Source coding with feedforward: Gaussian sources

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I. Introduction

With the recent emergence of applications related to sensor networks, efficient encoding of information signals in the presence of side information has received special attention. In such situations, the underlying information field may be traveling over space from the encoder to the decoder, for example, a seismic wave traveling in the direction of the receiver from the encoder. The side information at the receiver is a noisy and/or delayed version of the signal observed at the encoder. Motivated by this application, we consider the following source representation problem. Consider a stationary discrete memoryless source X with a probability distribution $p(x)$ with some alphabet $\mathcal{X}$, and a reconstruction alphabet $\hat{\mathcal{X}}$. Associated with the source, there is a distortion measure $d : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow \mathbb{R}^+$. The encoder is a mapping from the l-product source alphabet to an index set: $f : \mathcal{X}^l \rightarrow \{1, 2, \ldots, M\}$, where l denotes the block-length in encoding and $\frac{1}{l}\log M$ denotes the rate in bits/sample. The distortion measure for a pair of sequences of length l is the average of the distortions of l samples. To reconstruct any source sample, the decoder has access to all the past source samples. Thus the decoder is a sequence of mappings $g_i : \{1, 2, \ldots, M\} \times \mathcal{X}^{i-1} \rightarrow \hat{\mathcal{X}}$ for $i = 1, 2, \ldots, l$. Let $\mathbf{g}(\mathbf{x})$ denote the l-length vector reconstruction of the l-length source vector $\mathbf{x}$. The goal is to minimize $E\left[d\left(\mathbf{X}, \mathbf{g}(\mathbf{X})\right)\right]$ for a given rate $R = (1/l)\log M$. We refer to this problem as source coding with feedforward. Let $R_{ff}(D)$ denotes the infimum of R over all encoder-decoder pairs such that $Ed \leq D$ for some $D > 0$. It can be shown that for stationary memoryless sources, $R_{ff}(D) = R(D)$, where $R(D)$ denotes the optimal Shannon rate-distortion function. For sources with memory, $R_{ff}(D) \leq R(D)$. Although the author arrived at this problem from the applications of sensor networks, and its duality with channel coding with feedback, as pointed out by one of the anonymous referees, this problem has also been considered in an independent work [3] in a different context of competitive prediction.

II. Main Results

In this paper we consider the case of stationary memoryless Gaussian source with zero-mean and variance σ^2, and with mean squared error as the distortion measure, where we give a deterministic scheme that achieves the optimal rate-distortion bound using simple uniform scalar quantizers.

Let $Y = -\sum_{k=2}^{l}\sqrt{\beta^2-1}\,\beta^{-(k+1)}X_k - \beta^{-1}X_1$, where $\beta > 1$ be some constant, The encoder quantizes Y using a uniform scalar quantizer bounded between $-\Delta/2$ and $\Delta/2$ with M levels, where Δ will be determined later. The index of the cell containing it is sent to the decoder. Let $\hat{Y}$ denotes the quantized version of Y, where the set of mid-points of quantization cells denote the alphabet of $\hat{Y}$. The decoder reconstruction is given by: $\hat{X}_i = \beta\hat{X}_{i-1} - (\beta^2-1)\beta^{-1}X_{i-1}$, for $i = 3, \ldots, l$ and $\hat{X}_1 = -\beta\hat{Y}$, $\hat{X}_2 = \sqrt{\beta^2-1}(\hat{X}_1 - X_1)$. Let $Q = \hat{Y} - Y$, and $D' = EQ^2$. It can be shown that

$$\frac{1}{l}\sum_{i=1}^{l} E(X_i - \hat{X}_i)^2 = \frac{D'\beta^{2l}}{l} + \frac{\sigma^2(l\beta^2 - \beta^2)}{l\beta^4}. \tag{1}$$

Let $D = \frac{1}{l}\sum_{i=1}^{l} E(X_i - \hat{X}_i)^2$, and $M = \beta^{l(1+\epsilon')}$ for some $\epsilon' > 0$. It was shown in [2] that for any continuous source U, the mean squared error D_Γ obtained in using an unbounded uniform scalar quantizer with step size Γ with mid-point as reconstruction satisfies: $D_\Gamma = \frac{\Gamma^2}{12} + o(\Gamma^2)$, where $o(x)$ refers to a function such that $o(x)/x \rightarrow 0$ as $x \rightarrow 0$. Let $\mathbb{E}$ denote the event that $|Y| > \Delta/2$, and let $D'' = E(Q^2|\mathbb{E}^c)$. Using the above identities, given the event $\mathbb{E}^c$, we have

$$D'' = \frac{\Delta^2}{12M^2}(1+\tau) = \frac{\Delta^2}{12(\beta^{2l}\beta^{2l\epsilon'})}(1+\tau). \tag{2}$$

Now choose $\Delta = \beta^{l\epsilon'}\sqrt{\frac{\beta^2+1}{\beta^4}}$. which results in $D'' = \frac{(\beta^2+1)}{12\beta^4\beta^{2l}}(1+\tau)$. Since $\beta > 1$, one can make τ arbitrarily small by choosing sufficiently large l. Note that $D' \leq P(\mathbb{E}^c)D'' + P(E)Var(Y) \leq D'' + P(\mathbb{E})\frac{\sigma^2(1+\beta^2)}{\beta^4}$.

In sequel we will show that $P(\mathbb{E})$ can be made to decay faster than $1/\beta^{2l}$. Noting that $R_{ff}(D) = R(D) = (1/2)\log(\sigma^2/D)$, we have

$$R = \frac{1}{2}\log\left[\frac{\sigma^2}{D}\right] + \epsilon, \tag{3}$$

where $\lim_{l\rightarrow\infty}\lim_{\epsilon'\rightarrow 0}\epsilon(\epsilon', l) = 0$.

Now let us analyze the probability that the absolute value of Y is greater than $\Delta/2$, i.e., the probability of the event $\mathbb{E}$. We have the following doubly exponential decay of this probability:

$$P(\mathbb{E}) \leq 2\mathrm{erfc}\left[\beta^{l\epsilon'}\frac{1}{4\sigma}\right] \leq c_1 e^{-c_2 e^{lc_3}}, \tag{4}$$

where

$$c_1 = \frac{8\sigma}{a\sqrt{2\pi}e^{\frac{lR\epsilon'}{(1+\epsilon')}}}, \quad c_2 = \frac{1}{8\sigma^2} \quad \text{and} \quad c_3 = \frac{2R\epsilon'}{(1+\epsilon')}. \tag{5}$$

This method is related to a similar deterministic approach of [1] for coding for certain channels with feedback.

References

[1] J. P. M. Schalkwijk and T. Kailath, "A coding scheme for additive noise channels with feedback- part II: Band-limited signals," *IEEE Trans. on Information Theory*, vol. IT-12, pp. 183–189, April 1966.

[2] D. Marco and D. L. Neuhoff, "On the distortion and additive noise of uniform scalar quantizers," *submitted to IEEE Trans. on Information Theory*, December 2003.

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