the optimal encoding problem from the first encoder's point of view is to find the optimal control actions Z_t^1 for the controlled Markov chain V_t when the cost function is of the form in (16). It is a well known result of Markov decision theory (see [3], Chapter 6) that there is an optimal control law of the form :

$$Z_t^1 = f_t^1(V_t) \tag{17}$$

or equivalently,

$$Z_t^1 = f_t^1(X_t^1, A, Z^{1,t-1}) \tag{18}$$

We can repeat the same argument for the second encoder to establish the structural result for the second encoder. □

B. Second structural Result

As mentioned before, the structural result of equation (9) suffers from the drawback that the domain of the encoding rules f_t^i, $(\mathcal{X}^i \text{ X } \mathcal{A} \text{ X } \mathcal{Z}^{i,t-1}), i = 1, 2$, keeps increasing with time. We prove a second structural result for the encoders that is free from this drawback.

Because of the first structural result on the encoder, we will only consider encoding rules of the form $Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1})$. Also, for any fixed pair of encoding rules, it is optimal to consider decoders of the form $\hat{X}_t = \tau(\psi_t)$ where ψ_t depends on the encoding rules used.

Fix an arbitrary encoding rule of the form $Z_t^2 = f_t^2(X_t^2, A, Z^{2,t-1})$ for the second encoder. We now consider the problem of finding an optimal encoding rule of the form in (9) for the first encoder. Any encoding rule of the form in (9) can be viewed as follows: at time t, a "pre-encoder" knows $Z^{1,t-1}$ and selects a function $w_t^1 : \mathcal{X}^1 \text{ X } \mathcal{A} \longrightarrow \mathcal{Z}^1$. Once w_t^1 is selected, the encoder observes X_t^1 and A and uses w_t^1 to find

$$Z_t^1 = w_t^1(X_t^1, A) \tag{19}$$

Thus, the optimization problem of finding Z_t^1 for any possible $X_t^1, A, Z^{1,t-1}$ is equivalent to finding the optimal w_t^1 for any possible $Z^{1,t-1}$. We now look at the problem from the "pre-encoder's" perspective and determine a qualitative property on the structure of the pre-encoder. In general, the pre-encoder uses a selection rule of the form :

$$w_t^1 = G_t^1(Z^{1,t-1}) \tag{20}$$

For the purpose of proving a structure on the pre-encoder, we define :

$$R_1 := (X_t^1, A) \tag{21}$$

$$R_t := (X_t^1, A, \xi_{t-1}^1) \tag{22}$$

where

$$\xi_{t-1}^1 = Pr(X_{t-1}/Z^{1,t-1}, w^{1,t-1}) \tag{23}$$

for t=2,3..T, and proceed as follows. We first obtain functional relations among different random variables of interest in *Claims 1* and *2* below. These relations are used to prove that R_t is a controlled Markov chain and the instantaneous cost of the system depends on R_t and w_t^1 (*Lemmas 1* and *2* below).

Claim 1: $\xi_t^1 = F_t(Z_t^1, w_t^1, \xi_{t-1}^1) = \hat{F}_t(R_t, w_t^1)$, where F_t, $\hat{F}_t$ are deterministic functions.
Proof : Consider

$$\xi_t^1 = Pr(X_t/Z^{1,t}, w^{1,t}) \tag{24}$$

Using Bayes' rule, it can be written as:

$$\xi_t^1 = Pr(X_t, Z_t^1/Z^{1,t-1}, w^{1,t}) / \sum_{x\in\mathcal{X}} Pr(x, Z_t^1/Z^{1,t-1}, w_t^1) \tag{25}$$

The numerator in (25) can be written as :

$$Pr(X_t, Z_t^1/Z^{1,t-1}, w^{1,t}) = Pr(Z_t^1/X_t, w_t^1).Pr(X_t/Z^{1,t-1}, w^{1,t}) \tag{26}$$

$$= Pr(Z_t^1/X_t, w_t^1). \sum_{x'\in\mathcal{X}} Pr(X_t/X_{t-1}=x').Pr(X_{t-1}=x'/Z^{1,t-1}, w^{1,t-1}) \tag{27}$$

Observe that the first term in (27) is either 1 or 0 since given X_t and w_t^1, Z_t^1 is exactly known. The first term in the summation is the source statistic known apriori. The second term in the summation is $\xi_{t-1}^1(x')$. The same holds true for each term in the summation in the denominator of the right hand side of (25) . Thus ξ_t^1 is a function of $Z_t^1, w_t^1, \xi_{t-1}^1$. Since Z_t^1 is simply $w_t^1(X_t^1, A)$, we conclude that :

$$\xi_t^1 = \hat{F}_t(X_t^1, A, w_t^1, \xi_{t-1}^1) \tag{28}$$

or equivalently,

$$\xi_t^1 = \hat{F}_t(R_t, w_t^1) \tag{29}$$

□

Claim 2: $\psi_t = H_t(X^{2,t}, R_t, w_t^1)$, where H_t are deterministic functions.
Proof:

$$\begin{aligned}\psi_t = \; & Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}) \\ = \; & Pr(X_t/Z^{1,t}, Z^{2,t}, w^{1,t}, f^{2,t})\end{aligned} \tag{30}$$

Note that ψ_t is the belief on the source formed by the receiver. As mentioned previously, for a given $Z^{1,t-1}, Z^{2,t-1}$, this depends only on the choice of encoding rules used. The equality in (30) follows because with a fixed $f^{2,t}$, this belief depends only on the mappings $w^{1,t}$ used by the pre-encoder. Also, observe that at any time, the pre-encoder selects the mapping based on $Z^{1,t-1}$- which the receiver also knows at that time. Hence, the receiver always knows exactly what w_t^1 is being used.
Now, using Bayes' rule, we have

$$\psi_t = Pr(X_t, Z^{2,t}/Z^{1,t}, w^{1,t}, f^{2,t}) / \sum_{x\in\mathcal{X}} Pr(x, Z^{2,t}/Z^{1,t}, w^{1,t}, f^{2,t}) \tag{31}$$

The numerator in right hand side of (31) can be written as

$$Pr(Z^{2,t}/Z^{1,t}, X_t, w^{1,t}, f^{2,t}).Pr(X_t/Z^{1,t}, w^{1,t}, f^{2,t}) = Pr(Z^{2,t}/X_t^2, A, f^{2,t}).Pr(X_t/Z^{1,t}, w^{1,t}) \tag{32}$$

where the first term on the right hand side (RHS) in (32) follows because of the conditional independence of $Z^{2,t}$ and $Z^{1,t}, X_t^1$ given A and in the second term, $f^{2,t}$ is irrelevant.

Since the second encoder is fixed, the first term in RHS of (32) is fixed apriori and depends only on $X^{2,t}$ and A. The second term is simply ξ_t^1 which by *Claim 1* is a function of R_t and w_t^1. The same is true for each term in the denominator's sum in RHS of (31). Hence ψ_t is a function of $X^{2,t}, A, R_t, w_t^1$ or equivalently (since R_t contains A),

$$\psi_t = H_t(X^{2,t}, R_t, w_t^1) \tag{33}$$

□

Lemma 1 : R_t is a controlled Markov chain with w_t^1 as control actions.
Proof:

$$Pr(R_{t+1}/R^t, w^{1,t}) = Pr(X_{t+1}^1, A, \xi_t^1/X^{1,t}, A, \xi^{1,t-1}, w^{1,t}) \tag{34}$$

$$= Pr(X_{t+1}^1, A/X^{1,t}, A, \xi^{1,t}, w^{1,t}). Pr(\xi_t^1/X^{1,t}, A, \xi^{1,t-1}, w^{1,t}) \tag{35}$$

$$= P(X_{t+1}^1, A/X^{1,t}, A, \xi^{1,t}, w^{1,t}, R_t) .Pr(\xi_t^1/X^{1,t}, A, \xi^{1,t-1}, w^{1,t}, R_t) \tag{36}$$

$$= Pr(X_{t+1}^1, A/\xi_t^1, w_t^1, R_t).Pr(\xi_t^1/w_t^1, R_t) \tag{37}$$

$$= Pr(X_{t+1}^1, A, \xi_t^1/w_t^1, R_t) \tag{38}$$

$$= Pr(R_{t+1}/R_t, w_t^1) \tag{39}$$

The equality in (36) follows because the variables in conditioning determine R_t exactly, so its inclusion in the conditioning does not alter the probability. In the first term of (37), because of the Markovian nature of source, one only needs X_t^1 and A in the conditioning (X_t^1 and A are present in R_t) while in the second term of (37) one only needs $w_t^1$and R_t in the conditioning because of *Claim 1*. Equation (39) proves the lemma. Thus R_t is a conditional Markov chain given w_t^1. □

Lemma 2 : For a fixed $f^{2,t}$ of the form in (9) and a decoder of the form in (6), the cost function can be written as :

$$J(f^{1,T}, f^{2,T}, g^T) = \sum_{t=1}^{T} E[\rho_t^*(R_t, w_t^1)] \tag{40}$$

where ρ_t^* is a deterministic function.
Proof: The cost function can be written as :

$$\begin{aligned}J(f^{1,T}, f^{2,T}, g^T) =& E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)] \\ =& \sum_{t=1}^{T} E[\rho_t(X_t, \hat{X}_t)] \\ =& \sum_{t=1}^{T} E[\rho_t(X_t, \tau(\psi_t)]] \quad (41) \\ =& \sum_{t=1}^{T} E[\rho_t(X_t^1, X_t^2, A, \tau(H_t(X^{2,t}, R_t, w_t^1)]] \quad (42) \\ =& \sum_{t=1}^{T} E[\hat{\rho}_t(X^{2,t}, R_t, w_t^1)] \quad (43) \\ =& \sum_{t=1}^{T} E[E[\hat{\rho}_t(X^{2,t}, R_t, w_t^1)/R_t, w_t^1]] \quad (44)\end{aligned}$$

where equality in (42) follows because of *Claim 2*, $\hat{\rho}_t$ in (43) is simply a different representation of the composite function in (42) and the equality in (44) uses the smoothing property of conditional expectations.
The inner expectation in (44) can be evaluated as:

$$\begin{aligned}&= \sum_{x^{2,t} \in (\mathcal{X}^2)^t} Pr(x^{2,t}/A).\hat{\rho}_t(x^{2,t}, R_t, w_t^1) \quad (45) \\ &= \rho_t^*(R_t, w_t^1) \quad (46)\end{aligned}$$

where we have used the conditionally independent nature of the source to get the first term in the summation in (45), and (46) follows because $Pr(X^{2,t}/A)$ is a known statistic. Thus we can write (44) as:

$$= \sum_{t=1}^{T} E[\rho_t^*(R_t, w_t^1)] \tag{47}$$

which proves the lemma. □

We now present the structural result for the pre-encoder.

Theorem 2: With a fixed encoding rule of the form $Z_t^2 = f_t^2(X_t^2, A, Z^{2,t-1})$ for the second encoder and a decoding rule of the form $\hat{X}_t = \tau(\psi_t)$, the pre-encoder can restrict attention to selection rules of the form $w_t^1 = G_t^1(\xi_{t-1}^1)$ without any loss of optimality.
Proof: The optimization problem from the pre-encoder's perspective can now be seen as follows. There is an underlying controlled Markov chain R_t for which the pre-encoder has to find the optimal control actions w_t^1 (Lemma 1). The expected cost of an action at time t (w_t^1) is $E[\rho_t^*(R_t, w_t^1)]$. At time t, the Markov chain is in state R_t, the pre-encoder takes an action w_t^1, it makes an observation Z_t^1 which depends only on the state R_t and the action w_t^1. The state then changes to R_{t+1} with the transition statistic depending only on R_t and w_t^1. This is a typical partially observed Markov decision problem. From Markov decision theory (see [3], [7]), we know that $\pi_t = Pr(R_t/Z^{1,t-1}, w^{1,t-1})$ is an information state appropriate for performance evaluation and there is an optimal policy of the form:

$$w_t^1 = G_t(\pi_t) \tag{48}$$

We now argue that ξ_{t-1}^1 is an equivalent information state. To show that, we need to show that a) ξ_{t-1}^1 is a function of the pre-encoder's previous observations ($Z^{1,t-1}$) and actions ($w^{1,t-1}$), b) ξ_t^1 can be obtained from the ξ_{t-1}^1, the action at time t (w_t^1) and the observation at time t (Z_t^1); and c) π_t can be obtained from ξ_{t-1}^1.
By (23), ξ_{t-1}^1 is a function of the pre-encoder's previous observations ($Z^{1,t-1}$) and actions ($w^{1,t-1}$).
Claim 1 establishes the required update, that is, $\xi_t^1 = F(Z_t^1, w_t^1, \xi_{t-1}^1)$.
Consider $\pi_t = Pr(R_t/Z^{1,t-1}, w^{1,t-1})$

$$= Pr((X_t^1, A, \xi_{t-1}^1)/Z^{1,t-1}, w^{1,t-1}) \tag{49}$$

Given $Z^{1,t-1}, w^{1,t-1}$, ξ_{t-1}^1 is known exactly, hence (49) can be written as :

$$\pi_t = Pr((X_t^1, A)/Z^{1,t-1}, w^{1,t-1}) \tag{50}$$

$$= \sum_{x \in \mathcal{X}} Pr((X_t^1, A)/X_{t-1} = x). \quad Pr(X_{t-1} = x/\xi_{t-1}^1, Z^{1,t-1}, w^{1,t-1}) \tag{51}$$

$$= \sum_{x \in \mathcal{X}} Pr((X_t^1, A)/X_{t-1} = x).\xi_{t-1}^1(x) \tag{52}$$

where (51) uses the Markov property of the source. Observe that the first term in (52) is a known source statistic and the second term depends only on ξ_{t-1}^1 . Thus, π_t is a deterministic function of ξ_{t-1}^1.
Hence, ξ_{t-1}^1 is an equivalent information state. Therefore, there exists an optimal control law of the form :

$$w_t^1 = G_t^1(\xi_{t-1}^1). \tag{53}$$

□

We can now state the desired structural result for the encoders.

Theorem 3: With a decoder of the form $\hat{X}_t = \tau(\psi_t)$, there is no loss in optimality in restricting attention to encoders of the form:

$$Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i) \tag{54}$$

where $\psi_t = Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}))$ and $\xi_{t-1}^i = Pr(X_{t-1}/Z^{i,t-1}, f^{i,t-1}), i = 1, 2$.

Proof: By Theorem 1, one can restrict attention to encoders of the form:

$$Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1}) \tag{55}$$

Consider a fixed encoding rule of the second encoder of the form in (55). Then, by Theorem 2, there exists an optimal selection rule of the first pre-encoder of the form :

$$w_t^1 = G_t^1(\xi_{t-1}^1)$$

With this selection rule, the encoded symbol at time t is given as :

$$\begin{aligned} Z_t^1 =& w_t^1(X_t^1, A) \\ =& G_t^1(\xi_{t-1}^1)(X_t^1, A) \qquad (56) \\ =& f_t^1(X_t^1, A, \xi_{t-1}^1) \qquad (57) \end{aligned}$$

where (57) is simply another representation of (56). Thus, there exists an optimal encoder of the form in (54) for the first encoder. Consequently, one can restrict attention to encoders of the form in (54) for the first encoder. Now observe that any encoder of the form in (54) is also of the form $Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1})$. Hence with the first encoder as in (54), we can repeat the same argument for the second encoder. Therefore, by only considering encoders of the form $Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i)$, we do not lose optimality. □

C. Discussion

It is worthwhile to compare our results with those obtained in [2] for a communication system with a single encoder. The results in [2] are also true for a noiseless channel under no feedback. The key result in [2] is a structural result on the encoder (Theorem 1 of [2]). With the help of this result, the authors have been able to formulate the problem of finding optimal real-time encoding and decoding rules as a centralized optimization problem for which they present an optimal solution by a dynamic program. This optimal solution has the structural property that the authors proposed in their Theorem 2 of [2].

Our first structural result in Theorem 1 is analogous to Theorem 1 in [2]. However, in spite of this result, we cannot formulate the problem of finding optimal encoding and decoding rules as a centralized optimization problem because of the following reason.

An essential feature of any centralized problem is that all decisions at time t must be made on the basis of the *same information*. In Theorem 2 of [2], the two decisions to be made at time t (a pre-encoding function at the encoder and the source estimate at the decoder) are both based on the same information which is the encoded symbols sent till time (t-1). This is crucial for the centralized formulation proposed in [2]. Now note that in the problem we consider in this paper, the two pre-encoding functions (w_t^1, w_t^2) and the source estimate are based on *different information*. In particular, w_t^1 is selected on the basis of $Z^{1,t-1}$ and w_t^2 on the basis of $Z^{2,t-1}$. This fact of making different decisions based on different information is unavoidable in our problem, and it gives our problem its decentralized nature. It must be emphasized here that the fact that the receiver knows the information of both encoders ($Z^{1,t-1}$ and $Z^{2,t-1}$) does not alter the decentralized nature of the problem. Even though the receiver can choose w_t^1 and w_t^2, it must do so on the basis of different information - $Z^{1,t-1}$ and $Z^{2,t-1}$, respectively. The mere fact that these two decisions could be made at the same location (the receiver) does not remove their informational separation and even from the receiver's perspective, the problem is still equivalent to one with two separate agents making decisions based on separate information.

The fact that this problem cannot be viewed as a centralized optimization problem has two important implications :

a) Firstly, we had to introduce an imaginary pre-encoder that essentially represents the common information between the receiver and one encoder. This enabled us to identify a structural result similar to Theorem 2 in [2].

b) More importantly, the decentralized nature of the problem makes the task of finding jointly optimal real-time encoding and decoding functions considerably more difficult than in [2]. The main difficulty is the identification of an information state that is sufficient for performance evaluation [9]. This difficulty is a fundamental conceptual issue for any decentralized optimization problem. Since there are multiple agents (the encoders and the decoder) taking actions based on different information, the usual information states from Markov decision theory [3] are not appropriate for our problem.

In the next section, we present an information state that is sufficient for performance evaluation and has a time-invariant domain. We then present the resulting sequential decomposition of the joint optimization problem.

IV. Joint Optimization

The structural results presented in Section III allow us to restrict the space in which one must look for optimal encoding and decoding rules. Now, we want to find jointly optimal strategies. Note that for any choice of encoding strategies, the decoder's structural result of (6) and (7) gives us the optimal decoder. Hence, we are looking for jointly optimal encoding strategies $f^{1,T}, f^{2,T}$ of the form in equation (54) that along with the corresponding optimal decoder of (6) give the best performance.

We propose a sequential decomposition of the problem since it reduces the complexity of the optimization problem. For that matter, we need to determine an information state appropriate for performance evaluation. Motivated by the approach in [4] and [5], we consider the problem from the point of a fictitious designer who has to select the strategies f_t^1 and $f_t^2, t = 1, 2..T$, without having access to any observations. An information state appropriate for performance evaluation for this designer should satisfy conditions of sequential update and sufficiency for cost evaluation. Specifically, if θ_t is an information state, then we want :

$$\theta_{t+1} = T_t(\theta_t, f_t^1, f_t^2) \tag{58}$$

and

$$E[\rho_t(X_t, \hat{X}_t)] = C_t(\theta_t, f_t^1, f_t^2) \tag{59}$$

We now present an information state for the designer and the resulting sequential decomposition of the problem. For that matter, we define:

$$\theta_t = Pr(X_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}) \tag{60}$$

We first need an update rule for ψ_t.

Claim 3: $\psi_t = \hat{T}_t(X_t, \psi_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, f_t^1, f_t^2)$, where $\hat{T}_t$ are deterministic transformations.

Proof: See Appendix B

We now show that θ_t satisfies (58) and (59).

Lemma 3 : $\theta_t = Pr(X_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1})$ satisfies equations (58) and (59).

Proof:

$$\theta_{t+1} = Pr(X_t, \xi_t^1, \xi_t^2, \psi_t)$$

Using *Claim 1* and *Claim 3*, we can write :

$$(\xi_t^1, \xi_t^2, \psi_t) = Q^{f_t^1, f_t^2}(X_t, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}) \tag{61}$$

where the transformation $Q^{f_t^1, f_t^2}$ is derived from the transformations $\hat{F}_t$ and $\hat{T}_t$ of *Claim 1* and *Claim 3*.($Q^{f_t^1, f_t^2}(X_t, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1})$ gives $\hat{F}_t(X_t^1, A, f_t^1)$, $\hat{F}_t(X_t^2, A, f_t^2)$ and $\hat{T}_t(X_t, \psi_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, f_t^1, f_t^2)$). Hence,

$$\begin{aligned}\theta_{t+1} &= Pr(X_t, \xi_t^1, \xi_t^2, \psi_t) \\ &= Pr(X_t, Q^{f_t^1, f_t^2}(X_t, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1})) \quad (62)\end{aligned}$$

$$= \tilde{T}_t(Pr(X_t, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}), f_t^1, f_t^2) \tag{63}$$

where (63) simply states that the probability of a function of random variables can be obtained from the joint probability of the random variables and the function[1].

$$\begin{aligned}\theta_{t+1} = \tilde{T}_t(&\sum_{x \in \mathcal{X}} Pr(X_t/X_{t-1} = x). \\ &Pr(X_{t-1} = x, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}), f_t^1, f_t^2) \quad (64)\end{aligned}$$

$$\begin{aligned}= \tilde{T}_t(\sum_{x \in \mathcal{X}} Pr(X_t/X_{t-1} = x).\theta_t(x, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}) \\ , f_t^1, f_t^2) \quad (65)\end{aligned}$$

[1]For the specific form of $\tilde{T}_t$, see [15]

Since $Pr(X_t/X_{t-1} = x)$ is given by the known statistical description of the source, (65) implies

$$\theta_{t+1} = T_t(\theta_t, f_t^1, f_t^2) \tag{66}$$

Now, consider

$$E[\rho_t(X_t, \hat{X}_t)] = E[\rho_t(X_t, \tau(\psi_t))] \tag{67}$$

The expectation in (67) is a function of the joint distribution of X_t and ψ_t -which is a marginal of $Pr(X_t, \xi_t^1, \xi_t^2, \psi_t)$. Hence,

$$\begin{aligned}E[\rho_t(X_t, \hat{X}_t)] =& \tilde{C}_t(Pr(X_t, \psi_t)) & (68)\\ =& \hat{C}_t(Pr(X_t, \xi_t^1, \xi_t^2, \psi_t)) & (69)\\ =& \hat{C}_t(\theta_{t+1}) & (70)\\ =& C_t(\theta_t, f_t^1, f_t^2) & (71)\end{aligned}$$

□

(For the specific form of the functions T_t and C_t, we refer the reader to [15].)

Theorem 4: The optimal encoding functions f_t^1, f_t^2 are given by the following optimality equations:

$$V_T(\theta) = 0 \tag{72}$$

$$V_t(\theta) = \inf_{f_t^1, f_t^2} [C_t(\theta, f_t^1, f_t^2) + V_{t+1}(T_t(\theta_t, f_t^1, f_t^2))] \tag{73}$$

for t =1,2..,T-1, where $f_t^i \in \mathcal{F}_t^i$ and $\mathcal{F}_t^i$ is the set of functions of the form $Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i)$

Proof: For a deterministically evolving system that is characterized by equations (58) and (59), the optimal f_t^1, f_t^2 are given by (72) and (73). This is a standard result (see [8], Chapter 2).

V. Conclusion

We have discovered the structure of optimal real-time encoders and decoders for the multi-terminal communication system considered in this paper. The structure of the Markov source, the nature of encoders' observations and the noiseless nature of the channel are critical in obtaining the results of Section III for the following reasons. In general, to determine its encoding rule at any time t, each encoder must form a belief about the information of the other encoder and the receiver's information. The conditional independence of X_t^1 and X_t^2 on A allows each encoder to use only the value of the random variable A to form a belief about the the other encoder's information. The noiseless nature of the communication channel allows each encoder at any time t to use the value of the random variable A and its previous transmissions (up to time t-1) to form its belief on the receiver's information. These considerations lead to the structural results of Theorem 1 and Theorem 3. The structural result of Theorem 3 plays an important role in identifying an information state that has a time-invariant domain and is appropriate for performance evaluation. Such an information state is appropriate for obtaining a sequential decomposition of the finite horizon joint optimization problem (considered in this paper) as well as the corresponding infinite horizon problem.

The problem of joint optimization is significantly more difficult than the one considered in [2] because of the following reason. The presence of two encoders with different information implies that encoding decisions have to be necessarily based on different information. It is this fact that makes our problem decentralized. Decentralized optimization problems are considerably more challenging than centralized problems and the information states appropriate for them are more complicated than their centralized counterparts. We presented an information state for the joint optimization problem and obtained a methodology that allows us to sequentially determine globally optimal real-time encoding and decoding strategies. Such a methodology reduces exponentially the complexity of determining globally optimal real-time encoding and decoding strategies. Even with a sequential decomposition, the problem is computationally difficult. The finite size of the source and encoding alphabets imply that there are only finitely many strategy choices available at each time, but the information state is a probability measure on an uncountable space.

The results of this paper can be extended to multi-terminal systems consisting of N encoders, communication with one receiver by noiseless channels, general distortion metrics and Markov sources of the form $X_t := (X_t^1, X_t^2, ..., X_t^N, A)$, where A does not change with time and conditioned on A, $X_t^1, X_t^2, ..., X_t^N$ form independent Markov chains.

Appendix A
Structural Result for the Decoder

Observe that with fixed encoding rules, minimizing

$J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)]$

is equivalent to minimizing $E[\rho_t(X_t, \hat{X}_t)]$ for each t. This can be minimized by minimizing $E[\rho_t(X_t, \hat{X}_t)/Z^{1,t}, Z^{2,t}]$ for all $Z^{1,t}, Z^{2,t}$. The structural property of the decoder then follows from the definition of ψ_t and τ in (7) and (8).

Appendix B
Proof of Claim 3

$$\begin{aligned}\psi_t =& Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}) \\ =& Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2) \end{aligned} \tag{74}$$

We can introduce ξ_{t-1}^1, ξ_{t-1}^2 in the conditioning in (74) since they are functions of the conditioning variables. By Bayes' rule, we have

$$\begin{aligned}\psi_t = & Pr(X_t, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \\ & \xi_{t-1}^1, \xi_{t-1}^2)/ \\ & \sum_{x\in\mathcal{X}} Pr(x, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}.f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2)\end{aligned} \tag{75}$$

We can write the numerator as:

$$\begin{aligned}& Pr(Z_t^1, Z_t^2/X^t, Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2). \\ & Pr(X_t/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2)\end{aligned} \tag{76}$$

$$\begin{aligned}&= Pr(Z_t^1, Z_t^2/X_t, f_t^1, f_t^2, \xi_{t-1}^1, \xi_{t-1}^2). \\ & \sum_{x\in\mathcal{X}} Pr(X_t/X_{t-1}=x). \\ & Pr(X_{t-1}=x/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2)\end{aligned} \tag{77}$$

The first term in (77) is because of the the structural result of the encoders and the second by the Markov nature of the source. Observe that the first term in (77) is either 1 or 0, the first term in the summation is the source statistic known apriori and the second term in the summation is $\psi_{t-1}(x)$. The same holds true for each term in the summation in the denominator of 75. Thus ψ_t^1 is a function of $X_t, f_t^1, f_t^2, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}$.That is,

$$\psi_t = \hat{T}_t(X_t, \psi_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, f_t^1, f_t^2) \tag{78}$$

□

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