On effectiveness of application-layer coding

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***Abstract*—The effectiveness of application-layer coding in a system with a large number of users is considered. The end users encode data packets before transmitting them. The effect of additional packets on the system performance is twofold: (i) additional packets increase the offered load, which results in higher drop probability, and (ii) some of dropped packets can be recovered at the receivers after decoding. The paper establishes an asymptotic regime in which systems with and without coding have the same performance. The space of all systems is partitioned into two regions where coding is beneficial and detrimental, respectively. Informally, it is argued that application-layer coding improves the performance only in systems with low loss probabilities (without coding), and employing such coding in systems with high loss probabilities only degrades the performance.**

I. Introduction

The primary reason for losses in packet networks is buffer overflow – each link in a network has finite capacity, and intermediate routers have limited memories to store packets. In general, there are two basic approaches to overcome this kind of packet losses:

- *Retransmission mechanism*. The source transmits its data packets to the receiver. Packets that have not been acknowledged (explicitly or implicitly) are retransmitted.
- *Application-layer coding*. The source encodes its data packets into coded packets and transmits them instead. The receiver reconstructs the original data packets by decoding received coded packets.

Application-layer coding allows end users to recover the original data packets from the received subset of coded packets by decoding. However, employing such coding results in a higher offered load, and, therefore, increases drop probability. From this perspective, it is unclear when application-layer coding is advantageous; application-layer coding was shown to be advantageous in certain cases [1]. Hence, it is of interest to investigate the effectiveness of such coding. As a first step, we study the effectiveness of coding in the baseline model consisting of a single link with a finite buffer. In particular, the paper considers a sequence of systems indexed by the number of users N. We first discuss an appropriate scaling of the system parameters for investigating the effectiveness of coding, and establish that the critical-load scaling is the relevant one. Under the critical-load scaling, system utilization and drop probability behave as $1-\Theta(1/\sqrt{N})$ and $\Theta(\sqrt{N})$, respectively, when the number of users N is large[1]. We then examine the loss probabilities in systems with and without coding. Our asymptotic analysis indicates that application-layer coding can be advantageous in under-loaded systems; in over-loaded systems, however, the overhead of coding exceeds its benefit, and coding only worsens the system performance. In addition, we demonstrate on examples that our asymptotic results render reasonable approximations for systems with a finite number of users.

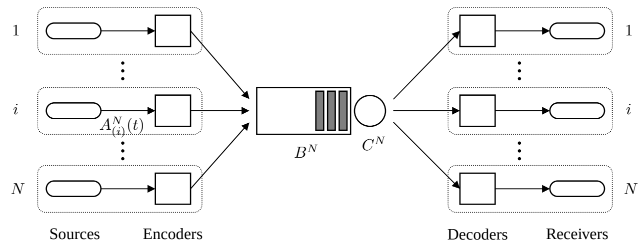

Fig. 1. The source i, $1 \le i \le N$, generates $A^N_{(i)}(t)$ packets in the time slot t. When application-layer coding is employed, source-generated packets are encoded into coded packets by individual encoders. All packets from the encoders of N sources are transmitted to a single link of capacity C^N with a buffer of size B^N. The packets that are not dropped from the buffer are first delivered to the decoder of each source and decoded into the original packets. End users receive the output packets of their own decoders.

The rest of the paper is organized as follows. In the next section, we describe a system model and assumptions that are used throughout the paper. We discuss a relevant scaling for investigating the effectiveness of coding in Section III. In Section IV we review erasure codes and their performance. Section V contains the analysis for the *loss* probability without coding. Then, we analyze the *drop* probability with coding and discuss the coding overhead due to the increased offered load in the following section. In Section VII we explore the *loss* probability with coding and establish the boundary where systems with and without coding have the same performance. A discussion on the system performance for a systematic minimum-distance-separable (MDS) code is presented in Section VIII. Concluding remarks and technical proofs can be found in Section IX and Section X, respectively.

II. System Model

A. Model

We consider a sequence of systems indexed by N, where N is the number of sources that transmit packets to a link with a finite buffer. Let C^N and B^N denote the link capacity and the buffer size, respectively. Time slotted operations are assumed. In addition, let $A^N_{(i)}(t)$, $1 \le i \le N$, $t \ge 1$, be the number of packets generated by the source i in the time slot t. The processes $\{A^N_{(i)}(t), t \ge 1\}$, $i = 1, 2, \cdots, N$, are assumed to be independent Bernoulli random processes with parameter λ. If present, application-layer coding is performed at each source (see Figure 1). All packets from the encoders are transmitted

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[1]Throughout the paper, we use the standard asymptotic notation, e.g., see [2, Sec. I.3].

to the queue consisting of a single link with a finite buffer. The packets that are not dropped from the queue are first delivered to the decoder of each source and decoded into the original packets. End users receive the output packets of their own decoders.

We examine loss probability as a measure of the system performance. The loss probability is defined as the long-term ratio of the number of lost packets to the total number of source-generated packets. A dropped packet is a packet that is discarded from the queue when the buffer is full, and a lost packet is a packet that is not delivered to the end users. In a system without coding, every dropped packet is also a lost packet since no dropped packets can be recovered. If a system utilizes coding, however, some of dropped packets can be recovered at the end users, and, therefore, we differentiate a lost packet (loss probability) from a dropped packet (drop probability) in this case. Even though the drop probability increases due to the additional offered load attributed to coding, the loss probability can decrease by means of coding if enough dropped packets are recovered.

B. Coding scheme

The queue, in which some packets are dropped when the buffer is full, can be thought of as an erasure channel, e.g., see [3], [4]. Two main features of our model are as follows:

- *Systematic linear block code*. We assume that a linear block code is used to generate additional α coded packets from M^N data packets in each coding block of the sources. These additional α packets are transmitted in the same time slot as the last data packet of the block[2]; α does not vary with N. Note that this coding scheme can easily be implemented. In addition, systematic codes have good delay properties since data packets are transmitted without delay and decoding delay is positive only in the presence of packet drops.
- *Non-priority queue*. It is assumed that all packets have the same priority in the queue and that they are served on the first-come, first-serve basis. Although giving priority to data packets can improve the performance of a system, we consider a system without priority since such a system is straightforward to implement and users have no incentive to mislabel their packets intentionally (cheat).

Let $H_{(i)}^N(t)$, $1 \le i \le N$, $t \ge 1$, denote the number of packet arrivals from the encoder of the source i in the time slot t. The arrival process $\{H_{(i)}^N(t), t \ge 1\}$ can be described by a Markov chain $\{(X_{(i)}^N(t), Y_{(i)}^N(t)), t \ge 1\}$ where $X_{(i)}^N(t) = 1\{H_{(i)}^N(t) > 0\}$, and $Y_{(i)}^N(t) \in \{0, 1, \ldots, M^N - 1\}$ is the number of data packets in the present coding block that arrived before the time slot t. Since the Markov chain is aperiodic, finite and irreducible, it has an unique stationary distribution $\pi(x, y)$, $x \in \{0, 1\}$, $y \in \{0, 1, \ldots, M^N - 1\}$, given by

$$\pi(x, y) = \begin{cases} (1-\lambda)/M^N, & x = 0, \\ \lambda/M^N, & x = 1. \end{cases} \tag{1}$$

[2]This assumption is not crucial and does not impact the nature of our main results. Other schemes are possible, e.g., additional packets can be transmitted in consecutive time slots.

III. Scaling

In this section, we discuss an appropriate scaling (as the number of users increases, $N \to \infty$) for investigating the effectiveness of application-layer coding. Recall that we assume that α additional packets are generated per each block of length M^N. The additional offered load due to coding is then equal to $\alpha\lambda N/M^N$ while the spare capacity of the link is $C^N - \lambda N$. Thus, we consider the block length M^N such that $\alpha\lambda N/M^N = \Theta(C^N - \lambda N)$, as $N \to \infty$, since this scaling allows one to examine both under- and over-loaded systems by adjusting appropriate constants (system parameters). Next we review three possible scalings for C^N and M^N[3]. Let p and p_D denote the loss probability without coding and the drop probability with coding, respectively.

- *Under-load scaling*: $C^N = \lfloor \lambda N + \beta N \rfloor$ and $M^N = \lfloor m\lambda \rfloor$ for $\beta > 0$ and $m > \alpha/\beta$. In this case, p and p_D are asymptotically given by $O(e^{-\theta N})$ and $O(e^{-\theta' N})$, respectively, as $N \to \infty$, for some positive constants θ and θ' ($\theta' < \theta$) (e.g., see [5, Ch. 12]). Despite the fact that drop probability increases due to coding, the expected number of dropped packets in a block is close to zero for large N. If at least one coded packet is added per block ($\alpha \ge 1$), we can recover most of the dropped packets. Therefore, coding improves the system performance in this case as suggested in [1].
- *Over-load scaling*: $C^N = \lfloor \lambda N + \beta \rfloor$ and $M^N = \lfloor m\lambda N \rfloor$ for $\beta > 0$ and $m > \alpha/\beta$. Due to the central limit theorem (CLT), both p and p_D are asymptotically given by $\Theta(1/\sqrt{N})$, as $N \to \infty$. In this case, the expected number of dropped packets in a block is $\Theta(\sqrt{N})$ since the block length is $\Theta(N)$. However, the maximum number of dropped packets that can be recovered in a block is only $\alpha = \Theta(1)$. For large N, hence, the possibility of recovering dropped packets is very small. Hence, in this scaling, coding worsens the system performance.
- *Critical-load scaling*: $C^N = \lfloor \lambda N + \beta\sqrt{N} \rfloor$ and $M^N = \lfloor m\lambda\sqrt{N} \rfloor$. Under this scaling, both p and p_D behave as $\Theta(1/\sqrt{N})$ in the limit as $N \to \infty$ (e.g., see [6, Ch. 10]). Since the block length is $\Theta(\sqrt{N})$, the expected number of dropped packets in a block is $\Theta(1)$, i.e., the numbers of dropped and additional packets are of the same order. Therefore, in this case, the effectiveness of coding depends on α and m for given system parameters β (capacity) and b (buffer size), and it is feasible to find the critical points where systems with and without coding have the same performance.

In the following sections, we demonstrate that the *critical-load* scaling is the relevant scaling as far as the effectiveness of coding is concerned. Under the critical-load scaling, the following scaled system parameters are useful in obtaining the drop and loss probabilities:

$$\hat{C}^N = (C^N - \lambda N)/\sqrt{N} \to \beta,$$
$$\hat{B}^N = B^N/\sqrt{N} \to b,$$

[3]Although other scalings are possible, these three cover the main tradeoffs between efficiency and quality.

as $N \to \infty$. The scaling for the buffer size B^N stems from the fact that if $B^N = o(\sigma^N)$, as $N \to \infty$, where σ^N denotes the standard deviation of the total arrival process, then the performance of the system is asymptotically equal to the one with $B^N = 0$ (as $N \to \infty$); on the other hand, if $B^N = \omega(\sigma^N)$, as $N \to \infty$, then the system behaves asymptotically as the one with $B^N = \infty$ (as $N \to \infty$). Hence, for evaluating the effect of the buffer size on the system performance, the relevant buffer size should satisfy $B^N = \Theta(\sigma^N)$, as $N \to \infty$. For the considered model, we have $\sigma^N = \Theta(\sqrt{N})$, as $N \to \infty$, and, thus, we let $B^N = \lfloor b\sqrt{N} \rfloor$ for $b \geq 0$.

IV. Erasure codes

In this section, we review erasure codes and their performance. The relevance of such codes is due to the fact that the finite-buffer queue can be thought of as an erasure channel, e.g., see [3], [4]. We consider $(M+\alpha, M)$ linear block codes – M data packets are used to generate $M+\alpha$ packets to be transmitted. Let $\mathbf{v} = [v_1 \ v_2 \ \cdots \ v_M]$ be the data packets in a single coding block, and let $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_{M+\alpha}]$ be the output packets encoded from these data packets. The output packets are generated from the data packets according to the following rule:

$$\mathbf{u} = \mathbf{v}\mathbf{G}, \qquad (2)$$

where $\mathbf{G}$ is a generator matrix that depends on a specific code. All arithmetic is over $GF(q)$ for some positive integer q (e.g., see [7, Ch. 5]). Next we examine various erasure codes.

A. Ideal block code

Let D denote the number of dropped packets among the $M+\alpha$ output packets from a single block, and let L denote the number of lost packets in the same block, i.e., L original data packets can not be reconstructed after decoding. We define the *ideal block code* as a code that satisfies the following property:

$$L = (D-\alpha)^+ = D - (D \wedge \alpha). \qquad (3)$$

Note that if D output packets are dropped, then a decoder can recover only $M+\alpha-D$ linear equations in (2) from the remaining output packets. From $M+\alpha-D$ linear equations, at most $M+(\alpha-D)^-$ data packets can be decoded correctly. Therefore, the ideal block code, if it exists, achieves the best performance among all linear block codes.

B. Systematic MDS code

A linear block code with minimum distance d can recover all of the original data packets in a block when the number of dropped packets in the block is less than d. If a $(M+\alpha, M)$ linear block code has minimum distance $d = \alpha+1$, we call such codes as MDS codes; these MDS codes achieve equality in the Singleton bound (e.g., see [7, Ch. 15]). Reed-Solomon codes belong to the class of MDS codes. When a code is systematic, the output packets from a block contain M original data packets and additional α coded packets, i.e., $u_i = v_i$ for $i = 1, 2, \ldots, M$. Given a block, let D_d and D_c denote the numbers of dropped packets among the M data packets and the additional α coded packets, respectively. If $D_d + D_c \leq \alpha$, then all M data packets can be reconstructed from (2). On the other hand, if $D_d + D_c > \alpha$, then no dropped data packets can be recovered from (2) and only $M - D_d$ data packets are obtained. Therefore, letting L be the number of lost packets after decoding leads to

$$L = D_d \cdot 1_{\{D_d + D_c > \alpha\}}. \qquad (4)$$

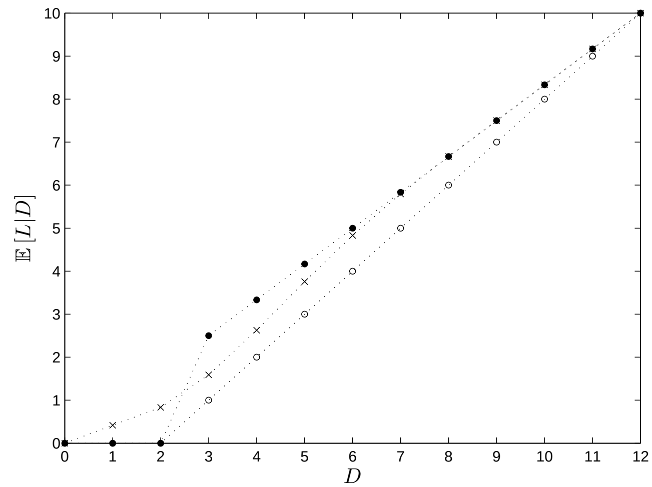

Fig. 2. The conditional expectation of the number of lost packets $\mathbb{E}[L|D]$ given the value of the number of dropped packets D in a block for the ideal block code ($\circ$), a systematic MDS code ($\bullet$) and partial coding with $\rho = 0.5$ ($\times$) when $M = 10$ and $\alpha = 2$. In this example, it is assumed that all packet drops are independent with the same drop probability.

C. Partial coding

Suppose that a systematic MDS code is applied to only ρ fraction of data packets in a block. That is, $\lfloor \rho M \rfloor$ data packets are used to generate $\lfloor \rho M \rfloor + \alpha$ output packets, and remaining $M - \lfloor \rho M \rfloor$ data packets are transmitted without any encoding. In this case, only the dropped packets from the ρ fraction of the block can potentially be recovered. Let $\tilde{D}_d$ and $\bar{D}_d$ denote the numbers of dropped packets among $\lfloor \rho M \rfloor$ data packets in the coding part and $M - \lfloor \rho M \rfloor$ data packets in the non-coding part, respectively. Moreover, let D_c denote the number of dropped packets among additional α coded packets in the coding part. Setting L to be the number of lost packets after decoding yields

$$L = \bar{D}_d + \tilde{D}_d \cdot 1_{\{\tilde{D}_d + D_c > \alpha\}} = D_d - \tilde{D}_d \cdot 1_{\{\tilde{D}_d + D_c \leq \alpha\}}, \qquad (5)$$

where $D_d = \tilde{D}_d + \bar{D}_d$.

D. Comparison

In Figure 2, we illustrate the difference between these three coding schemes on an example. In particular, we compare the conditional expectation of the number of lost packets given the value of the number of dropped packets in a block for the ideal block code, a systematic MDS code and partial coding with $\rho = 0.5$. The block length M and the number of additional coded packets α are set to be 10 and 2, respectively. Just for this example, all packet drops are assumed to be independent with the same drop probability. As expected, the ideal block code has the smallest expected value of the number of lost packets for a given value of the number of dropped packets. When a systematic MDS code is employed, all dropped data packets can be recovered if the number of dropped packets is at most α ($\alpha = 2$ in this example); otherwise, no dropped data packets can be recovered. When partial coding is used, even though the number of dropped packets is greater than α, the dropped data packets that belong to the coding part can be recovered if the number of dropped packets in the coding part is at most α; in this case, thus, partial coding has better performance than pure block coding. On the other hand, if the number of dropped packets is not greater than α, then partial coding underperforms pure block coding since the dropped data packets in the non-coding part can not be recovered.

E. Coding with overlapping blocks

In this subsection, we examine one particular scheme that utilizes overlapping blocks. Suppose that each half of a block overlaps with either one of its adjacent blocks and that $\alpha/2$ additional packets are generated from each block of length M; M and α are assumed to be even for simplicity. Note that the number of additional packets per block is halved for fair comparison to the scheme with non-overlapping blocks since the number of blocks is doubled. Let $\{v_i, i \in \mathbb{Z}\}$ be the sequence of data packets from a source. The nth coding block $\mathbf{v}_n$, $n \in \mathbb{Z}$, is given by $\mathbf{v}_n = [\dot{\mathbf{v}}_n \ \dot{\mathbf{v}}_{n+1}]$, where $\dot{\mathbf{v}}_n = [v_{(n-1)M/2+1} \ v_{(n-1)M/2+2} \ \dots \ v_{nM/2}]$. The output packets $\mathbf{u}_n$, which are generated from $\mathbf{v}_n$, include the data packets in $\dot{\mathbf{v}}_n$ and additional $\alpha/2$ coded packets. Observe that the data packets in $\dot{\mathbf{v}}_n$ are used to generate two sets of $\alpha/2$ coded packets. It is assumed that a systematic MDS code is used to encode each block, and each block is decoded independently, i.e., no dropped data packets are recovered if the number of dropped packets in a block is greater than $\alpha/2$.

Let $\dot{D}_d^n$ denote the number of dropped packets in $\dot{\mathbf{v}}_n$, and let $\dot{D}_c^n$ denote the number of dropped packets among the additional $\alpha/2$ coded packets that are generated from $\mathbf{v}_n$. In addition, let $\dot{L}^n$ denote the number of lost packets in $\dot{\mathbf{v}}_n$ after decoding. The following lemma characterizes the number of lost packets in one half of a block when the scheme with overlapping blocks is employed.

Lemma 1. *Suppose that the system is in stationarity. If $\{\dot{D}_d^n, n \in \mathbb{Z}\}$ and $\{\dot{D}_c^n, n \in \mathbb{Z}\}$ are two independent i.i.d. sequences, then*

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1-\xi(k))^2,$$

where $\xi(0) = 1$, $\xi(k) = 0$, $k > \alpha/2$, and $\xi(k)$, $1 \le k \le \alpha/2$, satisfies the following equation:

$$\begin{aligned}\xi(k) &= \mathbb{P}[\dot{D}_d^n + \dot{D}_c^n \le \alpha/2 - k] \\ &\quad + \sum_{i=1}^{\alpha/2} \xi(i)\mathbb{P}[\alpha/2 - k - i < \dot{D}_c^n \le \alpha/2 - k]\mathbb{P}[\dot{D}_d^n = i].\end{aligned}$$

Proof: See Subsection X-A. ∎

V. Loss probability without coding

This section discusses the loss probability due to buffer overflow in a system without coding. Since the link capacity is finite, if the number of packets generated by users exceeds the capacity, some packets should either be stored in the buffer, if possible, or be dropped from the queue. In a system without coding, every dropped packet is also a lost packet; thus, in this case, the loss probability is equal to the drop probability. We first study queue occupancy, i.e., the number of packets stored in the buffer, and, then, use it to analyze the loss probability in the following subsection.

A. Queue occupancy

Recall that $A_{(i)}^N(t)$, $1 \le i \le N$, $t \ge 1$, is the number of packet arrivals in the time slot t from the source i. Let $A^N(t)$, $t \ge 1$, denote the number of packets generated from all N sources in the time slot t:

$$A^N(t) = \sum_{i=1}^{N} A_{(i)}^N(t).$$

The queue occupancy $Q^N(t)$, $t \ge 0$, is defined to be the number of packets that remain in the buffer at the end of the time slot t. The packets that are transmitted in the time slot t include the packets that were in the buffer at the end of the previous time slot as well as newly arrived packets in the time slot t. Recall that the link is capable of transmitting C^N packets in one time slot and that at most B^N packets can be stored in the buffer. Therefore, the queue occupancy satisfies the following well-known equation:

$$Q^N(t) = (Q^N(t-1) + A^N(t) - C^N)^+ \wedge B^N. \tag{6}$$

The random variable $Q^N(t)$, $t \ge 1$, depends on $Q^N(t-1)$ and $A^N(t)$. Since the random variables $A^N(t)$, $t = 1, 2, \dots$, are i.i.d., the process $\{\hat{Q}^N(t), t \ge 0\}$ is a Markov chain with state space $\{0, 1, \dots, B^N\}$ and transition probabilities P_{ij}^N, $0 \le i, j \le B^N$, given by

$$P_{ij}^N = \begin{cases} \mathbb{P}\left[A^N \le C^N + j - i\right], & j = 0, \\ \mathbb{P}\left[A^N = C^N + j - i\right], & 1 \le j \le B^N - 1, \\ \mathbb{P}\left[A^N \ge C^N + j - i\right], & j = B^N, \end{cases} \tag{7}$$

where A^N is equal in distribution to $A^N(t)$. This Markov chain is aperiodic, finite and irreducible, and, therefore, it has an unique stationary distribution $\pi^N(i)$, $0 \le i \le B^N$. Assuming that all processes are in their stationary regimes, we have

$$\pi^N(i) = \mathbb{P}\left[Q^N(t) = i\right]. \tag{8}$$

Next we consider the scaled queue occupancy $\hat{Q}^N(t)$, $t \ge 0$, defined as

$$\hat{Q}^N(t) = Q^N(t)/\sqrt{N}.$$

Note that (6) can be rewritten in the following form:

$$\hat{Q}^N(t) = (\hat{Q}^N(t-1) + \hat{A}^N(t) - \hat{C}^N)^+ \wedge \hat{B}^N,$$

where $\hat{A}^N(t) = (A^N(t) - \lambda N)/\sqrt{N}$. Letting $F_{\hat{Q}^N}$ denote the distribution function of $\hat{Q}^N(t)$, it follows that (see (8))

$$F_{\hat{Q}^N}(x) = \sum_{j=0}^{\lfloor x\sqrt{N} \rfloor} \pi^N(j) = \sum_{i=0}^{B^N} \pi^N(i) \sum_{j=0}^{\lfloor x\sqrt{N} \rfloor} P_{ij}^N, \tag{9}$$

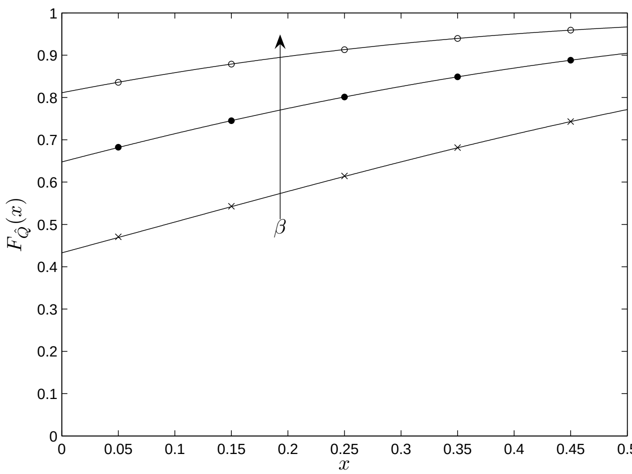

Fig. 3. The distribution function $F_{\hat{Q}}$ of the stationary scaled queue occupancy $\hat{Q}^N(t) = Q^N(t)/\sqrt{N}$ in the limit as $N \to \infty$ (see (12)) for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0.1, 0.3, 0.5\}$. Simulation results for $N = 100$ and $\beta = 0.1$ ($\times$), $\beta = 0.3$ ($\bullet$) and $\beta = 0.5$ ($\circ$) are also shown.

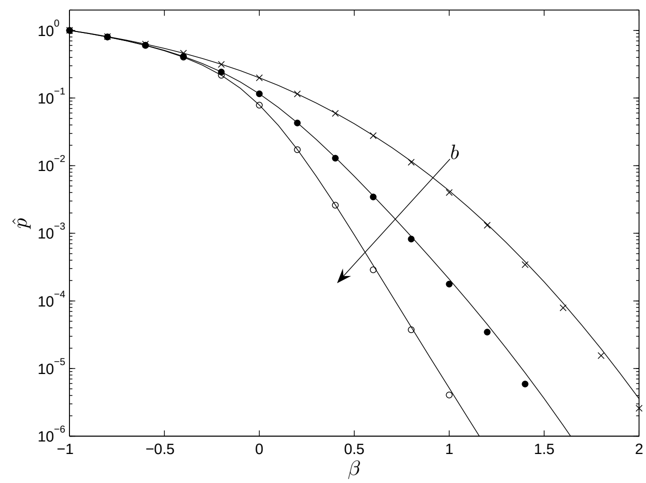

Fig. 4. The scaled loss probability without coding $\hat{p}^N = \lambda\sqrt{N}p^N$ in the limit as $N \to \infty$ (see (16)) for $\lambda = 0.5$ and $b \in \{0, 0.5, 1\}$. Simulation results for $N = 100$ and $b = 0$ ($\times$), $b = 0.5$ ($\bullet$) and $b = 1$ ($\circ$) are also shown.

and $\pi^N(i)$ can be represented by

$$\pi^N(i) = F_{\hat{Q}^N}(i/\sqrt{N}) - F_{\hat{Q}^N}((i-1)/\sqrt{N}). \tag{10}$$

Furthermore, (7) implies

$$\sum_{j=0}^{\lfloor x\sqrt{N}\rfloor} P_{ij}^N = \begin{cases} \mathbb{P}\left[\hat{A}^N - \hat{C}^N \le \frac{\lfloor x\sqrt{N}\rfloor - i}{\sqrt{N}}\right], & 0 \le x < \hat{B}^N, \\ 1, & x = \hat{B}^N, \end{cases} \tag{11}$$

where $\hat{A}^N = (A^N - \lambda N)/\sqrt{N}$. It can be shown that the distribution of the random variable $\hat{Q}^N(t)$ converges to the distribution of a random variable $\hat{Q}$, as $N \to \infty$. That is, if $F_{\hat{Q}}$ is the distribution function of $\hat{Q}$, then $F_{\hat{Q}^N}(x) \to F_{\hat{Q}}(x)$, as $N \to \infty$, for $x \neq 0$ and $x \neq b$ (e.g., see [8, Sec. 25]). Observe that the distribution of $\hat{A}^N$ in (11) tends to the normal distribution with zero mean and variance $\lambda(1-\lambda)$, as $N \to \infty$ (due to the CLT). Moreover, $\hat{C}^N \to \beta$ and $\hat{B}^N \to b$, as $N \to \infty$. Therefore, from (9), (10) and (11), the distribution function $F_{\hat{Q}}$ satisfies the following integral equation:

$$F_{\hat{Q}}(x) = \begin{cases} 0, & x < 0, \\ \int_{[0,b]} \Phi_{-\beta,\sigma^2}(x-y)dF_{\hat{Q}}(y), & 0 \le x < b, \\ 1, & x = b, \end{cases} \tag{12}$$

where $\Phi_{-\beta,\sigma^2}$ denotes the normal distribution function with mean $-\beta$ and variance $\sigma^2 = \lambda(1-\lambda)$. Note that $F_{\hat{Q}}$ has discontinuities at $x = 0$ and $x = b$.

Figure 3 shows the distribution functions of $\hat{Q}$ for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0.1, 0.3, 0.5\}$, which are numerically computed from (12). For a fixed value of x, $0 \le x < b$, the value of $F_{\hat{Q}}(x)$ increases as β increases since larger β implies a larger capacity. In addition, this figure includes the estimated values of $F_{\hat{Q}^N}(x)$ (by simulation) for $N = 100$, $B^N = 5$ ($b = 0.5$) and $C^N \in \{51, 53, 55\}$ ($\beta \in \{0.1, 0.3, 0.5\}$).

B. Loss probability

Let $L^N(t)$, $t \ge 1$, denote the number of lost packets in the time slot t. Without coding, a dropped packet is also a lost packet since no dropped packets can be recovered. Therefore, we have

$$L^N(t) = (Q^N(t-1) + A^N(t) - C^N - B^N)^+. \tag{13}$$

The loss probability p^N is defined to be the long-term ratio of the number of lost packets to the total number of arrivals from N sources, i.e.,

$$p^N = \lim_{T\to\infty} \frac{\sum_{t=1}^T L^N(t)}{\sum_{t=1}^T A^N(t)}.$$

Given that the system is in stationarity and it is ergodic, p^N can equivalently be represented by

$$p^N = \mathbb{E}[L^N(t)]/\mathbb{E}[A^N(t)]. \tag{14}$$

The preceding equality and (13) yield

$$p^N = \mathbb{E}\,(Q^N + A^N - C^N - B^N)^+/\lambda N, \tag{15}$$

where A^N is equal in distribution to $A^N(t)$, and Q^N has the stationary distribution of $Q^N(t)$; the random variables Q^N and A^N are independent. As discussed in Section III, the loss probability under the critical-load scaling behaves as $\Theta(1/\sqrt{N})$, as $N \to \infty$, and, thus, we define the scaled loss probability $\hat{p}^N$ by

$$\hat{p}^N = \lambda\sqrt{N}p^N = \mathbb{E}\,(\hat{Q}^N + \hat{A}^N - \hat{C}^N - \hat{B}^N)^+,$$

where $\hat{A}^N = (A^N - \lambda N)/\sqrt{N}$ and $\hat{Q}^N = Q^N/\sqrt{N}$. Furthermore, the limiting (as $N \to \infty$) scaled loss probability $\hat{p}$ is defined by

$$\hat{p} = \lim_{N\to\infty} \hat{p}^N = \mathbb{E}\,(\hat{Q} + \hat{A} - \beta - b)^+, \tag{16}$$

where $\hat{A}^N \Rightarrow \hat{A}$, $\hat{Q}^N \Rightarrow \hat{Q}$, as $N \to \infty$, and the random variables $\hat{A}$ and $\hat{Q}$ are independent.

Figure 4 shows $\hat{p}$ as a function of β for $\lambda = 0.5$ and $b \in \{0, 0.5, 1\}$. As expected, the loss probability decreases when β (capacity) or b (buffer size) increase. Moreover, this figure includes estimated values of $\hat{p}^N$ (by simulation) for $N = 100$

and $B^N \in \{0, 5, 10\}$ ($b \in \{0, 0.5, 1\}$). This example illustrates the applicability of our asymptotic analysis to systems with a finite number of users.

The loss probability can be approximated for large values of $|\beta|$. To this end, we have $\hat{p} \approx \mathbb{E}(\hat{A} - \beta - b)^+$ for $\beta \gg 0$ since the buffer is likely to be empty when the link capacity is larger than the offered load. In this case, it follows that

$$\hat{p} \approx \int_{\beta+b}^{\infty} (x - \beta - b)\varphi_{0,\sigma^2}(x)dx \approx \frac{\sigma^4}{(\beta+b)^2}\varphi_{0,\sigma^2}(\beta+b), \tag{17}$$

where φ_{0,σ^2} is the probability density function of the normal distribution with zero mean and variance $\sigma^2 = \lambda(1-\lambda)$; the approximation follows from $(x^{-1} - x^{-3})\varphi_{0,1}(x) < 1 - \Phi_{0,1}(x) < x^{-1}\varphi_{0,1}(x)$ (e.g., see [9, p.175]). On the other hand, if $\beta \ll 0$, then the buffer is likely to be full since the offered load is greater than the link capacity. Thus, in that case, we obtain

$$\hat{p} \approx \mathbb{E}(\hat{A} - \beta)^+ \approx -\beta. \tag{18}$$

In such an over-loaded system, all extra arrivals, which exceeds the capacity, are likely to be dropped from the queue since the buffer is full with high probability.

VI. Drop probability with coding

In this section, we examine the *drop* probability when coding is employed. When a system utilizes coding, the offered load is increased by additional coded packets, and, consequently, more packets are likely to be dropped from the buffer, compared to a system without coding. Note that in this case, the *drop* probability should be differentiated from the *loss* probability since some of the dropped packets can be recovered from the received subset of packets by decoding. We discuss the *loss* probability under coding in the next section.

Recall that $H_{(i)}^N(t)$, $1 \le i \le N$, $t \ge 1$, denotes the number of packet arrivals from the encoder of the source i to the buffer in the time slot t. Assuming that the system is in stationarity, (1) implies

$$\mathbb{P}\left[H_{(i)}^N(t) = h\right] = \begin{cases} 1-\lambda, & h = 0, \\ \lambda - \lambda/M^N, & h = 1, \\ \lambda/M^N, & h = 1+\alpha, \end{cases} \tag{19}$$

for $1 \le i \le N$, $t \ge 1$. The mean λ_*^N and the variance $(\sigma_*^N)^2$ of $H_{(i)}^N(t)$ are respectively given by

$$\begin{aligned} \lambda_*^N &= \lambda + \alpha\lambda/M^N, \\ (\sigma_*^N)^2 &= \lambda_*^N(1-\lambda_*^N) + \alpha(1+\alpha)\lambda/M^N; \end{aligned} \tag{20}$$

note that $\lambda_*^N \to \lambda$ and $(\sigma_*^N)^2 \to \lambda(1-\lambda)$, as $N \to \infty$. Let $H^N(t)$, $t \ge 1$, denote the total number of packets sent from the encoders of N sources to the buffer in the time slot t:

$$H^N(t) = \sum_{i=1}^{N} H_{(i)}^N(t).$$

The drop probability p_D^N is defined to be the long-term ratio of the number of dropped packets to the total number of arrivals from the encoders. Then, analogously to (15), we have

$$p_D^N = \mathbb{E}(Q_*^N + H^N - C^N - B^N)^+/\lambda_*^N N,$$

where H^N is equal in distribution to $H^N(t)$, and Q_*^N has the stationary distribution of the queue occupancy when coding is used. When coding is employed, original arrival processes are altered by additional coded packets as stated in Section II. Thus, the queue occupancy is also affected by the coding scheme. Under the critical-load scaling, the drop probability is $\Theta(1/\sqrt{N})$, as $N \to \infty$. Therefore, we consider the scaled drop probability $\hat{p}_D^N$ defined by

$$\begin{aligned} \hat{p}_D^N &= \lambda\sqrt{N}p_D^N \\ &= \frac{\lambda}{\lambda_*^N}\mathbb{E}(\hat{Q}_*^N + \hat{H}^N + (\lambda_*^N - \lambda)\sqrt{N} - \hat{C}^N - \hat{B}^N)^+, \end{aligned}$$

where $\hat{H}^N = (H^N - \lambda_*^N N)/\sqrt{N}$ and $\hat{Q}_*^N = Q_*^N/\sqrt{N}$; note that $(\lambda_*^N - \lambda)\sqrt{N} \to \alpha/m$, as $N \to \infty$. Next we define $\hat{p}_D$ as the limiting (as $N \to \infty$) scaled drop probability:

$$\hat{p}_D = \lim_{N\to\infty} \hat{p}_D^N = \mathbb{E}(\hat{Q}_* + \hat{H} + \alpha/m - \beta - b)^+, \tag{21}$$

where $\hat{H}^N \Rightarrow \hat{H}$, $\hat{Q}_*^N \Rightarrow \hat{Q}_*$, as $N \to \infty$, and the random variables $\hat{H}$ and $\hat{Q}_*$ are independent. It can be shown that $\hat{H}$ has the normal distribution with zero mean and variance $\lambda(1-\lambda)$ (due to the CLT) and that the distribution of $\hat{Q}_*$ satisfies (12) with β replaced by $\beta - \alpha/m$ (e.g., see [8, Sec. 25]).

We define the coding overhead ζ as a function of α/m:

$$\zeta(\alpha/m) = \frac{\hat{p}_D}{\hat{p}} = \frac{\mathbb{E}(\hat{Q}_* + \hat{H} + \alpha/m - \beta - b)^+}{\mathbb{E}(\hat{Q} + \hat{A} - \beta - b)^+}. \tag{22}$$

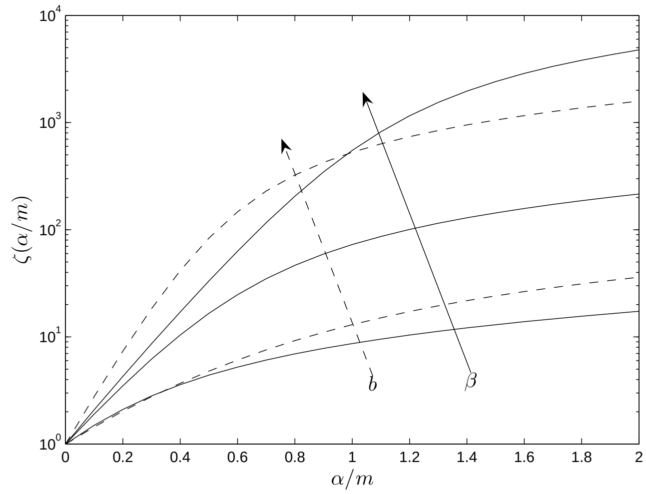

Fig. 5. The coding overhead $\zeta(\alpha/m) = \hat{p}_D/\hat{p}$ (see (22)) for $\lambda = 0.5$: solid lines for $b = 0.5$ and $\beta \in \{0, 0.5, 1\}$, and dashed lines for $\beta = 0.5$ and $b \in \{0, 1\}$.

In Figure 5, the solid lines show $\zeta(\alpha/m)$ for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0, 0.5, 1\}$. The dashed lines are for $\lambda = 0.5$, $\beta = 0.5$ and $b \in \{0, 1\}$. Since the additional offered load due to coding increases as α/m increases, ζ is an increasing function of α/m. The figure also illustrates that, for a fixed value of α/m, the value of ζ increases when β (capacity) or b (buffer size) increase. Note that $\hat{p}_D$ is exactly equal to $\hat{p}$ when β is replaced by $\beta - \alpha/m$ in (16). As seen in Figure 4, the larger the β is, the faster the $\hat{p}$ decreases as β increases. Thus, $\hat{p}$ decreases faster than $\hat{p}_D$ when β increases. Approximations given in (17) and (18) also support this observation. Namely, for $\alpha/m \gg \beta$, $\hat{p}_D$ decreases linearly when β increases while $\hat{p}$ decreases exponentially; when $\alpha/m \ll \beta$, both $\hat{p}$ and $\hat{p}_D$ decrease exponentially, but $\hat{p}$ decreases faster than $\hat{p}_D$ due to α/m term. Similar reasoning can be applied to the case of b.

Since decoding is performed on a per-block basis, the loss probability depends not only on the drop probabilities of individual packets but also on the distribution of the number of dropped packets in a block. Thus, in order to evaluate the loss probability, one needs to consider the behavior of the packet drops in a block. The following theorem characterizes the number of dropped packets in a block in the limit as $N \to \infty$.

Theorem 1. *Suppose that the system is in stationarity, and consider the critical-load scaling. Let D_d^N be the number of dropped packets among M^N data packets in a block. Then, in the limit as $N \to \infty$, D_d^N is Poisson:*

$$\mathbb{P}\left[D_d^N = k\right] \to \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D},$$

as $N \to \infty$, where $\hat{p}_D$ is the limiting scaled drop probability that satisfies (21)*. Furthermore, if D_c^N is the number of dropped packets among additional α coded packets in a block, then, as $N \to \infty$,*

$$\mathbb{P}\left[D_c^N = 0\right] \to 1.$$

Proof: See Subsection X-B. ∎

Informally, the theorem can be interpreted as follows. Consider a single block, and suppose that the packets in this block are dropped independently with drop probability equal to p_D^N. Then, the number of dropped packets in the block of length $M^N = \lfloor m\lambda\sqrt{N} \rfloor$ follows the binomial distribution:

$$\mathbb{P}\left[D_d^N = k\right] = \binom{\lfloor m\lambda\sqrt{N}\rfloor}{k} \left(p_D^N\right)^k \left(1 - p_D^N\right)^{\lfloor m\lambda\sqrt{N}\rfloor - k}.$$

It is straightforward to verify that this binomial distribution tends to the Poisson distribution with mean $m\hat{p}_D$ in the limit as $N \to \infty$. However, packet drops are not independent in a system with finite N. The drop probability of a packet in a fixed time slot t depends on the total number of arrivals from the encoders of N sources in the time slot t and the queue occupancy at the end of the time slot $t-1$. Since both the total arrival process and the queue occupancy have the Markov property, as discussed in Section II, packet drops have dependency across time. However, Theorem 1 shows that the effect of this time dependency becomes negligibly small in the limit as $N \to \infty$. Given that the drop probability in a fixed time slot t is $\Theta(1/\sqrt{N})$, the possibility that a packet is dropped shortly after another packet is dropped from the same block diminishes as $N \to \infty$. That is, when a packet is dropped, we can assure, with high probability, that enough time has elapsed for the system to enter its stationary regime.

Finally, Theorem 1 also indicates that the number of dropped packets in a block is $\Theta(1)$. Since the number of additional coded packets for each block is also $\Theta(1)$, the dropped packets can be recovered in some cases, as intended by means of coding. This result verifies the relevance of the considered critical-load scaling to the study of the effectiveness of application-layer coding.

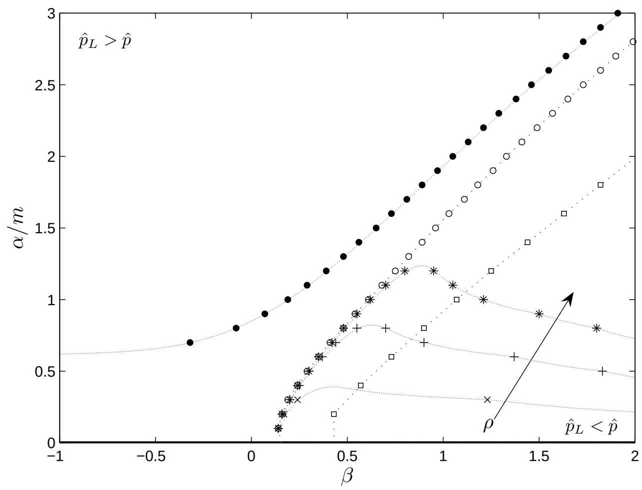

Fig. 6. The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for the ideal block code (●), a systematic MDS code (○), coding with overlapping blocks (□) and partial coding with $\rho = 0.9$ (×), $\rho = 0.99$ (+) and $\rho = 0.999$ (∗) when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that $\alpha/m \in \{0.1, 0.2, \ldots\}$ since $\alpha \in \mathbb{N}$.

VII. Loss probability with coding

Here, we consider the loss probability with coding for erasure codes discussed in Section IV. The loss probability p_L^N is defined as the long-term ratio of lost packets after decoding to the total number of data packets. Let L^N denote the number of lost packets among $M^N = \lfloor m\lambda\sqrt{N} \rfloor$ data packets in a block. Then, analogously to (14), we have

$$p_L^N = \mathbb{E}\, L^N / M^N.$$

The scaled loss probability $\hat{p}_L^N$ is given by

$$\hat{p}_L^N = \lambda\sqrt{N} p_L^N = \mathbb{E}\, L^N / m^N,$$

where $m^N = \lfloor m\lambda\sqrt{N} \rfloor / \lambda\sqrt{N}$, and the limiting (as $N \to \infty$) scaled loss probability $\hat{p}_L$ is defined by

$$\hat{p}_L = \lim_{N\to\infty} \hat{p}_L^N = \lim_{N\to\infty} \mathbb{E}\, L^N / m.$$

A. Ideal block code

From (3), the scaled loss probability $\hat{p}_L^N$ for the ideal block code satisfies

$$\hat{p}_L^N = \mathbb{E}\,(D_d^N + D_c^N - \alpha)^+ / m^N,$$

where the limiting distributions of D_d^N and D_c^N are given in Theorem 1. Letting $N \to \infty$ renders an expression for the limiting scaled loss probability $\hat{p}_L$:

$$\hat{p}_L = \mathbb{E}\,(D - \alpha)^+ / m, \tag{23}$$

where the random variable D has the Poisson distribution with mean $m\hat{p}_D$.

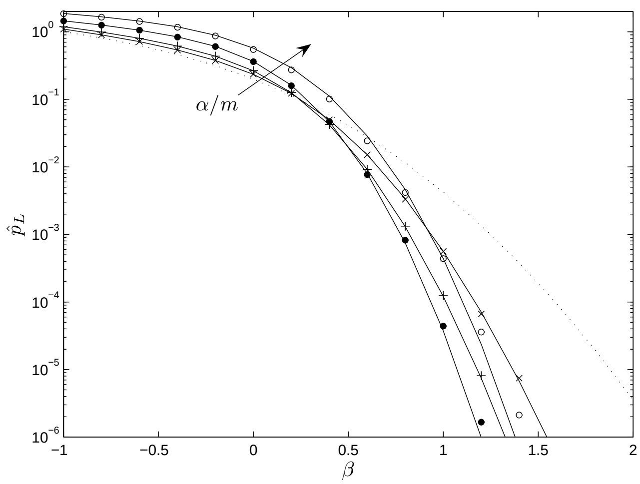

Fig. 7. The scaled loss probability with coding $\hat{p}_L^N = \lambda\sqrt{N}p_L^N$ in the limit as $N \to \infty$ (see (24)) for a systematic MDS code when $\lambda = 0.5$, $b = 0$, $m = 10$ and $\alpha \in \{1, 2, 5, 10\}$. The variances of arrival processes are adjusted to be $(\sigma_*^N)^2$ for $N = 900$ (see (20)) instead of the limiting value $\lambda(1-\lambda)$. Simulation results for $N = 900$ and $\alpha = 1$ ($\times$), $\alpha = 2$ ($+$), $\alpha = 5$ ($\bullet$) and $\alpha = 10$ ($\circ$) are also shown. The dotted line is for the scaled loss probability without coding.

By using (16), (21) and (23), one can compare the loss probabilities with and without coding for a given set of parameters (β, b, α, m). In Figure 6, we show the boundary where $\hat{p}_L = \hat{p}$ for the ideal block code when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that the boundary partitions the parameter space $(\beta, \alpha/m)$ into two regions: the upper left region where $\hat{p}_L > \hat{p}$ (no coding is preferable) and the lower right region where $\hat{p}_L < \hat{p}$ (coding is preferable). The ideal block code has the largest region where coding is advantageous among all linear block codes since it achieves the best performance among such codes. It is interesting to observe that employing even the ideal block code can be counter-productive for some set of system parameters. In particular, consider an over-loaded system ($\lambda N > C^N$ or, equivalently, $\beta < 0$). In this case, the expected number of dropped packets in a single block *due to coding overhead* is $(p_D - p) \cdot M^N \approx \alpha$. However, the number of dropped packets in a block is not a constant; this leads to a situation where more than α packets are dropped in some blocks and fewer than α packets are dropped in the others. In such a case, coding is not efficient in recovering dropped packets.

B. Systematic MDS code

We can obtain the scaled loss probability $\hat{p}_L^N$ for a systematic MDS code from (4):

$$\hat{p}_L^N = \mathbb{E}\big[D_d^N \cdot 1_{\{D_d^N + D_c^N > \alpha\}}\big]/m^N.$$

Then, due to Theorem 1, the limiting scaled loss probability $\hat{p}_L$ is given by

$$\hat{p}_L = \mathbb{E}\big[D \cdot 1_{\{D > \alpha\}}\big]/m, \tag{24}$$

where the random variable D has the Poisson distribution with mean $m\hat{p}_D$. We refer the reader to Section VIII for a further discussion on application-layer coding with a systematic MDS code.

Figure 7 shows $\hat{p}_L$ as a function of β for a systematic MDS code when $\lambda = 0.5$, $b = 0$, $m = 10$ and $\alpha \in \{1, 2, 5, 10\}$. Note that, just for this example, we use the variance $(\sigma_*^N)^2$ for $N = 900$ (see (20)) instead of the limiting value $\lambda(1-\lambda)$ when we compute $\hat{p}_D$, which determines $\hat{p}_L$. Since the loss probability is sensitive to the variances of arrival processes, this adjustment is needed to make our asymptotic result to be applicable for finite N. The figure also shows the estimated values of $\hat{p}_L^N$ (by simulation) for $N = 900$, $M^N = 150$ ($m = 10$) and $\alpha \in \{1, 2, 5, 10\}$. One can observe that simulation results agree with analytical results well in this example. For an over-loaded system ($\beta \ll 0$), $\hat{p}_L$ increases as α increases because the number of dropped packets in a block is likely to be beyond the number that can be recovered. Therefore, in this case, additional packets behave just as overhead. On the other hand, if a system is under-loaded ($\beta \gg 0$), then $\hat{p}_L$ decreases at first as α increases since the benefit of coding exceeds its overhead in this case. For some value of α, $\hat{p}_L$ is minimized, i.e., the coding benefit is maximized. If α is increased further, however, $\hat{p}_L$ starts increasing, and coding is not beneficial anymore. The dotted line represents the limiting scaled loss probability without coding $\hat{p}$ (see (16)). One can find a point where $\hat{p}_L = \hat{p}$ for each value of α/m. These points correspond to the boundary where schemes with and without coding have the same performance (shown in Figure 6).

C. Partial coding

The scaled loss probability $\hat{p}_L^N$ for the partial coding scheme can be computed from (5):

$$\hat{p}_L^N = \mathbb{E}\big[D_d^N - \tilde{D}_d^N \cdot 1_{\{\tilde{D}_d^N + D_c^N \le \alpha\}}\big]/m^N,$$

where $\tilde{D}_d^N$ denotes the number of dropped packets among $\lfloor \rho M^N \rfloor$ data packets in the coding part of a block. Similarly to Theorem 1, it can be shown that $\tilde{D}_d^N$ tends to Poisson with mean $\rho m \hat{p}_D$, as $N \to \infty$. Then, the limiting scaled loss probability $\hat{p}_L$ is given by

$$\hat{p}_L = \hat{p}_D - \mathbb{E}\big[\tilde{D} \cdot 1_{\{\tilde{D} \le \alpha\}}\big]/m,$$

where the random variable $\tilde{D}$ has the Poisson distribution with mean $\rho m \hat{p}_D$.

Figure 6 includes the boundary where $\hat{p}_L = \hat{p}$ for partial coding with $\rho \in \{0.9, 0.99, 0.999\}$ when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that the partial coding scheme with $\rho = 1$ is identical to the scheme with pure block coding. As seen in the figure, the region where coding is advantageous expands as ρ increases. Recall that the partial coding scheme might be beneficial only when the number of dropped packets in a block is greater than the number of additional packets in the block (see Section IV). In the region where coding is advantageous, however, the drop probability is so small that the number of dropped packets is not likely to be greater than the number of additional packets in a block. One can observe that partial coding is getting worse as β (capacity) increases. This result is consistent with the previous observation since larger β results in smaller drop probability.

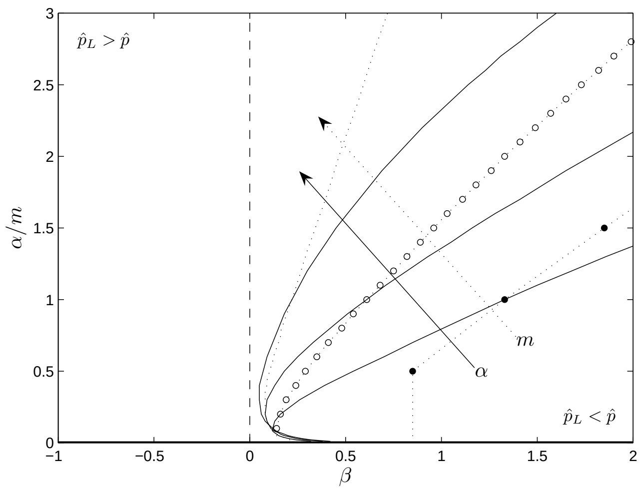

Fig. 8. The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for a systematic MDS code when $\lambda = 0.5$ and $b = 0.5$: solid lines for $\alpha \in \{2, 10, 50\}$, and dotted lines for $m = 2$ (●), $m = 10$ (○) and $m = 50$ (·).

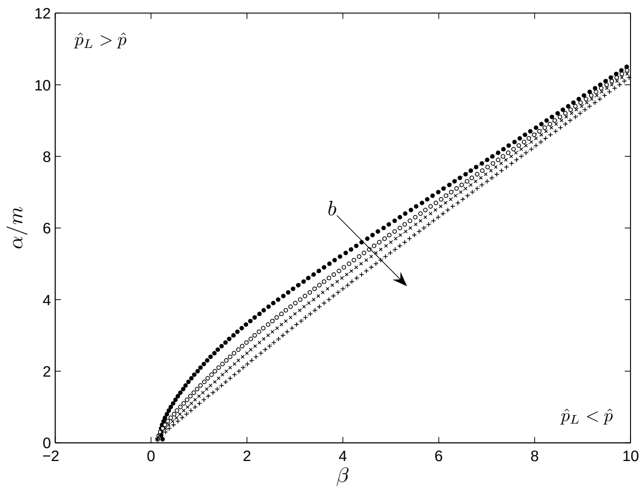

Fig. 9. The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for a systematic MDS code when $\lambda = 0.5$, $m = 10$ and $b \in \{0, 0.5, 1, 2\}$.

D. Coding with overlapping blocks

By combining Lemma 1 and Theorem 1, it can be shown that the limiting scaled loss probability $\hat{p}_L^N$ for the scheme with overlapping blocks satisfies

$$\hat{p}_L = 2\mathbb{E}\big[\dot{D}(1 - \xi(\dot{D}))^2\big]/m,$$

where the random variable $\dot{D}$ has the Poisson distribution with mean $m\hat{p}_D/2$, and $\xi(0) = 1$, $\xi(k) = 0$, $k > \alpha/2$, and $\xi(k)$, $1 \le k \le \alpha/2$, satisfies the following equation:

$$\xi(k) = \mathbb{P}[\dot{D} \le \alpha/2 - k] + \sum_{i=\alpha/2-k+1}^{\alpha/2} \xi(i)\mathbb{P}[\dot{D} = i];$$

recall that we only consider even values of α for simplicity.

In Figure 6, we plot the boundary where $\hat{p}_L = \hat{p}$ for coding with overlapping blocks. As seen in the figure, the described coding scheme with overlapping blocks underperforms compared to the one with non-overlapping blocks as far as probability of loss is concerned. This stems from the fact that the non-overlapping scheme can recover up to α dropped packets per block, while the overlapping version is capable of recovering only $\alpha/2$ dropped packets per half block. It should be noted, however, that the overlapping scheme might result in shorter decoding delays.

VIII. Discussion

In this section, we discuss the performance of application-layer coding with a systematic MDS code. Figure 8 shows the boundary where $\hat{p}_L = \hat{p}$ for a systematic MDS code when $\lambda = 0.5$, $b = 0.5$, $m \in \{2, 10, 50\}$ and $\alpha \in \{2, 10, 50\}$. If we increase the length of a block while increasing the number of additional packets as well, then the asymptotic drop probability does not change, but the number of possible packet drop patterns that can be recovered in a block increases. For example, suppose that $\alpha = 1$ and 1 dropped packet can be recovered in a block. If we double the length of a block and generate 2 coded packets per double-length block, then 2 dropped packets can be recovered in one half of a block provided that no packets are dropped in the other half. Note that larger m (and larger α for fixed α/m) implies a longer block. Hence, the region where coding is advantageous to no coding increases as m (and α for fixed α/m) increases. Note that this reasoning applies to a large class of block codes.

Figure 9 shows the boundary where $\hat{p}_L = \hat{p}$ for a systematic MDS code when $\lambda = 0.5$, $m = 10$ and $b \in \{0, 0.5, 1, 2\}$. Recall that Figure 5 indicates that the coding overhead $\zeta = \hat{p}_D/\hat{p}$ increases as b (buffer size) increases. Thus, larger b results in a smaller region where coding outperforms no coding. It is interesting to observe that the critical values of α/m for different values of b converge as β (capacity) increases. In particular, the boundary tends to $\alpha/m = \beta$ as β increases. As long as the system is under-loaded ($\alpha/(m\beta) < 1$), for large β (and, hence, large $\beta - \alpha/m$ for a fixed ratio of $\alpha/(m\beta)$), the buffer is likely to be empty with high probability; when the system is over-loaded ($\alpha/(m\beta) > 1$), however, the buffer is likely to be full. This behavior is not significantly impacted by the buffer size b. Thus, (for $\beta \gg 0$) the value of b only has a secondary effect on the loss probability, and, therefore, does not perform a significant role in determining the boundary.

In Figure 10, we plot the boundary where a buffer-less system with coding (using a systematic MDS code) and a system with a buffer but no coding have the same performance, i.e., $\min_\alpha \hat{p}_L$ for $b = 0$ and $\hat{p}$ for $b > 0$ are equal, for $\lambda = 0.5$ and $m \in \{10, 20, 50\}$. Similarly to Figure 8, this figure also illustrates that larger m (block length) results in a larger region where coding is advantageous. Moreover, the figure indicates that the boundary tends to $\beta = 0$ (the dashed line) in the limit as $m \to \infty$. Informally, from (24), we can derive

$$\hat{p}_L = \hat{p}_D \sum_{k=\alpha}^{\infty} \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D}.$$

In the limit as $m \to \infty$, the Poisson distribution tends to the normal distribution with mean $m\hat{p}_D$ and variance $m\hat{p}_D$:

$$\hat{p}_L \approx \hat{p}_D \left(1 - \Phi_{0,1}\left(\frac{\alpha - m\hat{p}_D}{\sqrt{m\hat{p}_D}}\right)\right);$$

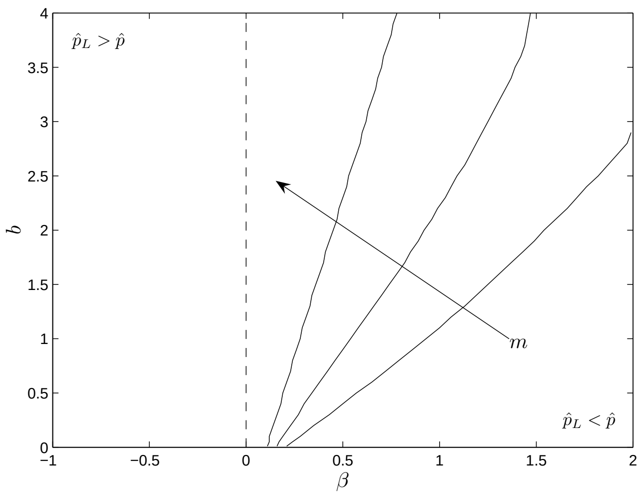

Fig. 10. The boundary where a buffer-less system with coding (using a systematic MDS code) and a system with a buffer but no coding have the same performance, i.e., $\min_\alpha \hat{p}_L^N$ for $b = 0$ and $\hat{p}^N$ for $b > 0$ are equal, as $N \to \infty$, for $\lambda = 0.5$ and $m \in \{10, 20, 50\}$. The figure also shows that the boundary tends to $\beta = 0$ (the dashed line) in the limit as $m \to \infty$.

this, in turn, implies (for large values of m)

$$\hat{p}_L \approx \begin{cases} 0, & \alpha/m > \hat{p}_D, \\ \hat{p}_D/2, & \alpha/m = \hat{p}_D, \\ \hat{p}_D, & \alpha/m < \hat{p}_D. \end{cases}$$

Now, for an under-loaded system ($\beta > 0$), there exists some α that satisfies $\alpha/m > \hat{p}_D$ for large m. Thus, it is possible to reduce $\hat{p}_L$ to an arbitrary small value by increasing m (block length). On the other hand, for an over-loaded system ($\beta < 0$), we have $\hat{p}_D \approx \alpha/m - \beta > \alpha/m$ from (18). In this case, $\hat{p}_L \approx \hat{p}_D \approx \alpha/m - \beta \geq -\beta > 0$ can not be made arbitrarily close to zero even if m is increased indefinitely.

IX. Concluding Remarks

In this paper, we investigated the effectiveness of application -layer coding for systems with a large number of users. The system consists of a single link with a finite buffer, and the loss probability was considered as the measure of the system performance. We first showed that the critical-load scaling is the relevant scaling to explore the effectiveness of coding. Next we examined the asymptotic behavior of the loss probabilities with and without coding, and established the boundary that partitions the system parameter space into two regions where coding is beneficial and detrimental. The asymptotic results showed that coding is advantageous for under-loaded systems with a certain set of system parameters; in over-loaded systems, however, coding is detrimental since the coding overhead exceeds its benefit. That is, application-layer coding enhances the performance in systems with low drop probabilities, but employing such coding in systems with high drop probabilities only worsens the performance. In addition, we illustrated in some simulation examples that our asymptotic results provide reasonable approximations for systems with a finite number of users.

Finally, we conclude this paper with a comment on systems with priority. As stated in Section II, a system can employ priority in order to improve its performance (implementing priority requires an extra level of system complexity). In systems with priority, data packets have priority over additional coded packets in the queue, and, thus, coded packets do not impact the drops of data packets. Therefore, in this case, coding does not harm the system performance, and one can generate, ideally, as many coded packets as needed to enhance the performance. Let $W^N(t)$, $t \geq 1$, denote the number of *received* coded packets in the time slot t by all N users. In the ideal setup (assuming that every user generates a large enough number of coded packets to fill up the spare capacity in every time slot), we have $W^N(t) = (C^N - A^N(t) - Q^N(t-1))^+$; for simplicity, it is assumed that coded packets are not buffered. Since there are N receivers in the system, the *receiving rate* of coded packets per receiver p_W^N is then given by $p_W^N = \mathbb{E}\,(C^N - A^N - Q^N)^+/N$. From this expression, we can derive the *critical-load scaling* for systems with priority that is similar to the one for systems with non-priority (see Section III). Namely, the link capacity and the buffer size are given by $C^N = \lfloor \lambda N + \beta\sqrt{N} \rfloor$ and $B^N = \lfloor b\sqrt{N} \rfloor$, respectively. Under this scaling, the drop probability of data packets and the receiving rate of coded packets are both $\Theta(1/\sqrt{N})$, as $N \to \infty$. Given a block of length $M^N = \lfloor m\lambda N \rfloor$, the expected numbers of dropped data packets and received coded packets are of the same order (i.e., $\Theta(1)$), and, therefore, it is feasible to find the boundary that partitions the system parameter space into two regions where employing coding is significantly and marginally beneficial.

X. Proofs

A. Proof of Lemma 1

Utilizing a systematic MDS code, we have either $\{\dot{L}^n = 0\}$ or $\{\dot{L}^n = k\}$ on the event $\{\dot{D}_d^n = k\}$. Therefore, it follows that

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1 - \mathbb{P}[\dot{L}^n = 0 | \dot{D}_d^n = k]). \tag{25}$$

Recall that the data packets in $\dot{\mathbf{v}}_n$ are included in both $\mathbf{v}_n$ and $\mathbf{v}_{n-1}$. Assuming that at least one packet from $\dot{\mathbf{v}}_n$ is dropped, define two events $\mathcal{E}_1^n = \{\dot{\mathbf{v}}_n \text{ is recovered by decoding } \mathbf{v}_n\}$ and $\mathcal{E}_2^n = \{\dot{\mathbf{v}}_n \text{ is recovered by decoding } \mathbf{v}_{n-1}\}$. For $k > 0$, we have

$$\begin{aligned}\mathbb{P}[\dot{L}^n = 0 | \dot{D}_d^n = k] &= \mathbb{P}[\mathcal{E}_1^n \cup \mathcal{E}_2^n | \dot{D}_d^n = k] \\ &= 1 - (1 - \mathbb{P}[\mathcal{E}_1^n | \dot{D}_d^n = k])(1 - \mathbb{P}[\mathcal{E}_2^n | \dot{D}_d^n = k]),\end{aligned}$$

where the second equality follows from the assumptions of the lemma. By combining the preceding equality with (25), one can obtain

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1 - \xi_1^n(k))(1 - \xi_2^n(k)), \tag{26}$$

where $\xi_1^n(k) = \mathbb{P}[\mathcal{E}_1^n | \dot{D}_d^n = k]$ and $\xi_2^n(k) = \mathbb{P}[\mathcal{E}_2^n | \dot{D}_d^n = k]$; for notational simplicity, we extended the definition of $\xi_1^n(k)$ and $\xi_2^n(k)$ for $k = 0$. Observe that if $\dot{D}_d^n = 0$ then $\dot{L}^n = 0$ since there are no dropped packets. Besides, if $\dot{D}_d^n > \alpha/2$ then $\dot{L}^n = \dot{D}_d^n$ since no dropped packets can be recovered in this case. Formally, we have

$$\xi_1^n(k) = \xi_2^n(k) = \begin{cases} 1, & k = 0, \\ 0, & k > \alpha/2. \end{cases} \tag{27}$$

Let D^n denote the number of dropped packets among the data packets in $\mathbf{v}_n$ and the additional $\alpha/2$ coded packets generated from $\mathbf{v}_n$, i.e., $D^n = \dot{D}_d^n + \dot{D}_d^{n+1} + \dot{D}_c^n$. If $D^n \le \alpha/2$, then all dropped packets in $\mathbf{v}_n$ can be recovered. Otherwise, the dropped packets in $\mathbf{v}_n$ can be recovered only when the dropped packets in $\dot{\mathbf{v}}_{n+1}$ are recovered by decoding the next block $\mathbf{v}_{n+1}$ and the number of (remaining) unrecovered dropped packets is at most $\alpha/2$. This argument leads to

$$\begin{aligned}\xi_1^n(k) &= \mathbb{P}[D^n \le \alpha/2 | \dot{D}_d^n = k] \\ &\quad + \mathbb{P}[D^n > \alpha/2,\, D^n - \dot{D}_d^{n+1} \le \alpha/2,\, \mathcal{E}_1^{n+1} | \dot{D}_d^n = k],\end{aligned}$$

for $1 \le k \le \alpha/2$. The assumptions of the lemma imply that the event $\mathcal{E}_1^{n+1}$ is independent of both $\dot{D}_d^n$ and $\dot{D}_c^n$. Thus, the second term on the right-hand side of the preceding equality can be expressed as

$$\begin{aligned}&\mathbb{P}[D^n > \alpha/2,\, D^n - \dot{D}_d^{n+1} \le \alpha/2,\, \mathcal{E}_1^{n+1} | \dot{D}_d^n = k] \\ &\qquad = \mathbb{P}[\alpha/2 - \dot{D}_d^{n+1} < \dot{D}_c^n + k \le \alpha/2,\, \mathcal{E}_1^{n+1}],\end{aligned}$$

and, if we represent this as the sum of conditional probabilities (conditioned on the event $\{\dot{D}_d^{n+1} = i\}$ for $1 \le i \le \alpha/2$), then we have

$$\begin{aligned}&\mathbb{P}[\alpha/2 - \dot{D}_d^{n+1} < \dot{D}_c^n + k \le \alpha/2,\, \mathcal{E}_1^{n+1}] \\ &\quad = \sum_{i=1}^{\alpha/2} \mathbb{P}[\alpha/2 - i < \dot{D}_c^n + k \le \alpha/2,\, \mathcal{E}_1^{n+1},\, \dot{D}_d^{n+1} = i].\end{aligned}$$

For notational simplicity, let $\dot{\alpha} = \alpha/2$. Then, one can derive the following equation from the preceding argument:

$$\begin{aligned}\xi_1^n(k) &= \mathbb{P}[\dot{D}_d^{n+1} + \dot{D}_c^n + k \le \dot{\alpha}] \\ &+ \sum_{i=1}^{\dot{\alpha}} \xi_1^{n+1}(i)\mathbb{P}[\dot{\alpha} - i < \dot{D}_c^n + k \le \dot{\alpha}]\mathbb{P}[\dot{D}_d^{n+1} = i], \quad (28)\end{aligned}$$

for $1 \le k \le \dot{\alpha}$. Likewise, it can be shown that

$$\begin{aligned}\xi_2^n(k) &= \mathbb{P}[\dot{D}_d^{n-1} + \dot{D}_c^{n-1} + k \le \dot{\alpha}] \\ &+ \sum_{i=1}^{\dot{\alpha}} \xi_2^{n-1}(i)\mathbb{P}[\dot{\alpha} - i < \dot{D}_c^{n-1} + k \le \dot{\alpha}]\mathbb{P}[\dot{D}_d^{n-1} = i]. \quad (29)\end{aligned}$$

for $1 \le k \le \dot{\alpha}$. Under steady-state, both (28) and (29) have the same solution $\xi(k) = \xi_1^n(k) = \xi_2^n(k)$. In this case, (26)–(29) yield

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1 - \xi(k))^2,$$

where $\xi(k)$, $k \ge 0$, is given as in the statement of the lemma. This concludes the proof of Lemma 1.

B. Proof of Theorem 1

Let $D^N(t)$, $t \ge 1$, denote the number of dropped packets in the time slot t:

$$D^N(t) = (Q_*^N(t-1) + H^N(t) - C^N - B^N)^+,$$

where $Q_*^N(t)$ denotes the queue occupancy at the end of the time slot t. Since all packets are assumed to have the same priority, the drop probability of a packet at the time slot t is given by

$$p_D^N(t) = D^N(t)/H^N(t). \quad (30)$$

Without loss of generality, consider the arrival process of the source 1. Let $\{\tau_s, 1 \le s \le M^N\}$ be the sequence of arrival times of data packets in a block of the source 1. Then, the probability that k data packets are dropped in a block is given by

$$\mathbb{P}[D_d^N = k] = \sum_{S \in \mathcal{S}^k} \mathbb{E}\left[\prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S}(1 - p_D^N(\tau_s))\right], \quad (31)$$

where $\mathcal{S}^k$ is the collection of all k-subsets of $\{1, 2, \ldots, M^N\}$.

Now define, for $l \in \mathbb{N}$,

$$\mathcal{S}_l^k = \{S : |i - j| > l,\, \forall i, j \in S \in \mathcal{S}^k\}. \quad (32)$$

The following lemma states that under the critical-load scaling, the intervals between packet drops in a block are asymptotically $\Omega(\log N)$, as $N \to \infty$.

Lemma 2. *Consider the critical-load scaling. Suppose that* $l = \lfloor a \log N \rfloor$ *for fixed* $a > 0$. *Then, as* $N \to \infty$,

$$\sum_{S \in \mathcal{S}^k \setminus \mathcal{S}_l^k} \mathbb{E}\left[\prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S}(1 - p_D^N(\tau_s))\right] \to 0.$$

Proof: See Subsection X-C ■

Lemma 3. *Consider the critical-load scaling. Suppose that* $l = \lfloor a \log N \rfloor$ *for fixed* $a > 0$. *Then, as* $N \to \infty$,

$$\sum_{S \in \mathcal{S}_l^k} \mathbb{E} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \to \frac{(m\hat{p}_D)^k}{k!},$$

where $\hat{p}_D$ *is the limiting scaled drop probability that satisfies* (21). *Moreover, we have*

$$\limsup_{N \to \infty} \sum_{S \in \mathcal{S}_l^k} \left(\mathbb{E} \prod_{s \in S} \left(\frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)}\right)^2\right)^{1/2} < \infty.$$

Proof: See Subsection X-D ■

Lemma 4. *Consider the critical-load scaling. Then*

$$\prod_{s=1}^{M^N} (1 - p_D^N(\tau_s)) \xrightarrow{\mathbb{P}} e^{-m\hat{p}_D},$$

as $N \to \infty$, *where* $\hat{p}_D$ *is the limiting scaled drop probability that satisfies* (21).

Proof: See Subsection X-E ■

Next we present a proof of Theorem 1.

Proof of Theorem 1: First consider D_d^N, which denotes the number of dropped data packets in a block of the source 1. The set $\mathcal{S}^k$ can be partitioned into two disjoint subsets $\mathcal{S}_l^k$ and $\mathcal{S}^k \setminus \mathcal{S}_l^k$. Thus, in view of (31), we have

$$\mathbb{P}[D_d^N = k] = \sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) + \sum_{S \in \mathcal{S}^k \setminus \mathcal{S}_l^k} \mathbb{E}\Pi^N(S). \quad (33)$$

where $\Pi^N(S) = \prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S}(1 - p_D^N(\tau_s))$. By letting

$$\Sigma^N = \sum_{S \in \mathcal{S}_l^k} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)},$$

$$\Gamma^N = \prod_{s=1}^{M^N} (1 - p_D^N(\tau_s)) - e^{-m\hat{p}_D},$$

and by using the triangular inequality, it is straightforward to show that

$$\left| \sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) - \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D} \right| \leq \left| \mathbb{E}[\Sigma^N \Gamma^N] \right| + e^{-m\hat{p}_D} \left| \mathbb{E}\Sigma^N - \frac{(m\hat{p}_D)^k}{k!} \right|. \quad (34)$$

For some $\epsilon > 0$, define an event $\mathcal{G}_\epsilon^N = \{|\Gamma^N| < \epsilon\}$. Then, we have

$$\begin{aligned} \left| \mathbb{E}[\Sigma^N \Gamma^N] \right| &\leq \mathbb{E}[|\Sigma^N \Gamma^N| \cdot 1_{\mathcal{G}_\epsilon^N}] + \mathbb{E}[|\Sigma^N \Gamma^N| \cdot 1_{\bar{\mathcal{G}}_\epsilon^N}] \\ &\leq \epsilon \mathbb{E}\Sigma^N + \mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}]; \end{aligned} \quad (35)$$

the first inequality is due to the Jensen's inequality and the second inequality follows from $|\Gamma^N| \leq 1$. Furthermore, the Cauchy-Schwarz inequality renders

$$\mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}] \leq \sum_{S \in \mathcal{S}_l^k} \left(\mathbb{E}\left[\prod_{s \in S} \left(\frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \right)^2 \right] \mathbb{P}[\bar{\mathcal{G}}_\epsilon^N] \right)^{1/2}.$$

Let $l = \lfloor a \log N \rfloor$ for fixed $a > 0$. Then, the second statement of Lemma 3, Lemma 4 and the preceding inequality imply $\mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}] \to 0$, as $N \to \infty$. Since $\lim_{N\to\infty} \mathbb{E}\Sigma^N < \infty$ (see the first statement of Lemma 3) and (35) holds for any $\epsilon > 0$, it follows that $|\mathbb{E}[\Sigma^N \Gamma^N]| \to 0$, as $N \to \infty$; combining this limit, the first statement of Lemma 3 and (34) yields

$$\sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) \to \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D},$$

as $N \to \infty$. Due to Lemma 2 and the preceding limit, the first statement of the theorem follows from (33).

Second consider D_c^N, i.e., the number of dropped packets among additional α coded packets in a block of the source 1 (recall that without loss of generality, we consider the arrival process of the source 1). Note that additional α coded packets are transmitted in the same time slot as the last data packet of the block. Moreover, all packets have the same priority in the system. Thus, it follows from the union bound:

$$\mathbb{P}[D_c^N > 0] \leq \alpha \mathbb{E} p_D^N(\tau_{M^N}), \quad (36)$$

where τ_{M^N} is the arrival time of the last data packet in the block. The drop probability $p_D^N(t)$ is bounded by

$$p_D^N(t) \leq \frac{(H^N(t) - C^N)^+}{H^N(t)} \leq \frac{(H^N(t) - C^N)^+}{C^N}; \quad (37)$$

this together with (36) leads to

$$\mathbb{P}[D_c^N > 0] \leq \alpha (\sqrt{N}/C^N) \mathbb{E}(\check{H}^N(\tau_{M^N}))^+, \quad (38)$$

where $\check{H}^N(t) = (H^N(t) - C^N)/\sqrt{N}$. Moreover, the relation $x^+ < e^x$ for $x \in \mathbb{R}$ implies

$$\mathbb{E}(\check{H}^N(\tau_{M^N}))^+ \leq \mathbb{E}e^{\check{H}^N(\tau_{M^N})}. \quad (39)$$

Since additional α packets are transmitted in the same time slot as the last data packet of the block, we have $H^N(\tau_{M^N}) = (1 + \alpha) + \sum_{i=2}^N H_{(i)}^N(\tau_{M^N})$. Assuming that processes are in their stationary regimes except the one corresponding to the source 1, the random variables $H_{(i)}^N(\tau_{M^N})$, $i = 2, 3, \ldots, N$, are i.i.d.. Thus, it follows that

$$\begin{aligned} \mathbb{E}e^{\check{H}^N(\tau_{M^N})} &= \mathbb{E}\prod_{i=1}^N e^{\check{H}_{(i)}^N(\tau_{M^N})} \\ &= e^{((1+\alpha) - C^N/N)/\sqrt{N}} \left(\mathbb{E}e^{\check{H}_{(2)}^N(\tau_{M^N})} \right)^{N-1}, \end{aligned}$$

where $\check{H}_{(i)}^N(t) = (H_{(i)}^N(t) - C^N/N)/\sqrt{N}$. From (19), it can be shown that

$$\mathbb{E}e^{\check{H}_{(2)}^N(t)} = 1 + \frac{\alpha/m - \beta}{N} + \frac{\lambda(1-\lambda)}{2N} + o\left(\frac{1}{N}\right),$$

as $N \to \infty$, and this further results in

$$\lim_{N\to\infty} \mathbb{E}e^{\check{H}^N(\tau_{M^N})} < \infty. \quad (40)$$

Finally, putting together (38)–(40) renders the second statement of the theorem. ■

C. Proof of Lemma 2

First we present a preliminary technical lemma.

Lemma 5. *If $l = o(M^N)$, as $M^N \to \infty$, then*

$$|\mathcal{S}_l^k| = ((M^N)^k/k!)(1 - o(1)),$$

and

$$|\mathcal{S}^k| - |\mathcal{S}_l^k| = O(l \cdot (M^N)^{k-1}),$$

as $M^N \to \infty$, for fixed $k \in \mathbb{N}$.

Proof: Observe that $|\mathcal{S}^k|$ is bounded above by

$$|\mathcal{S}^k| = \binom{M^N}{k} \leq \frac{(M^N)^k}{k!},$$

and $|\mathcal{S}_l^k|$ is bounded below by

$$|\mathcal{S}_l^k| \geq \frac{1}{k!} \prod_{i=0}^{k-1} (M^N - i(2l+1)) \geq \frac{(M^N - k(2l+1))^k}{k!}.$$

Under the assumption of the lemma, these two inequalities imply the first statement of the lemma. Furthermore, combining these two inequalities results in

$$|\mathcal{S}^k| - |\mathcal{S}_l^k| \leq \frac{(M^N)^k - (M^N - k(2l+1))^k}{k!},$$

and, then, the second statement of the lemma also follows due to the assumption of the lemma. ■

Now we provide a proof of Lemma 2.

Proof of Lemma 2: The lemma holds for $k = 0$ trivially; hence, we consider $k \geq 1$. From (37), it follows that

$$\begin{aligned} \prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S} (1 - p_D^N(\tau_s)) &\leq \prod_{s \in S} p_D^N(\tau_s) \\ &\leq (\sqrt{N}/C^N)^k \prod_{s \in S} (\check{H}^N(\tau_s))^+. \end{aligned} \quad (41)$$

Moreover, due to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}\prod_{s \in S} (\check{H}^N(\tau_s))^+ &\leq \prod_{s \in S} \left(\mathbb{E}((\check{H}^N(\tau_s))^+)^k \right)^{1/k} \\ &\leq \prod_{s \in S} \left(\mathbb{E}e^{k\check{H}^N(\tau_s)} \right)^{1/k}; \end{aligned} \quad (42)$$

the second inequality follows from the relation $x^+ < e^x$ for $x \in \mathbb{R}$. Recall that τ_s, $1 \le s \le M^N$, is the arrival time of the sth data packet in a block of the source 1. Thus, we have $H^N(\tau_s) = 1 + \alpha \cdot 1_{\{s=M^N\}} + H^N_{(-1)}(\tau_s)$, where $H^N_{(-1)}(t) = \sum_{i=2}^N H^N_{(i)}(t)$. Given that all processes are in their stationary regimes except the one corresponding to the source 1, the random variables $H^N_{(-1)}(\tau_s)$, $s = 1, 2, \ldots, M^N$, are equal in distribution, and, analogously to (40), it can be shown that

$$\lim_{N\to\infty} \mathbb{E} e^{k\check{H}^N(\tau_s)} < \infty, \tag{43}$$

for all τ_s, $1 \le s \le M^N$. Under the assumptions of the lemma, Lemma 5 implies $|\mathcal{S}^k| - |\mathcal{S}_l^k| \le c\lfloor a \log N\rfloor (M^N)^{k-1}$ for some finite constant c. Combining this and (41)–(43) leads to the statement of Lemma 2. ∎

D. Proof of Lemma 3

Consider $H^N_{(-1)}(t) - A^N_{(-1)}(t)$, i.e., the number of additional coded packets generated by the encoders of $N-1$ sources (except the source 1) in the time slot t, where $H^N_{(-1)}(t)$ and $A^N_{(-1)}(t)$ are respectively given by

$$H^N_{(-1)}(t) = \sum_{i=2}^N H^N_{(i)}(t), \quad A^N_{(-1)}(t) = \sum_{i=2}^N A^N_{(i)}(t).$$

The following lemma indicates that the number of such coded packets per time slot can be approximated by $(\alpha/m)\sqrt{N}$ during an entire block.

Lemma 6. *Consider the critical-load scaling. Then*

$$\sup_{\tau_1 \le t \le \tau_{M^N}} \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m} \right| \xrightarrow{\mathbb{P}} 0,$$

as $N \to \infty$.

Proof: For some $\epsilon > 0$, define an event $\mathcal{E}^N_\epsilon$ as

$$\mathcal{E}^N_\epsilon = \left\{ \sup_{\tau_1 \le t \le \tau_{M^N}} \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right| \ge \epsilon \right\},$$

where $m^N = \sqrt{N} M^N / (\lambda(N-1))$. Since $m^N \to m$, as $N \to \infty$, it is sufficient to show that for any $\epsilon > 0$,

$$\mathbb{P}[\mathcal{E}^N_\epsilon] \to 0, \tag{44}$$

as $N \to \infty$. The event $\mathcal{E}^N_\epsilon$ satisfies $\mathcal{E}^N_\epsilon \subseteq \bigcup_{t=\tau_1}^{\tau_{M^N}} \mathcal{E}^N_\epsilon(t)$, where

$$\mathcal{E}^N_\epsilon(t) = \left\{ \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right| \ge \epsilon \right\}.$$

Thus, the union bound renders

$$\mathbb{P}[\mathcal{E}^N_\epsilon] \le \mathbb{E} \sum_{t=\tau_1}^{\tau_{M^N}} \mathbb{P}[\mathcal{E}^N_\epsilon(t)]. \tag{45}$$

Furthermore, the Markov's inequality results in

$$\begin{aligned} \mathbb{P}[\mathcal{E}^N_\epsilon(t)] &= \mathbb{P}\left[\left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right|^4 \ge \epsilon^4 \right] \\ &\le (\epsilon\sqrt{N})^{-4} \mathbb{E}\left(\sum_{i=2}^N U^N_{(i)}(t) \right)^4, \end{aligned} \tag{46}$$

where $U^N_{(i)}(t) = H^N_{(i)}(t) - A^N_{(i)}(t) - \alpha\lambda/M^N$. Note that the random variables $U^N_{(i)}(t)$, $i = 2, 3, \ldots, N$, are independent since packets are generated and encoded by individual sources and their encoders independently. Furthermore, provided that the system is in stationarity, the random variables $U^N_{(i)}(t)$, $i = 2, 3, \ldots, N$, are i.i.d. with zero mean and

$$\mathbb{P}[U^N_{(i)}(t) = u] = \begin{cases} 1 - \lambda/M^N, & u = -\alpha\lambda/M^N, \\ \lambda/M^N, & u = \alpha - \alpha\lambda/M^N, \end{cases} \tag{47}$$

for all $t \in [\tau_1, \tau_{M^N}]$; this stems from the fact that all processes except the one corresponding to the source 1 are in steady-state at the time slot τ_1. Therefore, it follows that

$$\mathbb{E}\left(\sum_{i=2}^N U^N_{(i)}(t) \right)^4 \le N \mathbb{E}(U^N_{(2)}(t))^4 + 3N^2 (\mathbb{E}(U^N_{(2)}(t))^2)^2, \tag{48}$$

for all $t \in [\tau_1, \tau_{M^N}]$. From (47), one can obtain

$$\begin{aligned} \mathbb{E}(U^N_{(2)}(t))^k = (1 - \lambda/M^N)(-\alpha\lambda/M^N)^k \\ + (\lambda/M^N)(\alpha - \alpha\lambda/M^N)^k, \end{aligned}$$

for $k \ge 2$, and this together with (46) and (48) leads to

$$\mathbb{P}[\mathcal{E}^N_\epsilon(t)] = O(1/N), \tag{49}$$

as $N \to \infty$, for all $t \in [\tau_1, \tau_{M^N}]$. Observe that $\tau_{M^N} - \tau_1 = \sum_{i=2}^{M^N} (\tau_i - \tau_{i-1})$, where $(\tau_i - \tau_{i-1})$, $i = 2, 3, \ldots, M^N$, are i.i.d. geometric random variables with mean $1/\lambda$ (since the sources are Bernoulli). Hence, combining (45) and (49) yields (44), and this concludes the proof of the lemma. ∎

Next we introduce an additional technical lemma. For some $\epsilon > 0$, consider two systems that have the same link capacity C^N and buffer size B^N; however, assume that input processes are respectively given by $A^N_{-\epsilon}(t) = A^N_{(-1)}(t) + (\alpha/m - \epsilon)\sqrt{N}$ and $A^N_{+\epsilon}(t) = A^N_{(-1)}(t) + (1+\alpha) + (\alpha/m + \epsilon)\sqrt{N}$, instead of $H^N(t)$. Formally, the queue occupancies of these systems $Q^N_{\pm\epsilon}(t)$, $t \ge 1$, satisfy the following recursion:

$$Q^N_{\pm\epsilon}(t) = (Q^N_{\pm\epsilon}(t-1) + A^N_{\pm\epsilon}(t) - C^N)^+ \wedge B^N, \tag{50}$$

and the numbers of dropped packets $D^N_{\pm\epsilon}(t)$, $t \ge 1$, are respectively given by

$$D^N_{\pm\epsilon}(t) = (A^N_{\pm\epsilon}(t) + Q^N_{\pm\epsilon}(t-1) - C^N - B^N)^+. \tag{51}$$

Observe that the processes $\{Q^N_{\pm\epsilon}(t), t \ge 0\}$ are Markov chains since $\{A^N_{\pm\epsilon}(t), t \ge 1\}$ are i.i.d. processes.

The following lemma provides an upper and lower bound on the number of dropped packets.

Lemma 7. *Consider the critical-load scaling. For any $\epsilon > 0$, if $Q^N_{-\epsilon}(\tau_1 - 1) = Q^N_{+\epsilon}(\tau_1 - 1) = Q^N_*(\tau_1 - 1)$, then, as $N \to \infty$,*

$$\mathbb{P}[\mathcal{D}^N_\epsilon] \to 1,$$

where $\mathcal{D}^N_\epsilon = \{D^N_{-\epsilon}(t) \le D^N(t) \le D^N_{+\epsilon}(t), \forall t \in [\tau_1, \tau_{M^N}]\}$.

Proof: For some $\epsilon > 0$, define an event $\mathcal{A}^N_\epsilon$ as

$$\mathcal{A}^N_\epsilon = \left\{ \sup_{\tau_1 \le t \le \tau_{M^N}} \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m} \right| < \epsilon \right\}, \tag{52}$$

Given the event $\mathcal{A}_\epsilon^N$, the following bound holds:

$$A_{-\epsilon}^N(t) \le H^N(t) \le A_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]; \tag{53}$$

this further implies (due to the monotonicity in (50))

$$Q_{-\epsilon}^N(t) \le Q_*^N(t) \le Q_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]. \tag{54}$$

From (51), (53) and (54), on the event $\mathcal{A}_\epsilon^N$, we also have

$$D_{-\epsilon}^N(t) \le D^N(t) \le D_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]. \tag{55}$$

Then, the statement of the lemma follows from Lemma 6. ■

Next we present a proof of Lemma 3

Proof of Lemma 3: First consider the first statement of the lemma. Note that the statement holds for $k = 0$ trivially; hence, we consider $k \ge 1$. The proof consists of three parts.

Part I. Observe that $H^N(t) \wedge C^N \le H^N(t) - D^N(t) \le C^N + B^N$; this implies (see (30))

$$\frac{D^N(t)}{C^N + B^N} \le \frac{p_D^N(t)}{1 - p_D^N(t)} \le \frac{D^N(t)}{H^N(t) \wedge C^N} = \frac{D^N(t)}{C^N}; \tag{56}$$

the equality is due to the fact that the event $\{H^N(t) < C^N\}$ implies $\{D^N(t) = 0\}$. Then, it is sufficient to show that for all $S \in \mathcal{S}_l^k$,

$$\mathbb{E} \prod_{s \in S} \hat{D}^N(\tau_s) \to (\hat{p}_D)^k, \tag{57}$$

as $N \to \infty$, where $\hat{D}^N(t) = D^N(t)/\sqrt{N}$. In particular, note that (56) and the preceding limit imply

$$(C^N/\sqrt{N})^k \mathbb{E} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \to (\hat{p}_D)^k,$$

as $N \to \infty$, for all $S \in \mathcal{S}_l^k$; then, the statement of the lemma follows from the first statement of Lemma 5. For $k = 1$, the limit (57) is straightforward (see (21)); thus, we consider $k \ge 2$ from now on.

Part II. The proof is based on a coupling argument (e.g., see [10, Sec. 4.1.2]). Given $\epsilon > 0$, define two events $\mathcal{C}_{-\epsilon,0}^N(t_1, t_2)$ and $\mathcal{C}_{-\epsilon,b}^N(t_1, t_2)$ for time slots t_1 and t_2 ($t_1 < t_2$):

$$\mathcal{C}_{-\epsilon,0}^N(t_1, t_2) = \{\exists t \in [t_1, t_2) : A_{-\epsilon}^N(t) < C^N - B^N\},$$
$$\mathcal{C}_{-\epsilon,b}^N(t_1, t_2) = \{\exists t \in [t_1, t_2) : A_{-\epsilon}^N(t) > C^N + B^N\},$$

and consider the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$ given by

$$\mathcal{C}_{-\epsilon}^N(t_1, t_2) = \mathcal{C}_{-\epsilon,0}^N(t_1, t_2) \cap \mathcal{C}_{-\epsilon,b}^N(t_1, t_2). \tag{58}$$

The events $\{A_{-\epsilon}^N(t) < C^N - B^N\}$ and $\{A_{-\epsilon}^N(t) > C^N + B^N\}$ imply $\{Q_{-\epsilon}^N(t) = 0\}$ and $\{Q_{-\epsilon}^N(t) = B^N\}$, respectively, regardless of the queue occupancy of the previous time slot. On the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$, thus, the buffer becomes empty and full at least once during the time interval $[t_1, t_2)$. Now consider a queue occupancy process $\{\dot{Q}_{-\epsilon}^N(t), t \ge t_1 - 1\}$ with the same arrival process $\{A_{-\epsilon}^N(t), t \ge t_1\}$ but possibly different initial distribution at $t = t_1 - 1$. If there exists $t^0 \ge t_1$ such that $Q_{-\epsilon}^N(t^0) \le \dot{Q}_{-\epsilon}^N(t^0)$, then $Q_{-\epsilon}^N(t) \le \dot{Q}_{-\epsilon}^N(t)$ for all $t \ge t^0$. On the other hand, if there exists $t^b \ge t_1$ such that $Q_{-\epsilon}^N(t^b) \ge \dot{Q}_{-\epsilon}^N(t^b)$, then $Q_{-\epsilon}^N(t) \ge \dot{Q}_{-\epsilon}^N(t)$ for all $t \ge t^b$. Hence, the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$ implies that these two queue occupancy processes couple before the time slot t_2, i.e., $Q_{-\epsilon}^N(t_2) = \dot{Q}_{-\epsilon}^N(t_2)$, regardless of their initial distributions at $t = t_1 - 1$.

Let $S = \{s_1, s_2, \ldots, s_k\} \in \mathcal{S}_l^k$, where $s_1 < s_2 < \cdots < s_k$. The assumption of the lemma implies $|s_i - s_j| > \lfloor a \log N \rfloor$ for all $s_i, s_j \in S$ (see (32)); this further results in $(\tau_{s_i} - \tau_{s_{i-1}}) > \lfloor a \log N \rfloor$ for all i, $2 \le i \le k$. Therefore, we have

$$\mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \le (q_0^N)^{\lfloor a \log N \rfloor} + (q_b^N)^{\lfloor a \log N \rfloor},$$

for all i, $2 \le i \le k$, where $q_0^N = \mathbb{P}[A_{-\epsilon}^N(t) \ge C^N - B^N]$ and $q_b^N = \mathbb{P}[A_{-\epsilon}^N(t) \le C^N + B^N]$. Since $q_0^N \to q_0 \in (0,1)$ and $q_b^N \to q_b \in (0,1)$, as $N \to \infty$ (due to the CLT), it follows that

$$\mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{59}$$

as $N \to \infty$, for all i, $2 \le i \le k$. The union bound and the preceding limit yield

$$\mathbb{P}\left[\bigcup_{i=2}^k \bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})\right] \le \sum_{i=2}^k \mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{60}$$

as $N \to \infty$. One can define a corresponding event $\mathcal{C}_{+\epsilon}^N(t_1, t_2)$ (as in (58)) for the case "$+\epsilon$", and it can be shown that

$$\mathbb{P}[\bar{\mathcal{C}}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{61}$$

for all i, $2 \le i \le k$, and

$$\mathbb{P}\left[\bigcup_{i=2}^k \bar{\mathcal{C}}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})\right] \to 0, \tag{62}$$

as $N \to \infty$.

Part III. For some $\epsilon > 0$ and $S = \{s_1, s_2, \ldots, s_k\} \in \mathcal{S}_l^k$, $s_1 < s_2 < \cdots < s_k$, let

$$\mathcal{C}_{\epsilon,i}^N(S) = \mathcal{C}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i}) \cap \mathcal{C}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i}),$$

for $2 \le i \le k$, and consider the event $\mathcal{C}_\epsilon^N(S) = \bigcap_{i=2}^k \mathcal{C}_{\epsilon,i}^N(S)$. It is straightforward to show that

$$\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)}] = \mathbb{E}\left[\hat{D}_{-\epsilon}^N(\tau_{s_1}) \prod_{i=2}^k \left(\hat{D}_{-\epsilon}^N(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}\right)\right].$$

where $\Psi_{-\epsilon}^N(S) = \prod_{s \in S} \hat{D}_{-\epsilon}^N(\tau_s)$ and $\hat{D}_{-\epsilon}^N(t) = D_{-\epsilon}^N(t)/\sqrt{N}$. For each i, $2 \le i \le k$, consider a queue occupancy process $\{Q_{-\epsilon,i}^{*N}(t), \tau_{s_{i-1}} \le t \le \tau_{s_i} - 1\}$ that is in stationarity at the time slot $t = \tau_{s_{i-1}}$ and follows the same recursion (50) for $t \in [\tau_{s_{i-1}} + 1, \tau_{s_i} - 1]$. In particular, we assume that the initial queue occupancies $Q_{-\epsilon,i}^{*N}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are i.i.d. and that they do not depend on the arrival process $\{A_{-\epsilon}^N(t), t \ge 1\}$. In addition, let $D_{-\epsilon,i}^{*N}(t)$, $\tau_{s_{i-1}} + 1 \le t \le \tau_{s_i}$, $2 \le i \le k$, denote the number of dropped packets that corresponds to the queue occupancy $Q_{-\epsilon,i}^{*N}(\cdot)$:

$$D_{-\epsilon,i}^{*N}(t) = (A_{-\epsilon}^N(t) + Q_{-\epsilon,i}^{*N}(t-1) - C^N - B^N)^+. \tag{63}$$

Due to the coupling argument given in the previous part, we have, for all i, $2 \le i \le k$,

$$D_{-\epsilon}^N(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)} = D_{-\epsilon,i}^{*N}(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}. \tag{64}$$

Note that each random variable $D_{-\epsilon,i}^{*N}(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}$, $2 \le i \le k$, is determined by the initial queue occupancy $Q_{-\epsilon,i}^{*N}(\tau_{s_{i-1}})$ and the random variables $A_{-\epsilon}^N(t)$, $t \in [\tau_{s_{i-1}}+1, \tau_{s_i}]$. Since the random variables $A_{-\epsilon}^N(t)$, $t = 1, 2, \ldots$, are i.i.d. and the queue occupancies $Q_{-\epsilon,i}^{*N}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are assumed to be i.i.d. with stationary distribution (and they do not depend on the arrival process), it follows that

$$\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)}] = \mathbb{E}[\hat{D}_{-\epsilon}^N(\tau_{s_1})] \prod_{i=2}^{k} \mathbb{E}[\hat{D}_{-\epsilon,i}^{*N}(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}], \tag{65}$$

where $\hat{D}_{-\epsilon,i}^{*N}(t) = D_{-\epsilon,i}^{*N}(t)/\sqrt{N}$. From the bound $D_{-\epsilon,i}^{*N}(t) \leq (A_{-\epsilon}^N(t) - C^N)^+$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbb{E}[\hat{D}_{-\epsilon,i}^{*N}(\tau_{s_i}) 1_{\bar{\mathcal{C}}_{\epsilon,i}^N(S)}] &\leq \mathbb{E}[(\check{A}_{-\epsilon}^N(\tau_{s_i}))^+ 1_{\bar{\mathcal{C}}_{\epsilon,i}^N(S)}] \\ &\leq \left(\mathbb{E}[((\check{A}_{-\epsilon}^N(\tau_{s_i}))^+)^2] \mathbb{P}[\bar{\mathcal{C}}_{\epsilon,i}^N(S)]\right)^{1/2}, \end{aligned}\tag{66}$$

for all i, $2 \leq i \leq k$, where $\check{A}_{-\epsilon}^N(t) = (A_{-\epsilon}^N(t) - C^N)/\sqrt{N}$. By using the similar steps as in (39) and (40), one can obtain

$$\limsup_{N\to\infty} \mathbb{E}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2 < \infty. \tag{67}$$

Furthermore, due to (59) and (61), we have $\mathbb{P}[\mathcal{C}_{\epsilon,i}^N(S)] \to 1$, as $N \to \infty$, for all i, $2 \leq i \leq k$. Then, from (66) and (67), it follows that, as $N \to \infty$,

$$\mathbb{E}[\hat{D}_{-\epsilon,i}^{*N}(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}] - \mathbb{E}\hat{D}_{-\epsilon,i}^{*N}(\tau_{s_i}) \to 0, \tag{68}$$

for all i, $2 \leq i \leq k$. Recall that the initial queue occupancies $Q_{-\epsilon,i}^{*N}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are assumed to have the stationary distribution; this implies $\mathbb{E}\hat{D}_{-\epsilon,i}^{*N}(\tau_{s_i}) = \mathbb{E}\hat{D}_{-\epsilon}^N(\tau_{s_1})$, for all i, $2 \leq i \leq k$. Thus, combining (65) and (68) renders

$$\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)}] - (\mathbb{E}\hat{D}_{-\epsilon}^N(\tau_{s_1}))^k \to 0, \tag{69}$$

as $N \to \infty$.

The bound $D_{-\epsilon}^N(t) \leq (A_{-\epsilon}^N(t) - C^N)^+$ and the Cauchy-Schwarz inequality yield (see Lemma 7)

$$\begin{aligned}\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)} 1_{\bar{\mathcal{D}}_\epsilon^N}] &\leq \mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\bar{\mathcal{D}}_\epsilon^N}] \\ &\leq \left(\mathbb{E}\left[\prod_{s\in S}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2\right] \mathbb{P}[\bar{\mathcal{D}}_\epsilon^N]\right)^{1/2}. \end{aligned}\tag{70}$$

Since random variables $\check{A}_{-\epsilon}^N(\tau_s)$, $s \in S$, are i.i.d., we have

$$\mathbb{E}\prod_{s\in S}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2 = \prod_{s\in S}\mathbb{E}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2.$$

Then, by combining (67), (70) and Lemma 7, it follows that $\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)} 1_{\bar{\mathcal{D}}_\epsilon^N}] \to 0$, as $N \to \infty$; this limit and (69) further result in

$$\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)} 1_{\mathcal{D}_\epsilon^N}] - (\mathbb{E}\hat{D}_{-\epsilon}^N(\tau_{s_1}))^k \to 0, \tag{71}$$

as $N \to \infty$. Likewise, one can derive a similar limit for the case "$+\epsilon$", i.e.,

$$\mathbb{E}[\Psi_{+\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)} 1_{\mathcal{D}_\epsilon^N}] - (\mathbb{E}\hat{D}_{+\epsilon}^N(\tau_{s_1}))^k \to 0, \tag{72}$$

as $N \to \infty$, where $\Psi_{+\epsilon}^N(S) = \prod_{s\in S}(D_{+\epsilon}^N(\tau_s)/\sqrt{N})$. For any $\epsilon > 0$, Lemma 7 implies

$$\Psi_{-\epsilon}^N(S) 1_{\mathcal{Z}_\epsilon^N(S)} \leq \Psi^N(S) 1_{\mathcal{Z}_\epsilon^N(S)} \leq \Psi_{+\epsilon}^N(S) 1_{\mathcal{Z}_\epsilon^N(S)}, \tag{73}$$

where $\mathcal{Z}_\epsilon^N(S) = \mathcal{C}_\epsilon^N(S) \cap \mathcal{D}_\epsilon^N$ and $\Psi^N(S) = \prod_{s\in S}\hat{D}^N(\tau_s)$. In addition, due to continuity, it can be shown that

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\mathbb{E}\hat{D}_{\pm\epsilon}^N(\tau_{s_1}) = \hat{p}_D.$$

Hence, from (71)-(73) and the preceding limit, one can derive

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\mathbb{E}[\Psi^N(S) 1_{\mathcal{Z}_\epsilon^N(S)}] = (\hat{p}_D)^k. \tag{74}$$

On the other hand, the bound $D^N(t) \leq (H^N(t) - C^N)^+$ and the Cauchy-Schwarz inequality lead to

$$\begin{aligned}&\mathbb{E}[\Psi^N(S) 1_{\bar{\mathcal{Z}}_\epsilon^N(S)}] \\ &\leq \left(\mathbb{E}\left[\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2\right] \mathbb{P}[\bar{\mathcal{Z}}_\epsilon^N(S)]\right)^{1/2}. \end{aligned}\tag{75}$$

Similarly to (42) and (43), one can obtain

$$\limsup_{N\to\infty}\mathbb{E}\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2 < \infty. \tag{76}$$

Moreover, (60), (62) and Lemma 7 imply $\mathbb{P}[\mathcal{Z}_\epsilon^N(S)] \to 1$, as $N \to \infty$. Then, from (75) and (76), we have

$$\mathbb{E}[\Psi^N(S) 1_{\bar{\mathcal{Z}}_\epsilon^N(S)}] \to 0,$$

as $N \to \infty$. The preceding limit and (74) imply (57), and the first statement of the lemma follows.

Finally consider the second statement of the lemma. From (56) and the bound $D^N(t) \leq (H^N(t) - C^N)^+$, we have

$$\mathbb{E}\prod_{s\in S}\left(\frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)}\right)^2 \leq (\sqrt{N}/C^N)^{2k}\mathbb{E}\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2.$$

Then, since $|\mathcal{S}_l^k| = O((M^N)^k)$, as $N \to \infty$ (see Lemma 5), the second statement of the lemma follows from (76). ■

E. *Proof of Lemma 4*

The proof consists of two parts.

Part I. For some $\delta > 0$, it can be shown that

$$\begin{aligned}&\mathbb{P}\left[\left|\prod_{s=1}^{M^N}(1 - p_D^N(\tau_s)) - e^{-m\hat{p}_D}\right| \geq \delta\right] \\ &\leq \mathbb{P}\left[\left|-\sum_{s=1}^{M^N}\log(1 - p_D^N(\tau_s)) - m\hat{p}_D\right| \geq \delta'\right], \end{aligned}\tag{77}$$

where $\delta' = \log(1 + \delta/e^{-m\hat{p}_D})$. The relation $x \leq -\log(1-x) \leq x/(1-x)$ for $0 \leq x < 1$ renders

$$p_D^N(t) \leq -\log(1 - p_D^N(t)) \leq p_D^N(t)/(1 - p_D^N(t)). \tag{78}$$

Given $\epsilon > 0$, the event $\mathcal{A}_\epsilon^N$ implies (see (52), (53) and (55))

$$p_D^N(t) \geq \frac{D_{-\epsilon}^N(t)}{A_{+\epsilon}^N(t)}, \quad \frac{p_D^N(t)}{1 - p_D^N(t)} \leq \frac{D^N(t)}{C^N} \leq \frac{D_{+\epsilon}^N(t)}{C^N},$$

for all $t \in [\tau_1, \tau_{M^N}]$; the second inequality follows from (56). Due to (78) and the preceding inequalities, it follows that on the event $\mathcal{A}_\epsilon^N$,

$$\sum_{s=1}^{M^N} P_{-\epsilon}^N(\tau_s) \leq -\sum_{s=1}^{M^N}\log(1 - p_D^N(\tau_s)) \leq \sum_{s=1}^{M^N} P_{+\epsilon}^N(\tau_s), \tag{79}$$

where $P_{-\epsilon}^N(t) = D_{-\epsilon}^N(t)/A_{+\epsilon}^N(t)$ and $P_{+\epsilon}^N(t) = D_{+\epsilon}^N(t)/C^N$. Due to Lemma 6, we have $\mathbb{P}[\mathcal{A}_\epsilon^N] \to 1$, as $N \to \infty$, for any $\epsilon > 0$; hence, from (77) and (79), it is sufficient to show that

$$\mathbb{P}\left[\left|\sum_{s=1}^{M^N} P_{\pm\epsilon}^N(\tau_s) - m\hat{p}_D\right| \geq \delta\right] \to 0, \tag{80}$$

as $N \to \infty$, for any $\delta > 0$ and all sufficiently small $\epsilon > 0$.

Part II. Consider the case "$-\epsilon$". For some $\epsilon > 0$ and $\delta > 0$, define an event $\mathcal{B}_{\epsilon,\delta}^N$ as

$$\mathcal{B}_{\epsilon,\delta}^N = \left\{\left|\sum_{s=1}^{M^N} P_{-\epsilon}^N(t) - M^N \mathbb{E} P_{-\epsilon}^N(t)\right| \geq \delta\right\}.$$

For notational simplicity, let $\bar{P}_{-\epsilon}^N(t) = P_{-\epsilon}^N(t) - \mathbb{E}P_{-\epsilon}^N(t)$. The Chebyshev's inequality yields

$$\mathbb{P}[\mathcal{B}_{\epsilon,\delta}^N] \leq \delta^{-2}\mathbb{E}\left(\sum_{s=1}^{M^N} \bar{P}_{-\epsilon}^N(\tau_s)\right)^2. \tag{81}$$

Recall that the set $\mathcal{S}^2$ is the collection of all 2-subsets of $\{1, 2, \ldots, M^N\}$. Thus, it follows that

$$\mathbb{E}\left(\sum_{s=1}^{M^N} \bar{P}_{-\epsilon}^N(\tau_s)\right)^2 = 2\sum_{S\in\mathcal{S}^2} \mathbb{E}\prod_{s\in S} \bar{P}_{-\epsilon}^N(\tau_s) + M^N\mathbb{E}(\bar{P}_{-\epsilon}^N(\tau_1))^2. \tag{82}$$

From the following bound:

$$P_{-\epsilon}^N(t) \leq \frac{(A_{-\epsilon}^N(t) - C^N)^+}{A_{+\epsilon}^N(t)} \leq \frac{(A_{-\epsilon}^N(t) - C^N)^+}{C^N}, \tag{83}$$

we have

$$\begin{aligned}\mathbb{E}(\bar{P}_{-\epsilon}^N(\tau_1))^2 &\leq \mathbb{E}(P_{-\epsilon}^N(\tau_1))^2 \\ &\leq (\sqrt{N}/C^N)^2\mathbb{E}((\check{A}_{-\epsilon}^N(\tau_1))^+)^2.\end{aligned} \tag{84}$$

Then, due to (67), it can be shown that, as $N \to \infty$,

$$M^N\mathbb{E}(\bar{P}_{-\epsilon}^N(\tau_1))^2 \to 0. \tag{85}$$

For notational simplicity, define $\Lambda_{-\epsilon}^N(S) = \prod_{s\in S} \bar{P}_{-\epsilon}^N(\tau_s)$. The set $\mathcal{S}^2$ can be partitioned into two disjoint subsets $\mathcal{S}_l^2$ and $\mathcal{S}^2 \setminus \mathcal{S}_l^2$ (see (32)), and, therefore, we have

$$\sum_{S\in\mathcal{S}^2} \mathbb{E}\Lambda_{-\epsilon}^N(S) = \sum_{S\in\mathcal{S}_l^2} \mathbb{E}\Lambda_{-\epsilon}^N(S) + \sum_{S\in\mathcal{S}^2\setminus\mathcal{S}_l^2} \mathbb{E}\Lambda_{-\epsilon}^N(S). \tag{86}$$

The Cauchy-Schwarz inequality and the bound (84) render

$$\left|\mathbb{E}\Lambda_{-\epsilon}^N(S)\right| \leq (\sqrt{N}/C^N)^2\prod_{s\in S}\left(\mathbb{E}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2\right)^{1/2},$$

and this together with (67) leads to $|\mathbb{E}[\Lambda_{-\epsilon}^N(S)]| = O(1/N)$, as $N \to \infty$. Let $l = \lfloor a\log N\rfloor$ for fixed $a > 0$. In this case, Lemma 5 implies $|\mathcal{S}^2| - |\mathcal{S}_l^2| \leq c\lfloor a\log N\rfloor M^N$ for some finite constant $c > 0$; hence, we have, as $N \to \infty$,

$$\sum_{S\in\mathcal{S}^2\setminus\mathcal{S}_l^2} \mathbb{E}\Lambda_{-\epsilon}^N(S) \to 0. \tag{87}$$

Next consider the event $\mathcal{C}_{-\epsilon}^N(S) = \mathcal{C}_{-\epsilon}^N(\tau_{s_1}+1, \tau_{s_2})$ for some $S = \{\tau_{s_1}, \tau_{s_2}\} \in \mathcal{S}_l^2$ (see (58)). Recall that this event implies $D_{-\epsilon}^N(\tau_{s_2}) = D_{-\epsilon,2}^{*N}(\tau_{s_2})$, where $D_{-\epsilon,2}^{*N}(t)$, $\tau_{s_1} + 1 \leq t \leq \tau_{s_2}$, denotes the number of dropped packets that corresponds to the queue occupancy $Q_{-\epsilon,2}^{*N}(\cdot)$ (see (63) and (64)). In a similar manner as in (65), it can be shown that

$$\mathbb{E}[\Lambda_{-\epsilon}^N(S)1_{\mathcal{C}_{-\epsilon}^N(S)}] = \mathbb{E}[\bar{P}_{-\epsilon}^N(\tau_{s_1})]\mathbb{E}[\bar{P}_{-\epsilon}^{*N}(\tau_{s_2})1_{\mathcal{C}_{-\epsilon}^N(S)}] = 0, \tag{88}$$

where $\bar{P}_{-\epsilon}^{*N}(t) = D_{-\epsilon,2}^{*N}(t)/A_{+\epsilon}^N(t) - \mathbb{E}[D_{-\epsilon,2}^{*N}(t)/A_{+\epsilon}^N(t)]$; note that we used the fact that $\mathbb{E}\bar{P}_{-\epsilon}^N(\tau_{s_1}) = 0$. On the other hand, the Cauchy-Schwarz inequality renders

$$\left|\mathbb{E}[\Lambda_{-\epsilon}^N(S)1_{\bar{\mathcal{C}}_{-\epsilon}^N(S)}]\right| \leq \left(\mathbb{E}\left[\prod_{s\in S}(\bar{P}_{-\epsilon}^N(\tau_s))^2\right]\mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(S)]\right)^{1/2}. \tag{89}$$

From the bound (83), we have

$$\begin{aligned}&\mathbb{E}\prod_{s\in S}(\bar{P}_{-\epsilon}^N(\tau_s))^2 \\ &\leq (\sqrt{N}/C^N)^4\mathbb{E}\prod_{s\in S}\left((\check{A}_{-\epsilon}^N(\tau_s))^+ + \mathbb{E}(\check{A}_{-\epsilon}^N(\tau_s))^+\right)^2 \\ &\leq (\sqrt{N}/C^N)^4\prod_{s\in S} 4\mathbb{E}((\check{A}_{-\epsilon}^N(\tau_s))^+)^2;\end{aligned} \tag{90}$$

the second inequality follows from the fact that the random variables $\check{A}_{-\epsilon}^N(\tau_s)$, $s \in S$, are i.i.d. and also from the Jensen's inequality. Note that Lemma 5 implies $|\mathcal{S}_l^2| = O((M^N)^2)$, as $N \to \infty$. Thus, from (59), (67), (89) and (90), it follows that

$$\sum_{S\in\mathcal{S}_l^2} \mathbb{E}[\Lambda_{-\epsilon}^N(S)1_{\bar{\mathcal{C}}_{-\epsilon}^N(S)}] \to 0,$$

as $N \to \infty$. Putting together (81), (82), (85)–(88) and the preceding limit yields

$$\mathbb{P}[\mathcal{B}_{\epsilon,\delta}^N] \to 0, \tag{91}$$

as $N \to \infty$. By using the fact that $A_{+\epsilon}^N(t)/N \to \lambda$, almost surely, as $N \to \infty$ (due to the SLLN), it can be shown that $\lim_{\epsilon\downarrow 0}\lim_{N\to\infty} M^N\mathbb{E}P_{-\epsilon}^N(t) = m\hat{p}_D$ (due to continuity), and combining this with (91) renders (80) for the case "$-\epsilon$". In a similar manner, one can derive (80) for the case "$+\epsilon$" as well. This concludes the proof of Lemma 4.

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