Geometric Proof of Rate-Distortion Function of Gaussian Sources with Side Information at the Decoder

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Abstract — **The achievability of the Wyner-Ziv theorem [1] for the Gaussian case is shown using only geometric arguments. The motivation for this is to inspire the construction of practical codes based on this [2].**

I. Introduction

Distributed source coding deals with the encoding of correlated sources that do not communicate with one another. The concept of distributed source coding has been considered for continuous sources with a fidelity criterion in [1] where X and Y are sources such that the decoder has (lossless) access to Y, and the encoder compresses X using the fact that the decoder knows Y. It was shown that when X and Y are jointly Gaussian with mean squared error as the fidelity criterion, it is possible to compress X as efficiently as the case when both the encoder and the decoder have access to Y i.e. $R_{x|y}(D) = R^*(D)$ [1]. In [2], a construction of generalized coset codes has been proposed based on the channel coding principles. Using geometrical arguments we establish the achievability of the following rate distortion bound with side information for X:

$$R = \frac{1}{2}\log\left(\frac{\sigma_n^2}{(1+\sigma_n^2)d^*}\right), \tag{1}$$

where X is *i.i.d.* Gaussian with zero mean and unit variance, d^* is the reconstruction fidelity, and the side information is given by $Y = X + N$, where N is *i.i.d.* Gaussian with zero mean and with variance σ_n^2 and independent of X

II. Geometric Derivation of Wyner-Ziv bound

We encode the source X in blocks of length L. Randomly distribute 2^{LR_1} codewords independently and uniformly on the surface of an L-dimensional sphere $\mathcal{S}_L\left(\sqrt{1-\theta^2}\right)$ of radius $\sqrt{1-\theta^2}$ where R_1 is a real number which depends on θ ($\theta = \sqrt{\frac{d^*\sigma_n^2}{\sigma_n^2-d^*}}$). Let $\mathcal{C}$ denotes this set of codewords. Randomly choose 2^{LR_2} codewords independently and uniformly from this set $\mathcal{C}$, and give an index to this subset called a coset. Keep repeating this until all the codewords are exhausted. Thus, there are $2^{L(R_1-R_2)} = 2^{LR}$ indices.

Encoding involves finding a codeword from the space of codewords satisfying a distance criterion (within distance θ) and sending the index of the coset containing the encoded codeword to the decoder. If none of the codewords satisfies the criterion, then an error is declared. We derive a lower bound on R_1. The basic idea behind the proof is the following: encoder chooses a codeword to represent X, and nature chooses the side information Y independent of the encoder using a "channel" $p(y|x)$ on X. Yet there exists a strong correlation between the quantized codeword and the side information, which needs to be exploited. This key result (also known as the Markov Lemma [1]) used in the geometric arguments is given by the following theorem.

Theorem: For any $\epsilon > 0$ (sufficiently close to 0), $\exists\ L_1(\epsilon)$ such that for any $L > L_1(\epsilon)$, $P\left[\left|\|\tilde{\mathbf{Y}}\|^2 - \theta^2 - \sigma_n^2\right| > \epsilon\right] < \epsilon$, where $\tilde{\mathbf{Y}} = \mathbf{Y} - \mathbf{W}$, $\mathbf{Y}, \mathbf{W}$ are side information and quantized codeword vectors respectively.

The first part of the decoder involves finding a "suitable" codeword in the given coset. If either more than one or none of the codewords satisfies the distance criterion, then an error is declared. We derive an upper bound on R_2. The second part of the decoder then gets the reconstruction vector as a function of the encoded codeword and the side information vector. It is shown geometrically that the probabilities of the error events can be made arbitrarily small for large L. The geometry for the ideal case of very large L is shown in Fig. 1.

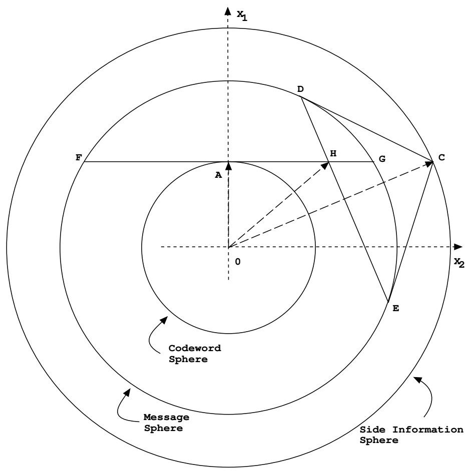

Figure 1: OA, OC and OH represent the codeword and observed side information and the reconstruction vectors respectively. A, H and C are collinear. OH gives the minimum distortion given that OA and OC are codeword and the side information respectively. FG and DE represent two spheres.

References

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[2] S. S. Pradhan and K. Ramchandran, "Group-theoretic construction and analysis of generalized coset codes for symmetric/asymmetric distributed source coding," *Proc. Conf. on Info. Sci. and Sys. (CISS)*, Princeton, March 2000.

[1] This work was supported by in part by DARPA Grant F29601-99-1-0169 and NSF (CAREER) Grant MIP 97-03181.