# Diversity Gain Region for MIMO Fading Multiple Access Channels

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**Abstract**

In this work, we introduce the notion of the diversity gain region for a multi-user channel. This region specifies the set of diversity-gain vectors that are simultaneously achievable by all users in a multi-user channel. This is done by associating different probabilities of error for different users, contrary to the traditional approach where a single *probability of system error* is considered. We derive an inner bound (achievable region) and an outer bound for the diversity gain region of a MIMO fading multiple access channel.

## 1 Introduction

It is well-known that the error exponent for a single-user channel provides the rate of exponential decay of the average probability of error as a function of the block length of the codebooks [1, 2]. The concept of the error exponent was extended to a Gaussian multiple access channel (MAC) in [3, 4], where an upper bound on the *probability of system error* (i.e., the probability that any user is in error) was derived for random codes. Recently, Zheng *et al.* considered error exponents in high signal-to-noise ratio (SNR) approximation, called diversity gains, for multi-input-multi-output (MIMO) fading single-user channels [5], and considered equal diversity gains for MIMO fading multiple access channels [6].

In many applications of multi-user networks, different users might have different reliability requirements. For instance, in an uplink (or downlink) of a cellular system, a user running an FTP application might have more stringent reliability requirements than a user running a multimedia application which is designed for graceful degradation. Based on the traditional approaches [3, 4, 6] which consider a single probability of system error, a network can only be designed to satisfy the most stringent reliability requirement. This might result in a mismatch of resources allocation, and thus, it is inherently suboptimal.

Motivated by the above observation, we consider a new approach in analyzing the users' performance in a multi-user scenario. In addition to the rate vs. performance tradeoffs that exist in traditional approaches, our approach realizes new degrees of freedom that enable a richer tradeoff among users' performance. Our approach hinges on the following two observations.

First, one can define a probability of error for each user, which, in general, may be different for different users. Therefore, there are multiple error exponents, one for each

user, for a given multi-user channel. Second, in contrast to a single-user channel where the error exponent is fixed for a given rate, in a multi-user channel one can tradeoff the error exponents among different users even for fixed rates. In [7, 8], we formalized these ideas by introducing the notion of error exponent region (EEE), and derived EER inner and outer bounds for Gaussian broadcast channels and Gaussian multiple access channels. It is our intention in this paper to extend these results to MIMO fading multiple access channels.

The rest of the paper is structured as follows. In Section 2, we introduce the notion of multiplexing gain region (MGR) and diversity gain region (DGR). In Section 3, we derive the DGR inner bound using two strategies - channel-splitting and superposition. In Section 4, we derive the DGR outer bound. We conclude our work in Section 5.

# 2 Multiplexing Gain Region and Diversity Gain Region

Consider a MIMO fading multiple access channel with $m_1$, $m_2$ transmit antennas for user 1 and user 2 and $n$ receive antennas. The channel model is

$$\mathbf{Y} = \sqrt{\frac{SNR}{m_1}}\mathbf{H}_1\mathbf{X}_1 + \sqrt{\frac{SNR}{m_2}}\mathbf{H}_2\mathbf{X}_2 + \mathbf{Z}. \tag{1}$$

The channel fading matrices between the transmitter 1 and the transmitter 2 and the receiver are represented by an $n \times m_1$ matrix $\mathbf{H}_1$ and an $n \times m_2$ matrix $\mathbf{H}_2$. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block length $l$, and change to a new independent realization in the next block length $l$. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry has a complex Gaussian distribution $\mathcal{CN}(0,1)$. We assume that fading matrices are known by the receiver but unknown to the transmitters. The channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ are $m_1 \times l$ and $m_2 \times l$ matrices and are normalized such that the average transmit power at each antenna is one. The noise $\mathbf{Z}$ is an $n \times l$ matrix with i.i.d. entries $\mathcal{CN}(0,1)$. The channel output $\mathbf{Y}$ is an $n \times l$ matrix.

Before introducing the notion of diversity gain region, let's review the notion of error exponent region (EER) introduced in [7, 8]. For a given operating (rate) point, an error exponent region consists of all achievable error exponents when the channel is operated at that point. For example, the error exponent region for a single-user channel operated at rate $R$ is a line segment from the origin to the error exponent $E(R)$ (see Fig. 1(a)). For a two-user channel, the error exponent region is a two-dimensional region which depends on the operating (rate) point $(R_1, R_2)$ (see Fig. 1(b)). The reader should not be confused by the concept of error exponent region with the concept of channel capacity region (CCR). In EER, it is possible to increase user 1's error exponent by decreasing user 2's error exponent. This is similar to the idea of increasing the data rate of user 1 by reducing the data rate of user 2 in CCR. However, there is a fundamental difference between CCR and EER. For a given channel, there is only one CCR. One the other hand, an EER depends on the channel operating point, and for a given channel, there are numerous EERs depending on which operating point we consider. Therefore, when we refer to an EER, we need to specify the channel operating point.

When we work on a MIMO fading channel in this paper, we consider this channel only in high SNR scenario. In [5, 6], an encoding scheme $\mathcal{C}(SNR)$ (a family of codes) in a MIMO fading single-user channel is said to achieve multiplexing gain $r$ and diversity

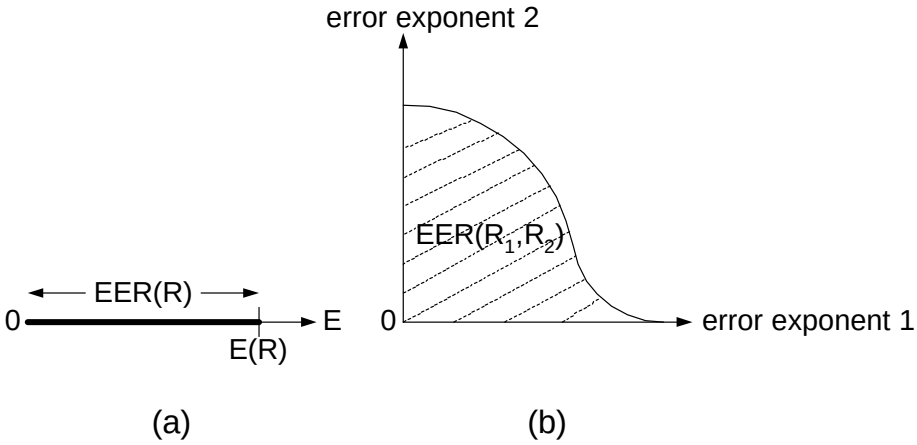



Figure 1: Error exponent region (a) single-user channel, (b) multi-user channel.

gain $d$ if

$$\lim_{SNR\to\infty} \frac{R(SNR)}{\log SNR} = r,$$
$$\lim_{SNR\to\infty} -\frac{\log P_e(SNR)}{\log SNR} = d. \tag{2}$$

Define $R(SNR) \cong r \ln SNR$ and $P_e(SNR) \doteq SNR^{-d}$ if equalities hold in the limit, and $\widetilde{\geq}$, $\widetilde{\leq}$, $\dot{\geq}$, $\dot{\leq}$ are defined similarly. Following the same notations in [5, 6], we define an encoding scheme $\mathcal{C}(SNR)$ to achieve multiplexing gain pair $(r_1, r_2)$ and diversity gain pair $(d_1, d_2)$ in a MIMO fading multiple access channel if

$$\lim_{SNR\to\infty} \frac{R_1(SNR)}{\log SNR} = r_1, \lim_{SNR\to\infty} \frac{R_2(SNR)}{\log SNR} = r_2,$$
$$\lim_{SNR\to\infty} -\frac{\log P_{e1}(SNR)}{\log SNR} = d_1, \lim_{SNR\to\infty} -\frac{\log P_{e2}(SNR)}{\log SNR} = d_2, \tag{3}$$

where $R_1(SNR)$, $R_2(SNR)$, $P_{e1}(SNR)$, $P_{e2}(SNR)$ are the rates and the probabilities of error for user 1 and user 2. Multiplexing gain region (MGR) is thus defined as the set of all achievable multiplexing gain pair $(r_1, r_2)$ for all encoding schemes. The MGR is the CCR in high SNR approximation. For a multiple access channel, the CCR is the closure of the convex hull of all $(R_1, R_2)$ satisfying

$$R_1 \leq I(X_1; Y|X_2) \tag{4}$$
$$R_2 \leq I(X_2; Y|X_1) \tag{5}$$
$$R_1 + R_2 \leq I(X_1, X_2; Y). \tag{6}$$

Thus the MGR for a MIMO fading multiple access channel with $m_1, m_2$ transmit antennas and $n$ receive antennas is

$$r_1 \leq \min(m_1, n) \tag{7}$$
$$r_2 \leq \min(m_2, n) \tag{8}$$
$$r_1 + r_2 \leq \min(m_1 + m_2, n). \tag{9}$$

An EER depends on the operating point $(R_1, R_2)$ [7, 8]. Similarly, given a multiplexing gain pair $(r_1, r_2)$, we define the diversity gain region (DGR) as the set of all achievable diversity gain pair $(d_1, d_2)$. The diversity gain region is the EER in high SNR approximation.

# 3 Achievable Diversity Gain Region

For the MIMO fading multiple access channel considered in this paper, we propose two encoding strategies - superposition and channel-splitting. For the superposition encoding, the channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ have i.i.d. entries $\mathcal{CN}(0,1)$. In the receiver side, we decode users' messages using *joint* maximum likelihood (ML) decoding, i.e., decoding users' messages based on the pair $(i,j)$ maximizing $P(Y|X_1(i), X_2(j))$, where $X_1(i)$ and $X_2(j)$ are the transmitted codewords from user 1 and user 2. There are three types of error events [3]. Type 1 error occurs when user 1 codeword is decoded erroneously, but user 2 codeword is decoded correctly. Type 2 error occurs when user 2 codeword is decoded erroneously, but user 1 codeword is decoded correctly. Type 3 error occurs when both users' codewords are decoded as wrong codewords. Denote $P_{e,t1}$, $P_{e,t2}$, and $P_{e,t3}$ the probabilities of type 1, type 2, and type 3 error events. Define $d^{out}_{m,n}(r)$ as the outage diversity gain for a MIMO fading single-user channel with $m$ transmit antennas and $n$ receive antennas, i.e., $d^{out}_{m,n}(r)$ is the piecewise linear function connecting the points $(k, d^{out}_{m,n}(k)) = (k, (m-k)(n-k)), k \in \mathbb{Z}^+$ [5]. Assuming the block length $l \geq m + n_1 + n_2 - 1$, it can be shown by random coding argument that there exist codebooks for user 1 and user 2 such that

$$P_{e,t1} \dot{\leq} SNR^{-d^{out}_{m_1,n}(r_1)} \tag{10}$$

$$P_{e,t2} \dot{\leq} SNR^{-d^{out}_{m_2,n}(r_2)} \tag{11}$$

$$P_{e,t3} \dot{\leq} SNR^{-d^{out}_{m_1+m_2,n}(r_1+r_2)}. \tag{12}$$

The probabilities of error for user 1 ($P_{e1}$) and user 2 ($P_{e2}$) can be upper bounded by

$$\begin{aligned} P_{e1} = P_{e,t1} + P_{e,t3} &\dot{\leq} SNR^{-d^{out}_{m_1,n}(r_1)} + SNR^{-d^{out}_{m_1+m_2,n}(r_1+r_2)} \\ &\doteq SNR^{-\min\{d^{out}_{m_1,n}(r_1), d^{out}_{m_1+m_2,n}(r_1+r_2)\}} \end{aligned} \tag{13}$$

$$\begin{aligned} P_{e2} = P_{e,t2} + P_{e,t3} &\dot{\leq} SNR^{-d^{out}_{m_2,n}(r_2)} + SNR^{-d^{out}_{m_1+m_2,n}(r_1+r_2)} \\ &\doteq SNR^{-\min\{d^{out}_{m_2,n}(r_2), d^{out}_{m_1+m_2,n}(r_1+r_2)\}}. \end{aligned} \tag{14}$$

Thus, we can derive achievable diversity gains for user 1 and user 2 in a MIMO fading multiple access channel as

$$d^s_1 = \min\{d^{out}_{m_1,n}(r_1), d^{out}_{m_1+m_2,n}(r_1+r_2)\} \tag{15}$$

$$d^s_2 = \min\{d^{out}_{m_2,n}(r_2), d^{out}_{m_1+m_2,n}(r_1+r_2)\}, \tag{16}$$

where the superscript "s" denotes superposition.

The MGR of a MIMO fading multiple access channel can be divided into four regions $r_{12}$, $r_{13}$, $r_{23}$, and $r_3$ depending whether type 1 error, type 2 error ,or type 3 error dominates (see Fig. 2). In the region $r_{12}$, $d^{out}_{m_1,n}(r_1) \leq d^{out}_{m_1+m_2,n}(r_1+r_2)$ and $d^{out}_{m_2,n}(r_2) \leq d^{out}_{m_1+m_2,n}(r_1+r_2)$. The achievable diversity gains are $d^s_1 = d^{out}_{m_1,n}(r_1)$ and $d^s_2 = d^{out}_{m_2,n}(r_2)$. In this region, we can not increase the diversity gains for either of the users, since each user attains his

single-user diversity gain. In the region $r_{13}$, $d^{out}_{m_1,n}(r_1) \leq d^{out}_{m_1+m_2,n}(r_1+r_2) \leq d^{out}_{m_2,n}(r_2)$. The achievable diversity gains are $d^s_1 = d^{out}_{m_1,n}(r_1)$ and $d^s_2 = d^{out}_{m_1+m_2,n}(r_1+r_2)$. In this region, although we can not increase the first user's achievable diversity gain by reducing the second user's achievable diversity gain, it is possible to increase the second user's achievable diversity gain by reducing the first user's achievable diversity gain because the dominant error for user 2 is a type 3 error. A similar result also holds for region $r_{23}$ by exchanging the roles of user 1 and user 2 in the $r_{13}$ region. In the region $r_3$, $d^{out}_{m_1+m_2,n}(r_1+r_2) \leq d^{out}_{m_1,n}(r_1)$ and $d^{out}_{m_1+m_2,n}(r_1+r_2) \leq d^{out}_{m_2,n}(r_2)$. The achievable diversity gains are $d^s_1 = d^s_2 = d^{out}_{m_1+m_2,n}(r_1+r_2)$. In this region, type 3 error is dominant over both type 1 and type 2 errors, so it is possible to increase the first (second) user's achievable diversity gain by reducing the second (first) user's achievable diversity gain.

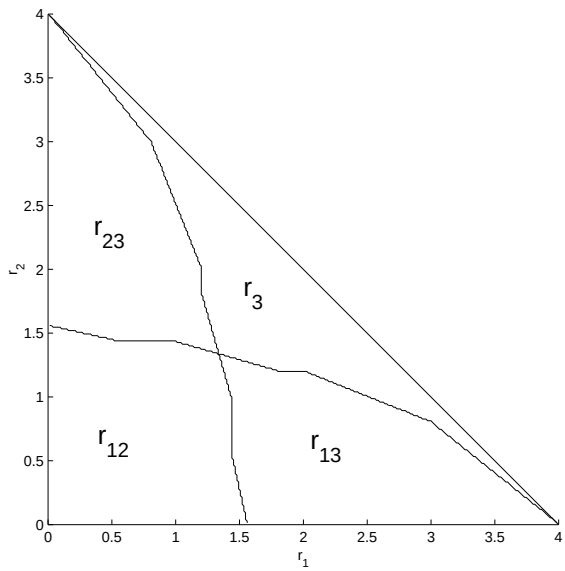



Figure 2: Multiplexing gain region for $m_1 = m_2 = n = 4$.

In Fig. 3(a), the solid curve is the boundary of the achievable DGR obtained by superposition. In addition to superposition encoding, we can also derive achievable diversity gains for user 1 and user 2 by channel-splitting. Before writing the achievable diversity gains by channel-splitting, we need to review the diversity-multiplexing tradeoff for a MIMO fading single-user channel derived in [5]. Define $(x)^+ = \max(x, 0)$ and $\mathbb{R}^n_+$ as the set of real n-vectors with nonnegative elements. The random coding diversity gain $d_{m,n,l}(r)$ is defined as

$$d_{m,n,l}(r) = \min_{\underline{\alpha} \in \mathcal{G}^c} \left\{ \sum_{i=1}^{\min(m,n)} (2i - 1 + |m-n|)\alpha_i + l \left[ \left( \sum_{i=1}^{\min(m,n)} (1-\alpha_i)^+ \right) - r \right] \right\}, \quad (17)$$

and

$$\mathcal{G} = \left\{ \underline{\alpha} \in \mathbb{R}^{\min(m,n)+} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0, \text{and} \sum_{i=1}^{\min(m,n)} (1-\alpha_i)^+ \leq r \right\} \quad (18)$$

$$\mathcal{G}^c = \mathbb{R}^{\min(m,n)+} - \mathcal{G}, \quad (19)$$

and the expurgated diversity gain $d^{ex}_{m,n,l}(r)$ is defined as

$$d^{ex}_{m,n,l}(r) = n \; d^{-1}_{m,l,n}(lr), \quad (20)$$

and $d^{-1}_{m,l,n}$ is the inverse function of $d_{m,l,n}(r)$. Both the random coding diversity gain $d_{m,n,l}(r)$ and the expurgated diversity gain $d^{ex}_{m,n,l}(r)$ are achievable in a MIMO fading single-user channel with $m$ transmit antennas, $n$ receive antennas, and block length $l$ [5]. Note that the random coding diversity gain $d_{m,n,l}(r)$ is equal to $d^{out}_{m,n}(r)$ for $l \geq m+n-1$.

In channel-splitting, we allocate $pl$ symbols to user 1 and $(1-p)l$ symbols to user 2 inside each block length $l$, where $p = \frac{k}{l}$ and $1 \leq k \leq l-1$, $k \in \mathbb{Z}$ (see Fig. 4). Thus, the achievable diversity gains are

$$d_1^{cs} = \max\{d_{m_1,n,pl}(\frac{r_1}{p}), d^{ex}_{m_1,n,pl}(\frac{r_1}{p})\} \tag{21}$$

$$d_2^{cs} = \max\{d_{m_2,n,(1-p)l}(\frac{r_2}{1-p}), d^{ex}_{m_2,n,(1-p)l}(\frac{r_2}{1-p})\}, \tag{22}$$

where the superscript "cs" denotes channel-splitting, and $p$ is the portion of time allocated to user 1 ($p = \frac{k}{l}$ and $1 \leq k \leq l-1$, $k \in \mathbb{Z}$). In Fig. 3(a), the dotted curve is the achievable DGR by channel-splitting. The union of the superposition achievable DGR and the channel-splitting achievable DGR is an inner bound for the DGR. We summarize all the results for the achievable DGR in the following theorem.

**Theorem 1** *For a MIMO fading multiple access channel with $m_1$, $m_2$ transmit antennas for user 1, user 2, $n$ receive antennas, and block length $l \geq m_1 + m_2 + n - 1$, an achievable DGR is $DGR(r_1, r_2) = DGR_s(r_1, r_2) \cup DGR_{cs}(r_1, r_2)$, where $DGR_s(r_1, r_2)$ and $DGR_{cs}(r_1, r_2)$ are given by*

$$\begin{aligned} DGR_{cs}(r_1, r_2) = \{(d_1, d_2) : p = \frac{k}{l}, 1 \leq k \leq l-1, k \in \mathbb{Z} \\ d_1 \leq \max\{d_{m,n_1,pl}(\frac{r_1}{p}), d^{ex}_{m,n_1,pl}(\frac{r_1}{p})\} \\ d_2 \leq \max\{d_{m,n_2,(1-p)l}(\frac{r_2}{1-p}), d^{ex}_{m,n_2,(1-p)l}(\frac{r_2}{1-p})\} \end{aligned} \tag{23}$$

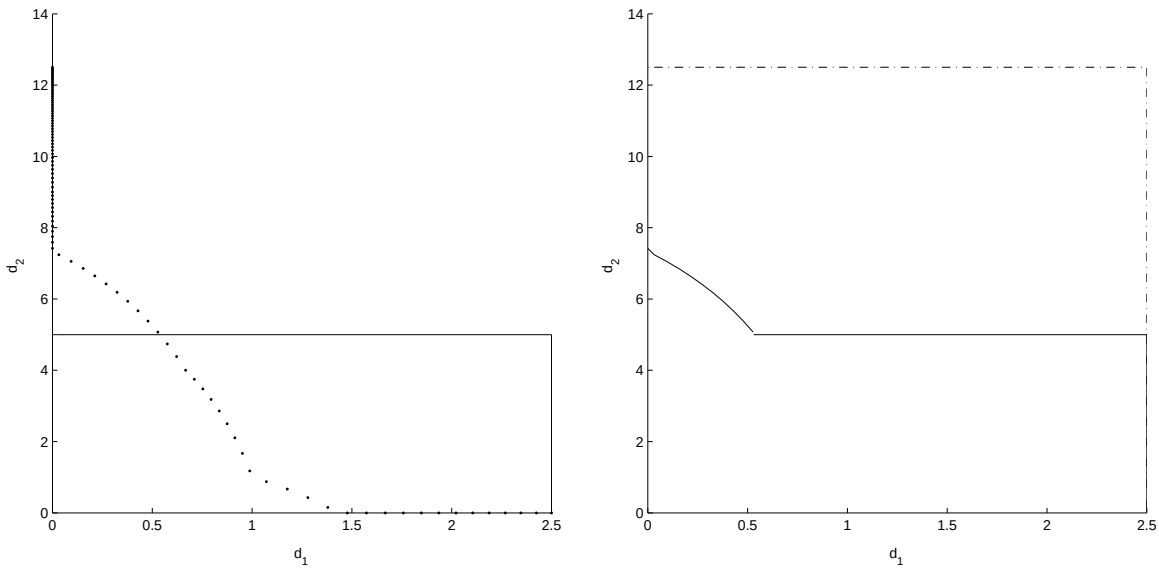
$$\begin{aligned} DGR_s(r_1, r_2) = \{(d_1, d_2) : \\ d_1 \leq \min\{d^{out}_{m_1,n}(r_1), d^{out}_{m_1+m_2,n}(r_1 + r_2)\} \\ d_2 \leq \min\{d^{out}_{m_2,n}(r_2), d^{out}_{m_1+m_2,n}(r_1 + r_2)\}\} \end{aligned} \tag{24}$$

□

Although we can not increase the achievable DGR by superposition for operating points inside region $r_{12}$, we would like to know for which operating points $(r_1, r_2)$ channel-splitting can enlarge the achievable DGR by superposition. Therefore, consider the following. In (21), (22), we choose the largest $p$ such that $d_2^{cs}$ is positive, then we get the largest achievable diversity gain $d^{cs}_{1,max}$ for user 1 by channel-splitting method. In Fig. 5(a), we have $d^{cs}_{1,max} > d_1^s$ inside the dotted region. Similarly, we can obtain another region with $d^{cs}_{2,max} > d_2^s$ by reducing $d_1^{cs}$ close to zero. The union of these two regions is the multiplexing gain region where superposition achievable DGR can be enlarged by channel-splitting (see Fig. 5(b)).

# 4 Outer Bound for Diversity Gain Region

For a MIMO fading multiple access channel, the probabilities of decoding error for user 1 and user 2 can always be lower bounded by the probabilities of decoding error for user 1




(a) (b)

Figure 3: Diversity gain region for $m_1 = m_2 = n = 4; r_1 = 2.5, r_2 = 0.5$ (a) channel-splitting(dotted) and superposition(solid), (b) inner bound(solid) and outer bound(dash-dotted).

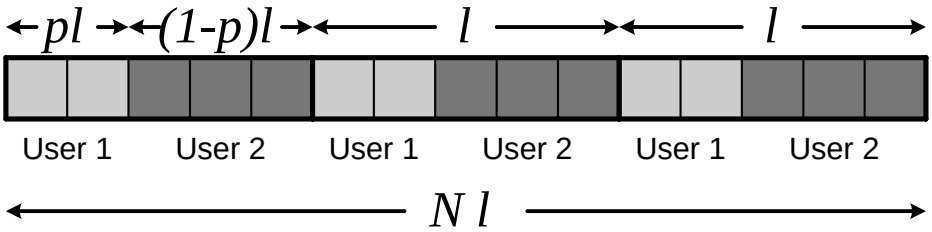



Figure 4: Channel-splitting

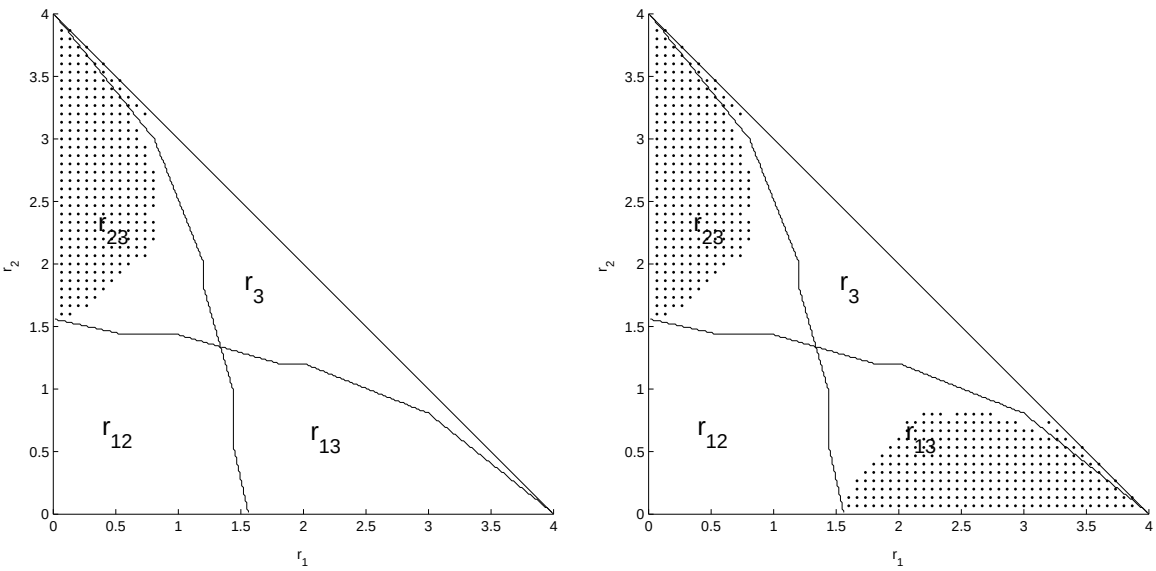
and user 2 operating over the point-to-point channels defined by $\mathbf{Y} = \sqrt{\frac{SNR}{m_i}}\mathbf{H_i X_i} + \mathbf{Z}$, for $i = 1, 2$. Further, if we allow the two transmitters in the MIMO fading multiple access channel to cooperate, we have a MIMO fading single-user channel with $m_1 + m_2$ transmit antennas and $n$ receive antennas, whose probability of error (using an optimal receiver), $P_e'$, should be less than or equal to the probability of system error $P_e$ in the original multiple access channel. Using the union bound, it is also easy to show that $P_e \leq 2\max\{P_{e1}, P_{e2}\}$, where $P_{ei}$ denotes the probability of error for user $i$ in the original multiple access channel. Collecting all these ideas, we have the following outer bound for the DGR

$$d_1 \leq d^{out}_{m_1,n}(r_1) \tag{25}$$

$$d_2 \leq d^{out}_{m_2,n}(r_2) \tag{26}$$

$$\min\{d_1, d_2\} \leq d^{out}_{m_1+m_2,n}(r_1 + r_2). \tag{27}$$

In Fig. 3(b), the solid curve is the achievable DGR, and the dashed-dotted curve is the outer bound for the DGR. We can also conclude from (25), (26) that all the operating points inside the region $r_{12}$ in Fig. 2 have tight DGR inner and outer bounds.




(a) (b)

Figure 5: Multiplexing gain region for $m_1 = m_2 = n = 4$ (a) $d^{cs}_{1,max} > d^s_1$, (b) $d^{cs}_{1,max} > d^s_1$ or $d^{cs}_{2,max} > d^s_2$.

# 5 Conclusion

In this work, we introduce the notion of multiplexing gain region and diversity gain region, which are the channel capacity region and the error exponent region in high SNR approximation. We derive the DGR inner bound and the DGR outer bound for a MIMO fading multiple channel. The concept of the MGR and the DGR is very general and can be applied to other multi-user channels, such as a MIMO fading broadcast channel. Currently the authors are investigating tighter DGR inner bounds and DGR outer bounds and practical schemes to achieve these bounds.

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