New Bounds on the Maximal Error Exponent for Multiple-Access Channels

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***Abstract*—The problem of bounding the reliability function of a multiple-access channel (MAC) is studied. An upper bound on the minimum Bhattacharyya distance between codeword pairs is derived. For a certain large class of two-user discrete memoryless (DM) MAC, a lower bound on the maximal probability of decoding error is derived as a consequence of the upper bound on Bhattacharyya distance. Further, an upper bound on the average probability of decoding error is studied. It is shown that the corresponding upper and lower bounds have a similar structure. Using a conjecture about the structure of the multi-user code, a tighter lower bound for the maximal probability of decoding error is derived and is shown to be tight at zero rates.**

I. Introduction

An interesting problem in network information theory is to determine the minimum probability of error which can be achieved on a discrete memoryless (DM), multiple-access channel (MAC). Ahlswede [1] and Liao [2] studied the capacity region for MAC. Later, stronger versions of their coding theorem, giving exponential upper and lower bounds on the error probability, have been derived by numerous other authors. Slepian and Wolf [3], Dyachkov [4], Gallager [5], Pokorny and Wallmeier [6], Liu and Hughes [7], and Nazari [8] have all studied *upper* bounds on the average probability of decoding error. Haroutunian [9] derived a *lower* bound on the optimal average error probability. Nazari et al. [10] derived a tighter lower bound that explicitly captures the separation of the encoders in the MAC. However, the bound in [10] is only valid for the maximal and not the average error probability.

In this paper, we derive a new lower bound on the maximal error probability for MAC. This bound is derived by establishing a link between minimum Bhattacharyya distance and maximal probability of decoding error; then, the upper bound on Bhattacharyya distance is used to infer the lower bound on probability of decoding error. Also, by the method of expurgation [8], an upper bound on the average probability of decoding error is derived. At zero rate pair, the upper and lower bounds have similar structure, however, they may not be equal. By using a conjecture about the structure of the code, we derive another bound on the Bhattacharrya distance, which results in a tighter lower bound on the maximal probability of decoding error. At zero rate pair, this bound is tight, i.e., it is asymptotically equal to the upper bound.

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The paper is organized as follows. Some preliminaries are introduced in section II. The main result of the paper, which is an upper bound on the reliability function of the channel, is obtained in section III. In section IV, a lower bound on the reliability function is developed and compared with the result of section III. Finally, in section V, a conjecture about the structure of all possible codes is proposed, and based on the conjecture, another upper bound on the reliability function of the channel is obtained. It is shown that this bound is always asymptotically tight at zero rate pair.

II. Preliminaries

For any alphabet $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ denotes the set of all probability distributions on $\mathcal{X}$. The *type* of a sequence $\mathbf{x} = (x_1, ..., x_n) \in \mathcal{X}^n$ is the distributions $P_{\mathbf{x}}$, on $\mathcal{X}$, defined by:

$$P_{\mathbf{x}}(x) \triangleq \frac{1}{n} N(x|\mathbf{x}), \qquad x \in \mathcal{X}, \tag{1}$$

where $N(x|\mathbf{x})$ denotes the number of occurrences of x in $\mathbf{x}$. Let $\mathcal{P}_n(\mathcal{X})$ denote the set of all types in $\mathcal{X}^n$, and define the set of all sequences in $\mathcal{X}^n$ of type P as

$$T_P \triangleq \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}. \tag{2}$$

The joint type of a pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ is the probability distribution $P_{\mathbf{x},\mathbf{y}}$ on $\mathcal{X} \times \mathcal{Y}$ defined by:

$$P_{\mathbf{x},\mathbf{y}}(x, y) \triangleq \frac{1}{n} N(x, y|\mathbf{x}, \mathbf{y}), \qquad (x, y) \in \mathcal{X} \times \mathcal{Y}, \tag{3}$$

where $N(x, y|\mathbf{x}, \mathbf{y})$ is the number of occurrences of (x, y) in $(\mathbf{x}, \mathbf{y})$.

Definition 1. *An (n, M, N) multi-user code is a set $\{(\mathbf{x}_i, \mathbf{y}_j, D_{ij}) : 1 \leq i \leq M, 1 \leq j \leq N\}$ with*

- $\mathbf{x}_i \in \mathcal{X}^n$, $\mathbf{y}_j \in \mathcal{Y}^n$, $D_{ij} \subset \mathcal{Z}^n$
- $D_{ij} \cap D_{i'j'} = \varnothing$ *for* $(i, j) \neq (i', j')$.

The average error probability of this code for the MAC, $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, is defined as

$$e(\mathcal{C}, W) \triangleq \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} W^n(D_{i,j}|\mathbf{x}_i, \mathbf{y}_j). \tag{4}$$

Similarly, the maximal error probability of this code for W is defined as

$$e_m(\mathcal{C}, W) \triangleq \max_{(i,j)} W^n(D_{i,j}|\mathbf{x}_i, \mathbf{y}_j). \tag{5}$$

Definition 2. *For the MAC, $W:\mathcal{X}\times\mathcal{Y}\to\mathcal{Z}$, the average and maximal error reliability functions, at rate pair (R_X,R_Y), are defined as:*

$$E^*_{av}(R_X,R_Y)\triangleq\lim_{n\to\infty}\max_{\mathcal{C}}\frac{1}{n}\log e(\mathcal{C},W) \tag{6}$$

$$E^*_{m}(R_X,R_Y)\triangleq\lim_{n\to\infty}\max_{\mathcal{C}}\frac{1}{n}\log e_m(\mathcal{C},W), \tag{7}$$

where the maximum is over all codes of length n and rate pair (R_X,R_Y).

Definition 3. *A code $\mathcal{C}_X=\{\boldsymbol{x}_i\in T_{P_X}:\ \ i=1,...,M_X\}$, for some P_X, is called a bad codebook, if*

$$\exists\,(i,j),\quad i\neq j\qquad \boldsymbol{x}_i=\boldsymbol{x}_j \tag{8}$$

A codebook which is not bad, is called a good one.

Definition 4. *A multi user code $\mathcal{C}=\mathcal{C}_X\times\mathcal{C}_Y$ is called a good multi user code, if both individual codebooks $\mathcal{C}_X$, $\mathcal{C}_Y$ are good codes.*

Definition 5. *For a good multi user code $\mathcal{C}=\mathcal{C}_X\times\mathcal{C}_Y$, and for a particular type $P_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})$, we define*

$$R(\mathcal{C},P_{XY})\triangleq\frac{1}{n}\log|\mathcal{C}\cap T_{P_{XY}}| \tag{9}$$

For a specified channel W, the Bhattacharyya distance between the channel input letter pairs (x,y), and $(\tilde{x},\tilde{y})$ is defined by

$$d_B\big((x,y),(\tilde{x},\tilde{y})\big)\triangleq-\log\Big(\sum_{z\in\mathcal{Z}}\sqrt{W(z|x,y)W(z|\tilde{x},\tilde{y})}\Big)$$

In this paper, we assume $d_B\big((x,y),(\tilde{x},\tilde{y})\big)\neq\infty$ for all (x,y) and $(\tilde{x},\tilde{y})$. A channel with this property is call as an *indivisible channel*. An indivisible channels for which the matrix $A_{(i,j),(k,l)}=2^{sd_B\big((i,j),(k,l)\big)}$ is nonnegative-definite for all $s>0$ is called a *nonnegative-definite* channel.

For a block channel W^n, the normalized Bhattacharyya distance between two channel input block pairs $(\mathbf{x},\mathbf{y})$, and $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is given by:

$$d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)=-\frac{1}{n}\log\Big(\sum_{\mathbf{z}\in\mathcal{Z}^n}\sqrt{W(\mathbf{z}|\mathbf{x},\mathbf{y})W(\mathbf{z}|\tilde{\mathbf{x}},\tilde{\mathbf{y}})}\Big)$$

If W is a memoryless channel, it can be easily shown that the Bhattacharyya distance between two pairs of codewords $(\mathbf{x},\mathbf{y})$ and $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, with joint empirical density $P_{XY\tilde{X}\tilde{Y}}$, is

$$d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)=\sum_{\substack{x,\tilde{x}\in\mathcal{X}\\ y,\tilde{y}\in\mathcal{Y}}}P_{XY\tilde{X}\tilde{Y}}(x,y,\tilde{x},\tilde{y})d_B\big((x,y),(\tilde{x},\tilde{y})\big)$$

As we see here, for a fixed channel, the Bhattacharyya distance between two pairs of words depends only on their joint composition. The minimum Bhattacharyya distance for a code $\mathcal{C}$ is defined as:

$$d_B(\mathcal{C})\triangleq\min_{\substack{(\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in\mathcal{C}\\ (\mathbf{x},\mathbf{y})\neq(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}}d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big). \tag{10}$$

Let us define

$$d^*_B\big(R_X,R_Y,n\big)\triangleq\max_{\mathcal{C}}d_B(\mathcal{C}) \tag{11}$$

Where the maximum is over all good codes of rate (R_X,R_Y), and blocklength n. Finally, we define

$$d^*_B(R_X,R_Y)\triangleq\lim_{n\to\infty}d^*_B\big(R_X,R_Y,n\big) \tag{12}$$

Note that, since any bad code has at least two identical codewords, we can conclude that the minimum distance of the code is equal to zero. Therefore, in order to find an upper bound for the best possible minimum distance, $d^*_B(R_X,R_Y)$, we only need to consider good codes (codes without any repetitions).

Now, consider any joint type $P_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})$. Using the structure of Bhattacharyya distance function, we can define spheres in $T_{P_{XY}}$. For any $(\mathbf{x},\mathbf{y})\in T_{P_{XY}}$, the sphere about $(\mathbf{x},\mathbf{y})$, of radius r, is given by

$$S\triangleq\{(\tilde{\mathbf{x}},\tilde{\mathbf{y}}):d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)\leq r\}$$

Every point, $(\mathbf{x},\mathbf{y})\in T_{P_{XY}}$, is surrounded by a set consisting of all pairs with which it shares some given joint type $V_{XY\tilde{X}\tilde{Y}}$. Basically, any pair of sequences, $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in T_{P_{XY}}$, sharing a common joint type with some given pair of sequences, $(\mathbf{x},\mathbf{y})\in T_{P_{XY}}$, belongs to the surface of a sphere with center $(\mathbf{x},\mathbf{y})$ and radius $r=d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)$. The set of these pairs is called a spherical collection about $(\mathbf{x},\mathbf{y})$ defined by $P_{\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}}}$.

III. Minimum Distance Upper Bound

Suppose the number of messages of the first source is $M_X=2^{nR_X}$, and the number of messages of the second source is $M_Y=2^{nR_Y}$. Suppose all the messages of any source are equiprobable and the sources are sending data independently. With these assumptions, all M_XM_Y pairs are occuring with the same probability. Thus, at the input of the channel, we can see all possible $2^{n(R_X+R_Y)}$ (an exponentially increasing function of n) pairs of input sequences. However, we also know that the number of possible types is a polynomial function of n. Thus, for at least one joint type, the number of pairs of sequences in the multi user code which have that particular type, should be an exponential function of n with the rate arbitrary close to the rate of the multi user code. We will look at these pairs of sequences as a subcode, and then try to find an upper bound for the minimum distance of this subcode. Clearly, this bound is still a valid upper bound for the minimum distance of the original multi user code.

Lemma 1. *For any $\delta>0$, and for any good multi user code $\mathcal{C}$ with rate pair (R_X,R_Y), as defined above, there exists $P_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})$ such that*

$$R(\mathcal{C},P_{XY})\geq R_X+R_Y-\delta \qquad \textit{for sufficiently large } n$$

Proof: The proof is provided in a more complete version [11]. ∎

Definition 6. *For a sequence of joint types $P^n_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})$, with marginal types P^n_X and P^n_Y, the sequence of type graphs,*

G_n, *is defined as follows. For every* n, G_n *is a bipartite graph, with its left vertices consisting of all* $x^n \in T_{P_X^n}$ *and the right vertices consisting of all* $y^n \in T_{P_Y^n}$. *A vertex on the left (say* $\tilde{x}^n$*) is connected to a vertex on the right (say* $\tilde{y}^n$*) if and only if* $(\tilde{x}^n, \tilde{y}^n) \in T_{P_{XY}^n}$.

Lemma 2. *For all sequences of nearly complete subgraphs of a particular type graph* $T_{P_{XY}}$, *the rates of the subgraph* (R_X, R_Y) *must satisfy*

$$R_X \leq H(X|U),\ R_Y \leq H(Y|U) \tag{13}$$

for some $P_{U|XY}$ *such that* $X - U - Y$.

Proof: The proof is provided in a more complete version [11]. ■

Consider any multiuser codebook $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ with dominant type P_{XY}. Consider any joint composition $V_{XY\tilde{X}\tilde{Y}}$ with marginal distributions $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$. In the following lemma, we find the average number of pairs of codewords in a spherical collection defined by joint type $V_{XY\tilde{X}\tilde{Y}}$ about an arbitrary pair of sequences $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$. For such $(\mathbf{x}, \mathbf{y})$, which is not necessarily a pair of codewords, let us define the following sets:

- $A_X(\mathbf{x}, \mathbf{y}) \triangleq \{(\mathbf{x}, \tilde{\mathbf{y}}) \in \mathcal{C} : (\mathbf{x}, \mathbf{y}, \mathbf{x}, \tilde{\mathbf{y}}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$
- $A_Y(\mathbf{x}, \mathbf{y}) \triangleq \{(\tilde{\mathbf{x}}, \mathbf{y}) \in \mathcal{C} : (\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \mathbf{y}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$
- $A_{XY}(\mathbf{x}, \mathbf{y}) \triangleq \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathcal{C} : (\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$

Note that, if $\mathbf{x} \notin \mathcal{C}_X$ or $X \neq \tilde{X}$, the first set would be empty. Similarly, if $\mathbf{y} \notin \mathcal{C}_Y$ or $Y \neq \tilde{Y}$, the second one would be an empty set.

Lemma 3. *Consider the multi-user code,* $\mathcal{C}$, *described above with dominant joint type* P_{XY}. *Additionally, consider any distribution* $V_{XY\tilde{X}\tilde{Y}} \in \mathcal{P}\big((\mathcal{X} \times \mathcal{Y})^2\big)$, *satisfying* $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$. *Then, there exists a pair of sequences* $(\boldsymbol{x}, \boldsymbol{y}) \in T_{P_{XY}}$ *such that*

$$|A_{XY}(\boldsymbol{x}, \boldsymbol{y})| \geq \exp\{n[R_X + R_Y - I(\tilde{X}\tilde{Y} \wedge XY)]\}. \tag{14}$$

Also, for any distribution $V_{XY\tilde{X}} \in \mathcal{P}\big(\mathcal{X} \times \mathcal{Y} \times \mathcal{X}\big)$ *satisfying* $V_{XY} = V_{\tilde{X}Y} = P_{XY}$, *and any* $\boldsymbol{y} \in \mathcal{C}_Y \cap T_{P_Y}$, *there exists some* $\boldsymbol{x} \in T_{P_X}$, *such that* $(\boldsymbol{x}, \boldsymbol{y}) \in T_{P_{XY}}$, *and*

$$|A_Y(\boldsymbol{x}, \boldsymbol{y})| \geq \exp\{n[R_X - I(\tilde{X} \wedge X|Y)]\},. \tag{15}$$

Similarly, for any distribution $V_{XY\tilde{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y})$ *satisfying* $V_{XY} = V_{X\tilde{Y}} = P_{XY}$, *and any* $\boldsymbol{x} \in \mathcal{C}_X \cap T_{P_X}$, *there exist some* $\boldsymbol{y} \in T_{P_Y}$ *such that* $(\boldsymbol{x}, \boldsymbol{y}) \in T_{P_{XY}}$, *and*

$$|A_X(\boldsymbol{x}, \boldsymbol{y})| \geq \exp\{n[R_Y - I(\tilde{Y} \wedge Y|X)]\}. \tag{16}$$

Proof: The proof is provided in a more complete version [11]. ■

Lemma 4. *Fix* $\epsilon > 0$. *Let* W *be a nonnegative-definite channel. Let* $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ *be any multi-user code with dominant composition* nP_{XY} *and rate pair* (R_X, R_Y). *Consider any distribution* $V_{XY\tilde{X}\tilde{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y})$ *satisfying the following constraints:*

- $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$
- $I_V(XY \wedge \tilde{X}\tilde{Y}) \leq R_X + R_Y - \epsilon$.

Then, $\mathcal{C}$ *has two pairs of codewords,* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ *and* $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$, *such that*

$$d_B\big((\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})\big) \leq (1 + \epsilon) E d_B\big((\tilde{X}, \tilde{Y}), (\hat{X}, \hat{Y})\big) \tag{17}$$

where the expectation is calculated based on $V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \in \mathcal{P}((\mathcal{X} \times \mathcal{Y})^3)$ *satisfying*

- $V_{XY} = V_{\tilde{X}\tilde{Y}} = V_{\hat{X}\hat{Y}} = P_{XY}$
- $\tilde{X}\tilde{Y} - XY - \hat{X}\hat{Y}$
- $V_{\tilde{X}\tilde{Y}|XY} = V_{\hat{X}\hat{Y}|XY}$
- $I_V(XY \wedge \tilde{X}\tilde{Y}) \leq R_X + R_Y - \epsilon$.

For any $V_{XY\tilde{X}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X})$ *satisfying the following constraints:*

- $V_{XY} = V_{\tilde{X}Y} = P_{XY}$
- $I_V(X \wedge \tilde{X}|Y) \leq R_X - \epsilon$.

$\mathcal{C}$ *has two pairs of codewords,* $(\tilde{\boldsymbol{x}}, \boldsymbol{y})$ *and* $(\hat{\boldsymbol{x}}, \boldsymbol{y})$, *such that*

$$d_B\big((\tilde{\boldsymbol{x}}, \boldsymbol{y}), (\hat{\boldsymbol{x}}, \boldsymbol{y})\big) \leq (1 + \epsilon) E d_B\big((\tilde{X}, Y), (\hat{X}, Y)\big) \tag{18}$$

where the expectation is calculated based on $V_{XY\tilde{X}\hat{X}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{X})$ *satisfying*

- $V_{XY} = V_{\tilde{X}Y} = V_{\hat{X}Y} = P_{UXY}$
- $\tilde{X} - XY - \hat{X}$
- $V_{\tilde{X}|XY} = V_{\hat{X}|XY}$
- $I_V(X \wedge \tilde{X}|Y) \leq R_X - \epsilon$.

Similarly, for any $V_{XY\tilde{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y})$ *satisfying the following constraints:*

- $V_{XY} = V_{\tilde{Y}} = P_{XY}$
- $I_V(Y \wedge \tilde{Y}|X) \leq R_Y - \epsilon$.

$\mathcal{C}$ *has two pairs of codewords,* $(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ *and* $(\boldsymbol{x}, \hat{\boldsymbol{y}})$, *such that*

$$d_B\big((\boldsymbol{x}, \tilde{\boldsymbol{y}}), (\boldsymbol{x}, \hat{\boldsymbol{y}})\big) \leq (1 + \epsilon) E d_B\big((X, \tilde{Y}), (X, \hat{Y})\big) \tag{19}$$

where the expectation is calculated based on $V_{XY\tilde{Y}\hat{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})$ *satisfying*

- $V_{XY} = V_{X\tilde{Y}} = V_{X\hat{Y}} = P_{XY}$
- $\tilde{Y} - XY - \hat{Y}$
- $V_{\tilde{Y}|XY} = V_{\hat{Y}|XY}$
- $I_V(Y \wedge \tilde{Y}|X) \leq R_Y - \epsilon$.

Proof: The proof is provided in a more complete version [11]. ■

Noting that the minimum distance between codeword pairs in $\mathcal{C}$ is smaller than the minimum distance of any subset of $\mathcal{C}$, we conclude the following result.

Theorem 1. *For any nonnegative-definite channel,* W, *the minimum distance of any multiuser code,* $\mathcal{C}$, *with rate pair* (R_X, R_Y) *satisfies*

$$d_B(\mathcal{C}) \leq E_U(R_X, R_Y, W) \tag{20}$$

where $E_U(R_X, R_Y, W)$ *is defined as*

$$\max_{P_{UXY}} \min_{\beta = X, Y, XY} E_U^{\beta}(R_X, R_Y, W, P_{XYU}) \tag{21}$$

The maximum is taken over all $P_{UXY} \in \mathcal{P}(\mathcal{U}\times\mathcal{X}\times\mathcal{Y})$ such that $X - U - Y$, and $R_X \le H(X|U)$ and $R_Y \le H(Y|U)$. The functions $E_U^{\beta}(R_X, R_Y, W, P_{XYU})$ are defined as follows:

$$E_U^X(R_X,R_Y,W,P_{XYU}) \triangleq \min_{V_{X\tilde{X}\hat{X}Y}\in\mathcal{V}_X^U} Ed_W\big((\hat{X},Y),(\tilde{X},Y)\big)$$

$$E_U^Y(R_X,R_Y,W,P_{XYU}) \triangleq \min_{V_{XY\tilde{Y}\hat{Y}}\in\mathcal{V}_Y^U} Ed_W\big((X,\hat{Y}),(X,\tilde{Y})\big)$$

$$E_U^{XY}(R_X,R_Y,W,P_{XYU}) \triangleq \min_{\substack{V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}}\\ \in\mathcal{V}_{XY}^U}} Ed_W\big((\hat{X},\hat{Y}),(\tilde{X},\tilde{Y})\big)$$

where

$$\begin{aligned}\mathcal{V}_X^U \triangleq \{&V_{X\tilde{X}\hat{X}Y} : V_{\tilde{X}Y} = V_{\hat{X}Y} = V_{XY} = P_{XY}\\ &\hat{X} - XY - \tilde{X}\\ &V_{\tilde{X}|XY} = V_{\hat{X}|XY}\\ &I(X\wedge\tilde{X}|Y) = I(X\wedge\hat{X}|Y) \le R_X\}\end{aligned} \tag{22}$$

$$\begin{aligned}\mathcal{V}_Y^U \triangleq \{&V_{XY\tilde{Y}\hat{Y}} : V_{X\tilde{Y}} = V_{X\hat{Y}} = V_{XY} = P_{XY}\\ &\hat{Y} - XY - \tilde{Y}\\ &V_{\tilde{Y}|XY} = V_{\hat{Y}|XY}\\ &I(Y\wedge\tilde{Y}|X) = I(Y\wedge\hat{Y}|X) \le R_Y\}\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{V}_{XY}^U \triangleq \{&V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} : V_{\tilde{X}\tilde{Y}} = V_{\hat{X}\hat{Y}} = V_{XY} = P_{XY}\\ &\hat{X}\hat{Y} - XY - \tilde{X}\tilde{Y}\\ &V_{\tilde{X}\tilde{Y}|XY} = V_{\hat{X}\hat{Y}|XY}\\ &I(XY\wedge\tilde{X}\tilde{Y}) = I(XY\wedge\hat{X}\hat{Y}) \le R_X + R_Y\}\end{aligned} \tag{24}$$

Theorem 2. *For any indivisible channel*

$$E_m^*(R_X,R_Y) \le d_B^*(R_X,R_Y) \tag{25}$$

where $E_m^*(R_X,R_Y)$ is the maximal error channel-rate reliability function at rate pair (R_X,R_Y).

Proof: The proof is very similar to [12]. ■

Therefore, by combining the result of theorem 1 and theorem 2, we can conclude the following result.

Theorem 3. *For any indivisible nonnegative-definite channel, W, the maximal error reliability function, $E_m^*(R_X,R_Y)$, must satisfy*

$$E_m^*(R_X,R_Y) \le E_U(R_X,R_Y,W) \tag{26}$$

The following observation will be used in section IV and V to compare the lower bound on the average error reliability function with the upper bounds on the maximal error reliability function at $R_X = R_Y = 0$.

Lemma 5. *If $\min\{R_X,R_Y\} = 0$, i.e., $R_X = 0$ or $R_Y = 0$,*

$$E_m^*(R_X,R_Y) = E_{av}^*(R_X,R_Y) \tag{27}$$

IV. An Expurgated Lower Bound

Theorem 4. *For every $\delta > 0$, $R_X \ge 0$, $R_Y \ge 0$, every finite set $\mathcal{U}$, every type $\mathcal{P}_{XYU} \in \mathcal{P}_n(\mathcal{X}\times\mathcal{Y}\times\mathcal{U})$, $X - U - Y$, satisfying $H(X|U) \ge R_X$ and $H(Y|U) \ge R_Y$, and $\boldsymbol{u} \in T_{P_U}^n$, there exists a multi-user code*

$$\mathcal{C} = \{(\boldsymbol{x}_i,\boldsymbol{y}_j,D_{ij}) : i = 1,\dots M_X^*, j = 1,\dots M_Y^*\} \tag{28}$$

with $\boldsymbol{x}_i \in T_{P_{X|U}}(\boldsymbol{u})$, $\boldsymbol{y}_j \in T_{P_{Y|U}}(\boldsymbol{u})$ for all i and j, $M_X^* \ge 2^{n(R_X-\delta)}$, and $M_Y^* \ge 2^{n(R_Y-\delta)}$, such that for every MAC $W : \mathcal{X}\times\mathcal{Y}\to\mathcal{Z}$

$$e(\mathcal{C},W) \le 2^{-n[E_L(R_X,R_Y,W,P_{XYU})-\delta]} \tag{29}$$

whenever $n \ge n_1(|\mathcal{Z}|,|\mathcal{X}|,|\mathcal{Y}|,|\mathcal{U}|,\delta)$, where

$$E_L(R_X,R_Y,W,P_{XYU}) \triangleq \min_{\beta=X,Y,XY} E_L^{\beta}(R_X,R_Y,W,P_{XYU}) \tag{30}$$

and $E_L^{\beta}(R_X,R_Y,W,P_{XYU})$, $\beta = X,Y,XY$ are defined respectively by

$$\begin{aligned}&E_L^X(R_X,R_Y,W,P_{XYU}) \triangleq\\ &\min_{V_{UXY\tilde{X}}\in\mathcal{V}_X} Ed_W\big((X,Y),(\tilde{X},Y)\big) + I_V(X\wedge Y|U)\\ &+ I(\tilde{X}\wedge X|YU) + I_V(\tilde{X}\wedge Y|U) - R_X\end{aligned} \tag{31}$$

$$\begin{aligned}&E_L^Y(R_X,R_Y,W,P_{XYU}) \triangleq\\ &\min_{V_{UXY\tilde{Y}}\in\mathcal{V}_Y} Ed_W\big((X,Y),(X,\tilde{Y})\big) + I_V(X\wedge Y|U)\\ &+ I(\tilde{Y}\wedge Y|XU) + I_V(X\wedge\tilde{Y}|U) - R_Y\end{aligned} \tag{32}$$

$$\begin{aligned}&E_L^{XY}(R_X,R_Y,W,P_{XYU}) \triangleq\\ &\min_{V_{UXY\tilde{X}\tilde{Y}}\in\mathcal{V}_{XY}} Ed_W\big((X,Y),(\tilde{X},\tilde{Y})\big) + I_V(X\wedge Y|U)\\ &+ I(\tilde{X}\tilde{Y}\wedge XY|U) + I_V(\tilde{X}\wedge\tilde{Y}|U) - R_X - R_Y\end{aligned} \tag{33}$$

where

$$\begin{aligned}&\mathcal{V}_X \triangleq \{V_{UXY\tilde{X}} : V_{XU} = V_{\tilde{X}U} = P_{XU}, V_{YU} = P_{YU}\\ &I_V(X\wedge Y|U), I_V(\tilde{X}\wedge Y|U) \le \min\{R_X,R_Y\} + 3\delta\\ &I_V(X\wedge Y|U) + I_V(\tilde{X}\wedge Y|U) + I_V(\tilde{X}\wedge X|UY)\\ &\qquad\le R_X + \min\{R_X,R_Y\} + 4\delta\}\end{aligned} \tag{34}$$

$$\begin{aligned}&\mathcal{V}_Y \triangleq \{V_{UXY\tilde{Y}} : V_{XU} = P_{XU}, V_{YU} = V_{\tilde{Y}U} = P_{YU}\\ &I_V(X\wedge Y|U), I_V(X\wedge\tilde{Y}|U) \le \min\{R_X,R_Y\} + 3\delta\\ &I_V(X\wedge Y|U) + I_V(X\wedge\tilde{Y}|U) + I_V(\tilde{Y}\wedge Y|UX)\\ &\qquad\le R_Y + \min\{R_X,R_Y\} + 4\delta\}\end{aligned} \tag{35}$$

$$\begin{aligned}&\mathcal{V}_{XY} \triangleq \{V_{UXY\tilde{X}\tilde{Y}} :\\ &V_{UXY\tilde{X}} \text{ satisfies all conditions in (34)}\\ &V_{UXY\tilde{Y}} \text{ satisfies all conditions in (35)}\\ &I_V(X\wedge\tilde{Y}|U) + I_V(\tilde{X}\wedge\tilde{Y}|U) + I_V(\tilde{X}\wedge X|U\tilde{Y})\\ &\qquad\le R_X + \min\{R_X,R_Y\} + 4\delta\\ &I_V(\tilde{X}\wedge Y|U) + I_V(\tilde{X}\wedge\tilde{Y}|U) + I_V(\tilde{Y}\wedge Y|U\tilde{X})\\ &\qquad\le R_Y + \min\{R_X,R_Y\} + 4\delta\\ &I_V(X\wedge Y|U) + I_V(\tilde{X}\wedge\tilde{Y}|U) + I_V(\tilde{X}\tilde{Y}\wedge XY|U)\\ &\qquad\le R_X + R_Y + \min\{R_X,R_Y\} + 5\delta\\ &I_V(\tilde{X}\wedge Y|U) + I_V(X\wedge\tilde{Y}|U) + I_V(X\tilde{Y}\wedge\tilde{X}Y|U)\\ &\qquad\le R_X + R_Y + \min\{R_X,R_Y\} + 4\delta\}\end{aligned} \tag{36}$$

Proof: Using the code whose existence is asserted in [8, lemma 3], we can conclude the result [11]. ■

Let us focus on the case where both codebooks have rate zero, $R_X = R_Y = 0$. One can easily show that,

$$E_L^X(0,0,P_{XYU}) = Ed_W\big((X,Y),(\tilde{X},Y)\big) \tag{37}$$
$$E_L^Y(0,0,P_{XYU}) = Ed_W\big((X,Y),(X,\tilde{Y})\big) \tag{38}$$
$$E_L^{XY}(0,0,P_{XYU}) = Ed_W\big((X,Y),(\tilde{X},\tilde{Y})\big) \tag{39}$$

where all the expectations in (37)-(39) are calculated based on

$$\begin{aligned} &P_{UXY\tilde{X}\tilde{Y}}(u,x,y,\tilde{x},\tilde{y}) \\ &= P_U(u)P_{X|U}(x|u)P_{Y|U}(y|u)P_{X|U}(\tilde{x}|u)P_{Y|U}(\tilde{y}|u). \end{aligned} \tag{40}$$

Similarly, at zero rate, E_U^X, E_U^Y, and E_U^{XY} would be equal to

$$E_U^X(0,0,P_{XYU}) = Ed_W\big((\hat{X},Y),(\tilde{X},Y)\big) \tag{41}$$
$$E_U^Y(0,0,P_{XYU}) = Ed_W\big((X,\hat{Y}),(X,\tilde{Y})\big) \tag{42}$$
$$E_U^{XY}(0,0,P_{XYU}) = Ed_W\big((\hat{X},\hat{Y}),(\tilde{X},\tilde{Y})\big) \tag{43}$$

where all the expectations in (41)- (43) are respectively calculated based on

$$P_{\hat{X}\tilde{X}Y}(\hat{x},\tilde{x},y) = P_{X|Y}(\hat{x}|y)P_{X|Y}(\tilde{x}|y)P_Y(y) \tag{44}$$
$$P_{X\hat{Y}\tilde{Y}}(x,\hat{y},\tilde{y}) = P_X(x)P_{Y|X}(\hat{y}|x)P_{Y|X}(\tilde{y}|x) \tag{45}$$
$$P_{\hat{X}\hat{Y}\tilde{X}\tilde{Y}}(\hat{x},\hat{y},\tilde{x},\tilde{y}) = P_{XY}(\hat{x},\hat{y})P_{XY}(\tilde{x},\tilde{y}). \tag{46}$$

V. A Conjectured Tighter Upper Bound

Conjecture 1. *For all sequences of nearly complete subgraphs of a particular type graph $T_{P_{XY}}$, the rates of the subgraph (R_X, R_Y) satisfy*

$$R_X \le H(X|U),\ R_Y \le H(Y|U) \tag{47}$$

for some $P_{U|XY}$ such that $X - U - Y$. Moreover, there exists $\boldsymbol{u} \in T_{P_U}$ such that the intersection of the fully connected subgraph with $T_{P_{XY|U}}(\boldsymbol{u})$ has the rate (R_X, R_Y).

Based on the result of previous lemma, and by following a similar argument as we did in lemma 3 and lemma 4, we can conclude the following result:

Theorem 5. *For any nonnegative-definite channel, W, the minimum distance of any multiuser code, $\mathcal{C}$, with rate pair (R_X, R_Y) satisfies*

$$d_B(\mathcal{C}) \le E_C(R_X, R_Y, W) \tag{48}$$

where $E_C(R_X, R_Y, W)$ is defined as

$$\max_{P_{UXY}} \min_{\beta=X,Y,XY} E_C^{\beta}(R_X, R_Y, W, P_{XYU}) \tag{49}$$

The maximum is taken over all $P_{UXY} \in \mathcal{P}(\mathcal{U}\times\mathcal{X}\times\mathcal{Y})$ such that $X - U - Y$, and $R_X \le H(X|U)$ and $R_Y \le H(Y|U)$. The functions $E_C^{\beta}(R_X, R_Y, W, P_{XYU})$ are defined as follows:

$$E_C^X(R_X,R_Y,W,P_{XYU}) \triangleq \min_{V_{UX\tilde{X}\hat{X}Y}\in\mathcal{V}_X^C} Ed_W\big((\hat{X},Y),(\tilde{X},Y)\big)$$
$$E_C^Y(R_X,R_Y,W,P_{XYU}) \triangleq \min_{V_{UXY\tilde{Y}\hat{Y}}\in\mathcal{V}_Y^C} Ed_W\big((X,\hat{Y}),(X,\tilde{Y})\big)$$
$$E_C^{XY}(R_X,R_Y,W,P_{XYU}) \triangleq \min_{\substack{V_{UXY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \\ \in\mathcal{V}_{XY}^C}} Ed_W\big((\hat{X},\hat{Y}),(\tilde{X},\tilde{Y})\big)$$

where

$$\begin{aligned} \mathcal{V}_X^C \triangleq \{&V_{UX\tilde{X}\hat{X}Y} : V_{U\tilde{X}Y} = V_{U\hat{X}Y} = V_{UXY} = P_{UXY} \\ &\hat{X} - UXY - \tilde{X} \\ &V_{\tilde{X}|XYU} = V_{\hat{X}|XYU} \\ &I(X\wedge\tilde{X}|YU) = I(X\wedge\hat{X}|YU) \le R_X\} \end{aligned} \tag{50}$$

$$\begin{aligned} \mathcal{V}_Y^C \triangleq \{&V_{UXY\tilde{Y}\hat{Y}} : V_{UX\tilde{Y}} = V_{UX\hat{Y}} = V_{UXY} = P_{UXY} \\ &\hat{Y} - UXY - \tilde{Y} \\ &V_{\tilde{Y}|XYU} = V_{\hat{Y}|XYU} \\ &I(Y\wedge\tilde{Y}|UX) = I(Y\wedge\hat{Y}|UX) \le R_Y\} \end{aligned} \tag{51}$$

$$\begin{aligned} \mathcal{V}_{XY}^C \triangleq \{&V_{UXY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} : V_{U\tilde{X}\tilde{Y}} = V_{U\hat{X}\hat{Y}} = V_{UXY} = P_{UXY} \\ &\hat{X}\hat{Y} - UXY - \tilde{X}\tilde{Y} \\ &V_{\tilde{X}\tilde{Y}|UXY} = V_{\hat{X}\hat{Y}|UXY} \\ &I(XY\wedge\tilde{X}\tilde{Y}|U) = I(XY\wedge\hat{X}\hat{Y}|U) \le R_X + R_Y\} \end{aligned} \tag{52}$$

Proof: The proof is provided in a more complete version [11]. ∎

Theorem 6. *At rate $R_X = R_Y = 0$,*

$$E_C^{\beta}(0,0,P_{XYU}) = E_L^{\beta}(0,0,P_{XYU}) \tag{53}$$

for $\beta = X, Y, XY$, and therefore $E_C = E_L$.

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