A characterization of functional duality between source and channel coding

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I. Introduction

Source coding under a rate-distortion (R-D) constraint and channel coding under a channel cost constraint have long been considered as duals of each other starting from Shannon [1]. We address this duality with the objective of providing a mathematical characterization of it in a *functional* sense. To be specific, given an optimal source (channel) coding scheme, we detail the conditions under which such a scheme is a functional dual to a channel (respectively source) coding scheme. We show that this functional duality is enabled through appropriate choices of metrics and constraints for the dual source (or channel) coding problem. Our inspiration for this is derived from a recent work on the optimality conditions for uncoded communications [2]. We then extend this duality formulation to source and channel coding with side information [3, 4].

II. Duality

To study these concepts in one framework, consider the following correspondence:

$$\text{source coding}\left\{\begin{array}{ccccc} \text{outcome} & X & \longleftrightarrow & Y & \text{output} \\ \text{reconstruction} & \hat{X} & \longleftrightarrow & X & \text{input}\end{array}\right\}\text{channel coding}$$

Theorem 1: For a source coding problem, with a given source $\bar{p}(x)$, input and reconstruction alphabets $\mathcal{X}$ and $\hat{\mathcal{X}}$ respectively, a distortion measure $d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}^+$, and a distortion constraint D, let the conditional distribution achieving rate-distortion optimality $R(D)$ be given by $p^*(\hat{x}|x)$, inducing the following distributions: $p^*(x|\hat{x}) \triangleq \frac{p^*(\hat{x}|x)\bar{p}(x)}{\sum_x \bar{p}(x)p^*(\hat{x}|x)}$, $p^*(\hat{x}) \triangleq \sum_x \bar{p}(x)p^*(\hat{x}|x)$. Then $\exists$ a *dual* channel coding problem for the channel $p^*(x|\hat{x})$, having input and output alphabets $\hat{\mathcal{X}}$ and $\mathcal{X}$ respectively, a cost measure $\omega : \hat{\mathcal{X}} \to \mathbb{R}^+$, and a cost constraint W, such that:

- the rate-distortion bound $R(D)$ is equal to the capacity-cost bound $C(W)$,
- the channel input distribution given by $p^*(\hat{x})$ achieves capacity-cost optimality.

where the cost function and cost constraint are given respectively by: $\omega(\hat{x}) \triangleq c_1 D(p^*(x|\hat{x}) \| \bar{p}(x)) + \theta$, $W \triangleq E_{p^*(\hat{x})}(\omega(\hat{X}))$, for arbitrary $c_1 > 0$ and θ. A similar result can be obtained for a given channel coding problem.

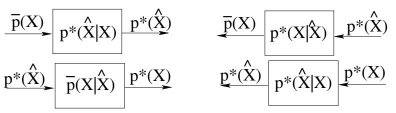

Figure 1: (a) Duality between source and channel coding.

For a given source coding problem, although the conditional distribution is $p^*(\hat{x}|x)$ with $\bar{p}(x)$ as input, the dual channel coding problem is defined for $p^*(x|\hat{x})$ with input distribution $p^*(\hat{x})$ in the reverse order as shown in Fig. 1. The test channel of the source coding problem given by $p^*(x|\hat{x})$ becomes the channel for the dual channel coding problem and vice versa.

III. Extension to coding with side information

This concept can be generalized to source and channel coding with side information (SCSI,CCSI) [3, 4]. Thus for a given problem of SCSI, we can associate a dual problem of CCSI with an appropriately chosen cost function, such that the encoder of the former is functionally identical to the decoder of the latter and vice versa.

Theorem 2: For a SCSI with source $X \sim \bar{p}(x|y)$ and side information $Y \sim \bar{p}(y)$, input, side information and reconstruction alphabets $\mathcal{X}, \mathcal{Y}$ and $\hat{\mathcal{X}}$ respectively, a distortion measure $d : \mathcal{X} \times \mathcal{Y} \times \hat{\mathcal{X}} \to \mathbb{R}^+$ and distortion constraint D, let $p^*(u|x), p^*(\hat{x}|u,y)$ achieve the rate-distortion optimality $R^*(D)$, inducing the following distributions: $p^*(x,y,\hat{x},u) \triangleq \bar{p}(y)\bar{p}(x|y)p^*(u|x)p^*(\hat{x}|u,y)$, $p^*(x|\hat{x},y) \triangleq \frac{\sum_u p^*(x,y,\hat{x},u)}{\sum_{u,x} p^*(x,y,\hat{x},u)}$, $p^*(u|y) \triangleq \frac{\sum_{x,\hat{x}} p^*(x,y,\hat{x},u)}{\bar{p}(y)}$, $p^*(\hat{x}|y) \triangleq \frac{\sum_{u,x} p^*(x,y,\hat{x},u)}{\bar{p}(y)}$. If $p^*(x,y,\hat{x},u)$ is such that $X \to (\hat{X},Y) \to U$, then $\exists$ a dual CCSI, with input, side information and output alphabets $\hat{\mathcal{X}}, \mathcal{X}$ and $\mathcal{Y}$ respectively, for the channel $p^*(x|\hat{x},y)$, with side information $\bar{p}(y)$, a cost measure $\omega : \hat{\mathcal{X}} \times \mathcal{Y} \to \mathbb{R}^+$ and a cost constraint W, such that:

- the rate-distortion bound $R^*(D)$ is equal to the capacity-cost bound $C^*(W)$.
- the distributions $p^*(u|y)$ and $p^*(\hat{x}|u,y)$ achieve the capacity-cost optimality.

where the cost function and cost constraint are given respectively by $\omega(\hat{x},y) \triangleq c_1 D(p^*(x|\hat{x},y) \| \bar{p}(x|y)) + \theta(y)$, $W \triangleq E_{\bar{p}(y)p^*(\hat{x}|y)}(\omega(\hat{X},Y))$, for arbitrary $c_1 > 0$ and $\theta(y)$. A similar result can be obtained for a given CCSI problem.

Example: Given a CCSI: $\mathcal{X} = \mathcal{Y} = \hat{\mathcal{X}} = \mathbb{R}$, $X = \hat{X} + S + Z$, $p(Z) = \mathcal{N}(0,D)$, S arbitrary, $\omega(\hat{x},s) = \hat{x}^2$, $W = N - D$ for some $N > D$. It was recently shown that, although S is arbitrary there is no rate-loss due to the presence of side information only at one end rather than at both ends. Its dual SCSI: $X = S + V$, $p(V) = \mathcal{N}(0,N)$ and $d(x,\hat{x},s) = (x - \hat{x} - s)^2$. It can be shown even here that there is no rate-loss due to the presence of side information only at one end.

References

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