Diversity Gain Regions for MIMO Fading Broadcast and Multiple Access Channels

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Abstract

In wireless communication systems, users with heterogeneous information content such as data, voice, and multimedia constrain the network by having different reliability requirements. In this paper an information-theoretic framework is proposed to study communication systems which provide heterogeneous reliabilities for the users. This is done by defining individual probabilities of error for the users in the network and obtaining their fundamental tradeoffs. Using this framework, a system can be realized, which can provide a tradeoff of reliabilities among the users for a fixed vector of users' rates. This adds a completely new dimension to the performance tradeoff in such networks, which is unique to multi-terminal communication systems, and is beyond what is given by the conventional performance versus rate tradeoff in single-user systems.

Although this is a very general concept and can be applied to any multi-terminal communication system, in this paper we consider multiple-input multiple-output (MIMO) fading broadcast and multiple access channels. In particular, we quantify the reliability tradeoff by introducing the notion of *diversity gain region* (DGR), which specifies the set of diversity gain vectors that are simultaneously achievable by the users for a fixed vector of users' multiplexing gains. We show the existence of a tradeoff among the users' diversity gains by deriving inner and outer bounds for the DGR.

Index Terms

Error exponent region, diversity gain region, multiplexing gain region.

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I. INTRODUCTION

Seamless, ubiquitous, and personal communication has become one of the essential services of the information society of ours. One of the crucial factors facilitating this untethered freedom of communication is wireless technology. This is supported by a multitude of network architectures such as mobile cellular networks, fixed access wireless networks, wireless indoor/outdoor local area networks and digital broadcast networks. At the same time, the heterogeneity of the networks in terms of performance measures such as bandwidth, data rate, quality-of-service (QoS), and content such as data, voice, multimedia, have grown tremendously in the last decade [1], [2], [3]. In addition, the heterogeneity in the user interfaces such as PDA, cell-phone cameras, wireless video phones, etc., have also grown steadily over the years.

Currently, in most of the wireless systems, all types of information are handled in the same fashion. But one of the important differences between data and multimedia is the ability of the latter to gracefully degrade in response to the effects of the channel impairments, whereas the former is much more rigid in terms of channel reliability requirements. As an illustration consider an uplink communication in a mobile cellular system with two users transmitting some information to the base station. This communication channel can be modeled as a multiple access channel (MAC) with the two users sharing the common interfering channel. Assume one of the users (the first) is transmitting data (e.g., running an FTP application) and the other (the second) is transmitting some multimedia (e.g., uploading a video stream from the cell-phone video camera). Let us also assume that their allocated data rates are the same in the system. The information content of these two users are very different. While the first user requires better reliability for data, the latter can easily cope up with channel degradation because of the nature of its content. A system which treats both of these users in an identical manner, i.e., a system which tries to guarantee the reliability requested by the most demanding user (the user having data to transfer) to all the users, is not optimal. That is so because the second user does not ask for better reliability and this will result in a mismatched resource allocation.

In order to see why a new framework is required to address this resource allocation problem, we need to understand the fundamental limits of achievable reliabilities in a point-to-point and a multi-user scenarios. In a point-to-point single-user channel, if the data rate of the user is small, then that user can be provided with better reliability, whereas at high data rates reliability is

compromised. As we move the data rate R away from the capacity C , the probability of error can be made to decay faster with block length. This reliability-rate tradeoff has been quantified by the error exponent [4], [5], [6], [7], [8], [9], which is also known as the reliability function. A straightforward extension of this concept can be realized in the multi-user setting by first defining the *probability of system error*. A system is considered to be in error if at least one user's codeword is decoded erroneously. For the study of the channel capacity of a multi-user channel, it is sufficient to show that this single performance measure, probability of system error, approaches zero as the block length approaches infinity. As expected, the probability of system error in a multi-user channel provides a reliability-rate tradeoff similar to the case in a point-to-point single-user channel. If the users are operating close to the boundary of the capacity region, then the probability of system error decays slowly as a function of the codeword length. Similarly, if the users are operating away from the capacity boundary, then the probability of system error decays faster with codeword length. Thus, one can quantify this tradeoff by defining the error exponent for a multi-user channel as the rate of exponential decay of the probability of system error [10]. This approach, however, does not solve the problem of mismatched resource allocation of the MACs that was mentioned earlier. That is, using the above idea, all the users in the system get the same reliability determined by the aforementioned probability of system error. Thus, the only way to guarantee the QoS requirement for the most demanding (first) user is to reduce the users' data rates (either individually or as a group) in order to meet the common QoS requirement.

The question to ask is the following: is it possible to simultaneously provide a better reliability for the first user and a reduced reliability for the second user, while keeping their rates the same in a MAC? More generally, for a fixed pair of data rates for the two users, is it possible to provide a set of choices of individual reliabilities for these two users? The main contribution of this paper is to provide a positive answer to these questions by formalizing these ideas in the context of information theory, studying the fundamental limits of such tradeoffs of individual reliabilities among the users for a fixed vector of data rates, and developing efficient transmission strategies that approach these limits. This is done by defining individual error probabilities for each user and studying the tradeoff of the corresponding error exponents. This tradeoff is quantified by introducing the concept of *error exponent region* (EER) of a multi-user channel. Although the idea proposed in this paper is very general, we present it in the context of multiple-input multiple-

output (MIMO) wireless fading multi-user channels (see [11], [12, Chap. 14]). Some earlier works related to the results in this paper are the work by Marton and Sgarro [13] which considers a broadcast channel with degraded message sets, and the work by Diggavi *et al.* [14], which considers a single-user channel with two different messages, i.e., a high- and a low-reliability message.

The rest of the paper is structured as follows. In Section II, we provide an overview of the reliability-rate tradeoff for single-user and multi-user channels as studied in the literature. The notion of error exponent region is introduced in Section III. We derive inner and outer bounds for these performance limits for MIMO fading broadcast channels in Section IV and for MIMO fading multiple access channels in Section V. We conclude our work in Section VI.

The following notation is used throughout this work. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the sets of real numbers, complex numbers, rational numbers and integers, respectively. $\mathbb{R}_+^n$ is the set of real n -vectors with nonnegative elements, and $(x)^+ \triangleq \max(x, 0)$, i.e., $(x)^+$ is defined as $\max(x, 0)$. We use boldface to denote a random variable $\mathbf{X}$, and lightface to denote a realization X of the random variable $\mathbf{X}$. The calligraphic letters $\mathcal{A}$, $\mathcal{B}$, etc., denote general sets or probability events. The abbreviation “i.i.d” stands for independent and identically distributed. The zero-mean, unit-variance, (real) Gaussian distribution is denoted by $\mathcal{N}(0, 1)$ and the zero-mean, unit-variance, circular symmetric, complex Gaussian distribution is denoted by $\mathcal{CN}(0, 1)$. We don’t distinguish between a scalar and a matrix in our notation.

II. BACKGROUND: ERROR EXPONENT AND DIVERSITY GAIN

It is well-known that the error exponent for a single-user channel provides the rate of exponential decay of the average probability of error as a function of the block length of the codebooks [15]. The concept of the error exponent was extended to a Gaussian MAC in [10], [16], where an upper bound on the *probability of system error* (i.e., the probability that any user is in error) was derived for random codes. Recently, Zheng and Tse considered error exponents under a high signal-to-noise ratio (SNR) approximation, called diversity gains, for MIMO fading single-user channels [17], and for MIMO fading MACs [18]. In the following, we briefly review some results regarding error exponents and diversity gains for single-user channels and MACs.

A. Error Exponent

Consider a discrete-time memoryless single-user channel. Let $P_e(N, R)$ denote the smallest probability of block decoding error, i.e., codeword error, of any code of block length N and rate R in this channel. The error exponent at rate R is defined as

$$E(R) \triangleq \lim_{N \rightarrow \infty} -\frac{\log P_e(N, R)}{N}, \quad (1)$$

where the limit in (1) (and throughout this paper) should be interpreted as $\limsup$ whenever the limit does not exist. The probability of error $P_e(N, R)$ can also be written as

$$P_e(N, R) = e^{-NE(R) + o(N)}, \quad (2)$$

where $o(N)$ is a term such that $\lim_{N \rightarrow \infty} \frac{o(N)}{N} = 0$.

Error exponents have been studied in detail for discrete memoryless channels and additive white Gaussian noise (AWGN) channels [7], [8], [9], [6]. Lower and upper bounds are known for the error exponent $E(R)$ in these channels. A lower bound, known as random coding exponent $E_r(R)$, was developed by Fano [7]. The random coding exponent is tightened at low rates by Gallager to yield the expurgated exponent $E_{ex}(R)$ [8]. An upper bound, known as the sphere packing exponent $E_{sp}(R)$ was developed by Shannon, Gallager, and Berlekamp [6], [9].

Error exponents have also been studied for MACs [10], [16], [18]. For a given MAC, let $P_{e,sys}(N, R_1, R_2)$ denote the smallest probability of block decoding system error of any code of block length N and rates R_1, R_2 for user 1, user 2, respectively. The error exponent for a MAC is defined as

$$E_{sys}(R_1, R_2) \triangleq \lim_{N \rightarrow \infty} -\frac{\log P_{e,sys}(N, R_1, R_2)}{N}. \quad (3)$$

In the following, we summarize the basic technique used by Gallager to provide an upper bound on the probability of system error in a MAC [10]. A variation of this method will be used later to provide similar upper bounds. Consider a codebook $CB_1 = \{C_{1,1}, C_{1,2}, \dots, C_{1,M_1}\}$ for user 1, where $C_{1,i}$ is the i^{th} codeword with length N ($1 \leq i \leq M_1$) and M_1 is the number of the codewords in the codebook CB_1 . Similarly, $CB_2 = \{C_{2,1}, C_{2,2}, \dots, C_{2,M_2}\}$ is a codebook for user 2. Gallager [10] derived the random coding exponent using *joint* maximum likelihood (ML) decoding, i.e., decoding users' messages based on the pair (i, j) maximizing $P(Y^N | C_{1,i}, C_{2,j})$,

where Y^N is the received sequence of length N . Denote $(\hat{i}, \hat{j})$ the indexes of the decoded codewords for user 1 and user 2. The probability of system error can be written as

$$\begin{aligned} P_{e,sys} &= Pr\{\hat{i} \neq i \text{ or } \hat{j} \neq j\} \\ &= Pr\{\hat{i} \neq i \text{ and } \hat{j} = j\} + Pr\{\hat{i} = i \text{ and } \hat{j} \neq j\} + Pr\{\hat{i} \neq i \text{ and } \hat{j} \neq j\} \\ &= P_{e,t1} + P_{e,t2} + P_{e,t3}, \end{aligned} \quad (4)$$

where we define

$$P_{e,t1} \triangleq Pr\{\hat{i} \neq i \text{ and } \hat{j} = j\} \quad (5a)$$

$$P_{e,t2} \triangleq Pr\{\hat{i} = i \text{ and } \hat{j} \neq j\} \quad (5b)$$

$$P_{e,t3} \triangleq Pr\{\hat{i} \neq i \text{ and } \hat{j} \neq j\}. \quad (5c)$$

Thus there are three types of error events while using joint ML decoding. Type 1 error occurs when user 1's codeword is decoded erroneously, but user 2's codeword is decoded correctly. Type 2 error occurs when user 2's codeword is decoded erroneously, but user 1's codeword is decoded correctly. Type 3 error occurs when both users' codewords are decoded as wrong codewords. Applying the random coding argument, it was shown in [10] that there exist codebooks CB_1 and CB_2 such that $P_{e,ti}$ can be upper bounded by

$$P_{e,t1} \leq e^{-NE_{t1}(R_1)} \quad (6a)$$

$$P_{e,t2} \leq e^{-NE_{t2}(R_2)} \quad (6b)$$

$$P_{e,t3} \leq e^{-NE_{t3}(R_1+R_2)}, \quad (6c)$$

where E_{ti} , $1 \leq i \leq 3$, is an exponent which accounts for type i error. The probability of system error can be upper bounded by

$$\begin{aligned} P_{e,sys} &= P_{e,t1} + P_{e,t2} + P_{e,t3} \\ &\leq e^{-NE_{t1}(R_1)} + e^{-NE_{t2}(R_2)} + e^{-NE_{t3}(R_1+R_2)} \\ &\leq 3e^{-N \min\{E_{t1}(R_1), E_{t2}(R_2), E_{t3}(R_1+R_2)\}}, \end{aligned} \quad (7)$$

and the system error exponent can be lower bounded by

$$E_{sys}(R_1, R_2) \geq \min\{E_{t1}(R_1), E_{t2}(R_2), E_{t3}(R_1 + R_2)\}. \quad (8)$$

B. Diversity Gain

Error exponents have also been studied for MIMO fading single-user channels in the high SNR regime [17]. Consider a MIMO fading single-user channel with m transmit antennas and n receive antennas. The channel model is

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}} \mathbf{H} \mathbf{X} + \mathbf{Z}. \quad (9)$$

The channel fading matrix between the transmitter and the receiver is represented by an $n \times m$ matrix $\mathbf{H}$. We assume that $\mathbf{H}$ remains constant over a block with length l , and changes to a new independent realization in the next block. $\mathbf{H}$ has i.i.d. entries distributed as $\mathcal{CN}(0, 1)$. We assume that the fading matrix $\mathbf{H}$ is known by the receiver but not known by the transmitter. $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average SNR at each receive antenna is SNR . The noise $\mathbf{Z}$ is an $n \times l$ matrix with i.i.d. entries $\mathcal{CN}(0, 1)$. The channel output $\mathbf{Y}$ is an $n \times l$ matrix.

In contrast to a single-antenna system, multiple antennas provide spatial diversity, which can be used to support higher data rate than a single-antenna system. The channel capacity of a MIMO fading channel can be written as

$$C(SNR) = \min\{m, n\} \log SNR + O(1) \quad (10)$$

at high SNR [19]. In addition to supporting higher data rate, multi-antenna channels can also improve the reliability of the link. Note that the probability of error goes to zero as SNR goes to infinity and this implies that the error exponent goes to infinity as SNR goes to infinity. Nevertheless, we can define a “normalized” error exponent, with respect to SNR , for a multi-antenna channel. To be specific, let the channel be operated at a rate $R = R(SNR)$ which is a fraction of the channel capacity at high SNR, i.e.,

$$\lim_{SNR \rightarrow \infty} \frac{R(SNR)}{\log SNR} = r. \quad (11)$$

The normalized error exponent $e(r)$ (with respect to SNR) is thus defined as

$$e(r) \triangleq \lim_{SNR \rightarrow \infty} \frac{E(R(SNR))}{\log SNR} = \lim_{SNR \rightarrow \infty} \left\{ \lim_{N \rightarrow \infty} -\frac{\log P_e(N, R(SNR))}{N \log SNR} \right\}. \quad (12)$$

If we define $d(N, r)$ as

$$d(N, r) \triangleq \lim_{SNR \rightarrow \infty} -\frac{\log P_e(N, R(SNR))}{N \log SNR}, \quad (13)$$

and if we can exchange the order of $\lim_{SNR \rightarrow \infty}$ and $\lim_{N \rightarrow \infty}$ in (12), then the normalized error exponent $e(r) = \lim_{N \rightarrow \infty} d(N, r)$. Define $f \doteq SNR^b$ if

$$\lim_{SNR \rightarrow \infty} \frac{\log f}{\log SNR} = b, \quad (14)$$

and $\dot{\geq}, \dot{\leq}$ are defined similarly. Thus (13) can be written as $P_e(N, R(SNR)) \doteq SNR^{-Nd(N, r)}$.

It was shown in [17] that in a MIMO fading single-user channel with m transmit antennas, n receive antennas, and block length l , both the random coding diversity gain $d_{m,n,l}(r)$ and the expurgated diversity gain $d_{m,n,l}^{ex}(r)$ are lower bounds of $d(r) \triangleq d(1, r)$, where $d(r)$ is used as a shorthand notation for $d(1, r)$, and r and $d(r)$ were referred to as the multiplexing gain and the diversity gain, respectively, in [17]. The random coding diversity gain $d_{m,n,l}(r)$ is defined as

$$d_{m,n,l}(r) = \min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} \left\{ \sum_{i=1}^{\min(m,n)} (2i - 1 + |m - n|) \alpha_i + l \left[\sum_{i=1}^{\min(m,n)} (1 - \alpha_i)^+ - r \right] \right\}, \quad (15)$$

with

$$\mathcal{B} = \left\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1 - \alpha_i)^+ < r \right\}, \quad (16)$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{\min(m,n)})$. The expurgated diversity gain $d_{m,n,l}^{ex}(r)$ is defined as

$$d_{m,n,l}^{ex}(r) = n \, d_{m,l,n}^{-1}(lr), \quad (17)$$

where $d_{m,l,n}^{-1}$ is the inverse function of $d_{m,l,n}(r)$. In addition, it was shown in [17] that $d(N, r)$ is upper bounded by the outage diversity gain $d_{m,n}^{out}(r)$, which is the piecewise linear function connecting the points $(k, d_{m,n}^{out}(k)) = (k, (m - k)(n - k))$, $k \in \{0, 1, \dots, \min(m, n)\}$. Finally, it was also shown in [17] that $d_{m,n,l}(r)$ and $d_{m,n}^{out}(r)$ coincide for $l \geq m + n - 1$.

In [18], these concepts were extended to MIMO fading MACs. In particular, the channel was considered to operate at a rate pair $R_1 = R_1(SNR)$, $R_2 = R_2(SNR)$, such that

$$\lim_{SNR \rightarrow \infty} \frac{R_1(SNR)}{\log SNR} = r_1, \quad \lim_{SNR \rightarrow \infty} \frac{R_2(SNR)}{\log SNR} = r_2, \quad (18)$$

and $d(N, r_1, r_2)$ was defined as

$$d(N, r_1, r_2) \triangleq \lim_{SNR \rightarrow \infty} - \frac{\log P_{e,sys}(N, R_1(SNR), R_2(SNR))}{N \log SNR}. \quad (19)$$

For a MIMO fading MAC with m transmit antennas for user 1 and user 2, n receive antennas, and block length $l \geq 2m + n - 1$, it was shown in [18], that $d(N, r_1, r_2) = d(1, r_1, r_2) = d(r_1, r_2) = \min\{d_{m,n}^{out}(r_1), d_{m,n}^{out}(r_2), d_{2m,n}^{out}(r_1 + r_2)\}$, where $d(r_1, r_2)$ is a shorthand notation for $d(1, r_1, r_2)$.

III. ERROR EXPONENT REGION

In this section, we introduce the notion of error exponent region (EER) for a multi-user channel [20], [21]. We only consider two kinds of multi-user channels in this paper - two-user multiple access channels or two-user broadcast channels, so the term “multi-user channel” is referred to either one of them in this paper. Recall that for a multi-user channel, the probability of system error (or equivalently, the corresponding system error exponent) is not sufficient to capture the different reliability requirements of the users. Our approach to addressing this issue hinges on the following two observations. First, one can define a separate probability of error for each user. Therefore, there can be multiple error exponents, one for each user. Second, in contrast to a single-user channel where the error exponent is fixed for a given rate, in a multi-user channel one can tradeoff the error exponents among different users even for a fixed vector of users’ rates. To illustrate this point, consider the capacity region of a multi-user channel as shown in Fig. 1(b). As expected, the error exponents for the two users are functions of both the operating point A and the channel capacity region. However, unlike the case in a single-user channel where the channel capacity boundary is a single point (Fig. 1(a)), in a multi-user channel we have multiple points on the capacity boundary (e.g. B , D in Fig. 1(b)). Thus one can expect to get different error exponents (and thus a tradeoff among them) depending on which target point on the capacity boundary is considered. For instance, consider an operating point A (corresponding to a rate pair (R_1, R_2)) obtained by backing off from a target point B on the capacity boundary in Fig. 1(b). It is expected that the error exponent for user 1 is smaller than that for user 2, since user 1 operates at rate R_1 which is very close to the corresponding capacity (determined by B , see Fig. 1(c)), while user 2 backs off significantly from the corresponding capacity (determined again by B , see Fig. 1(c)). On the other hand, if we consider point A as if it is obtained by backing off from a target point D on the capacity boundary in Fig. 1(b), we then expect the error exponent for user 1 to be larger than that for user 2 (see Fig. 1(d)). Therefore, a tradeoff of error exponents between users might be possible by considering different points on the capacity boundary.

This leads us to the notion of error exponent region (EER) for a multi-user channel. For a given operating point characterized by the rate pair (R_1, R_2) , the error exponent region consists of all achievable error exponent pairs for the two users. For example, the error exponent region

Fig. 1. Channel capacity region: (a) single-user channel, (b) multi-user channel, (c) users back off from point B to point A, (d) users back off from point D to point A.

for a channel operated at point A in Fig. 1 is a two-dimensional region which depends on rates R_1 and R_2 (see Fig. 2). Note that the concepts of EER and channel capacity region (CCR) are fundamentally different. For a given channel, there is only one CCR. On the other hand, an EER depends on the channel operating point (R_1, R_2) . Thus, for a given channel, there is one EER for every operating point inside the CCR.

In the same way that error exponents were extended to diversity gains for a MIMO fading single-user channel at high SNR, we can extend the notion of EER to the notion of diversity gain region (DGR) for a multi-user MIMO fading channel. The DGR is the set of all achievable diversity gain vectors.

Fig. 2. Error exponent region for a rate pair (R_1, R_2) .

IV. DIVERSITY GAIN REGIONS FOR MIMO FADING BROADCAST CHANNELS

A. Diversity Gain Region

Consider a MIMO fading broadcast channel with m transmit antennas and n_1 and n_2 receive antennas for user 1 and user 2, respectively. The channel model is

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{H}_1 \mathbf{X} + \mathbf{Z}_1 \quad (20a)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{H}_2 \mathbf{X} + \mathbf{Z}_2. \quad (20b)$$

The channel fading matrices between the transmitter and the receiver 1 and the receiver 2 are represented by an $n_1 \times m$ matrix $\mathbf{H}_1$ and an $n_2 \times m$ matrix $\mathbf{H}_2$, respectively. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block with length l , and change to a new independent realization in the next block. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry is distributed as $\mathcal{CN}(0, 1)$. We assume that the fading matrices are known to the receivers but unknown to the transmitter. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average power at each transmit antenna is 1, which means that the average SNR at each receive antenna is SNR . The noise $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $n_1 \times l$ and $n_2 \times l$ matrices with i.i.d. entries distributed as $\mathcal{CN}(0, 1)$. The channel outputs $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are $n_1 \times l$ and $n_2 \times l$ matrices, respectively.

We formally define the diversity gain region (DGR) for a MIMO fading broadcast channel in the following.

Definition 1 Consider the MIMO fading broadcast channel (20) operated at a rate pair $R_1 =$

$R_1(SNR)$, $R_2 = R_2(SNR)$ over N blocks such that

$$\lim_{SNR \rightarrow \infty} \frac{R_1(SNR)}{\log SNR} = r_1, \quad \lim_{SNR \rightarrow \infty} \frac{R_2(SNR)}{\log SNR} = r_2, \quad (21)$$

where r_1 and r_2 are referred to as the multiplexing gains for user 1 and user 2, respectively. Given a multiplexing gain pair (r_1, r_2) of the two users, we define the DGR (corresponding to coding over N blocks) as the set of all achievable diversity gain pairs $(d_1(N, r_1, r_2), d_2(N, r_1, r_2))$ for all encoding schemes, where $d_1(N, r_1, r_2)$ and $d_2(N, r_1, r_2)$ are defined as

$$d_1(N, r_1, r_2) \triangleq \lim_{SNR \rightarrow \infty} -\frac{\log P_{e1}(N, R_1(SNR), R_2(SNR))}{N \log SNR} \quad (22a)$$

$$d_2(N, r_1, r_2) \triangleq \lim_{SNR \rightarrow \infty} -\frac{\log P_{e2}(N, R_1(SNR), R_2(SNR))}{N \log SNR}, \quad (22b)$$

where $P_{e1}(N, R_1(SNR), R_2(SNR))$, $P_{e2}(N, R_1(SNR), R_2(SNR))$ are the probabilities of error for user 1 and user 2, respectively.

We use $d_1(r_1, r_2)$ and $d_2(r_1, r_2)$ as shorthand notations for $d_1(1, r_1, r_2)$ and $d_2(1, r_1, r_2)$, respectively. Note that if (d_1^*, d_2^*) is achieved by an encoding scheme over one single block, i.e., $(d_1^*, d_2^*) = (d_1(r_1, r_2), d_2(r_1, r_2))$ for some encoding scheme, then for any $N > 1$ there exists an encoding scheme over N blocks such that $d_1^* \leq d_1(N, r_1, r_2)$ and $d_2^* \leq d_2(N, r_1, r_2)$, since encoding over multiple blocks can only improve the diversity gains of the users. Therefore, a DGR inner bound (achievable region) achieved by an encoding scheme over one single block is also a valid DGR inner bound for an encoding scheme over N blocks ($N > 1$). Similarly, if a DGR outer bound is valid for all encoding schemes over sufficiently large N ($N \rightarrow \infty$), this DGR outer bound is also valid for any encoding scheme over a finite number of blocks.

B. Superposition Encoding, Maximum-Likelihood Decoding, and Naive Single-User Decoding

In superposition encoding, we construct two independent random codebooks CB_1 and CB_2 , of size M_1 and M_2 , respectively (see Fig. 3). Denote $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ the i^{th} and the j^{th} codewords in codebooks CB_1 and CB_2 , respectively. Note that $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ are $m \times l$ random matrices. The channel input $\mathbf{X}$ is equal to $\mathbf{C}_{1,i} + \mathbf{C}_{2,j}$. Denote $\mathbf{C}_{1,i}(q, k)$ and $\mathbf{C}_{2,j}(q, k)$ the k^{th} elements in the q^{th} transmit antenna in the codewords $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$, respectively. Each random variable $\mathbf{C}_{1,i}(q, k)$ is i.i.d. with $\mathcal{CN}(0, 1)$ for $1 \leq k \leq \beta l$, and is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $\beta l + 1 \leq k \leq l$, where $\beta = \frac{a}{l}$, for some $a \in \{0, 1, \dots, l\}$, and $0 \leq p_1 \leq 1$. Similarly, each

random variable $C_{2,j}(q, k)$ is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $1 \leq k \leq \beta l$ and is i.i.d. with $\mathcal{CN}(0, 1)$ for $\beta l + 1 \leq k \leq l$, where $0 \leq p_2 \leq 1$. Note that the average power per transmit antenna in superposition encoding is $1 + SNR^{-(1-p_2)}$ for the first βl transmissions and is $1 + SNR^{-(1-p_1)}$ for the remaining $(1 - \beta)l$ transmissions, which are slightly more than the power constraint 1. However, we are only interested in the high SNR approximation, so a power constraint of 1 or 2 makes no difference on the diversity gains.

Fig. 3. Random codebooks for user 1 and user 2 using superposition encoding

In the receivers, the optimum decoding strategy is *individual* ML decoding, which minimizes the probabilities of error for user 1 and user 2, i.e., decoding user 1's message based on the index i maximizing $P(\mathbf{Y}_1 | \mathbf{C}_{1,i}) = \sum_{j=1}^{M_2} P(\mathbf{Y}_1 | \mathbf{C}_{1,i} + \mathbf{C}_{2,j})P(\mathbf{C}_{2,j})$, and decoding user 2's message based on the index j maximizing $P(\mathbf{Y}_2 | \mathbf{C}_{2,j}) = \sum_{i=1}^{M_1} P(\mathbf{Y}_2 | \mathbf{C}_{1,i} + \mathbf{C}_{2,j})P(\mathbf{C}_{1,i})$. However, it is difficult to derive analytical, single-letter expressions for diversity gains using individual ML decoding, so in this paper we use *joint* ML decoding to analyze the performance. In joint ML decoding, decoding user 1's message is based on the pair (i, j) maximizing $P(\mathbf{Y}_1 | \mathbf{C}_{1,i} + \mathbf{C}_{2,j})$, and decoding user 2's message is based on the pair (i, j) maximizing $P(\mathbf{Y}_2 | \mathbf{C}_{1,i} + \mathbf{C}_{2,j})$. As it will become evident in the subsequent analysis, the performance bounds based on joint ML

decoding can be tightened by considering another decoding strategy, namely the *naive* single-user decoding. In naive single-user decoding, user 1 simply regards user 2 as noise, and similarly, user 2 regards user 1 as noise.

The achievable diversity gains using superposition encoding and a mixture of joint ML and naive single-user decoding are derived in the following subsection.

C. Inner Bound for Diversity Gain Region (Achievability)

The inner bound derived in this subsection is based on encoding over one block (i.e., $N = 1$), so it is also a valid inner bound for encoding over multiple blocks $N > 1$.

Before deriving the DGR inner bound, we derive two intermediate results. The first result is a nonuniform-power random coding diversity gain $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ for a MIMO fading single-user channel. The second result is a diversity gain $d_{m,n,l,p}^{ns}(r)$ for a MIMO fading broadcast channel assuming naive single-user decoding.

Regarding the nonuniform-power random coding diversity gain, consider a random codebook CB with M codewords (see Fig. 4). Denote $\mathbf{C}_i$ the i^{th} codeword in the codebook CB . Note that $\mathbf{C}_i$ is a $m \times l$ random matrix. Denote $\mathbf{C}_i(q, k)$ the k^{th} element in the q^{th} transmit antenna in the codeword $\mathbf{C}_i$. Each random variable $\mathbf{C}_i(q, k)$ is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $1 \leq k \leq \beta l$, and is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $\beta l + 1 \leq k \leq l$, where $\beta = \frac{a}{l}$, for some $a \in \{0, 1, \dots, l\}$, and $0 \leq p_1, p_2 \leq 1$.

Extending the derivation of the random coding diversity gain $d_{m,n,l}(r)$ in [17], we can derive a nonuniform-power random coding diversity gain $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ for a nonuniform-power random codebook CB . The result is summarized in the following lemma.

Lemma 1 *For a MIMO fading single-user channel operated at a multiplexing gain r with m transmit antennas, n receive antennas, and block length l , the optimal probability of detection error is upper-bounded by*

$$P_e \leq SNR^{-d_{m,n,l,p_1,p_2,\beta}^{np}(r)}, \quad (23)$$

Fig. 4. Nonuniform-power random codebook.

where

$$d_{m,n,l,p_1,p_2,\beta}^{np}(r) = \min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} \left\{ \sum_{i=1}^{\min(m,n)} (2i-1 + |m-n|) \alpha_i + l \left[\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+ - r \right] \right\}, \quad (24)$$

and

$$\mathcal{B} = \left\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\min(m,n)} \geq 0; \beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+ < r \right\}. \quad (25)$$

Proof: The derivation for $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ is a straight forward extension of the derivation for the random coding diversity gain $d_{m,n,l}(r)$ in [17] and is omitted. In Appendix A, an explicit expression for $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ is given in the form of a piecewise linear function. ■

Regarding the naive single-user diversity gain, consider the broadcast channel (20) using superposition encoding with $\beta = 1$. If we decode user 1's message using naive single-user decoding, i.e., user 1 simply regards user 2 as noise, we can derive an achievable diversity gain for user 1. We summarize the result in the following lemma.

Lemma 2 For the MIMO fading broadcast channel (20) using superposition encoding with $\beta = 1$, the optimal probability of detection error for user 1 using naive single-user decoding is upper-bounded by

$$P_{e1} \leq SNR^{-d_{m,n_1,l,p_2}^{ns}(r_1)}, \quad (26)$$

where

$$d_{m,n_1,l,p_2}^{ns}(r_1) = \min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n_1)} \setminus \mathcal{B}} \left\{ \sum_{i=1}^{\min(m,n_1)} (2i-1 + |m-n_1|)\alpha_i + l \left[\sum_{i=1}^{\min(m,n_1)} (1 - \alpha_i - (p_2 - \alpha_i)^+)^+ - r_1 \right] \right\}, \quad (27)$$

and

$$\mathcal{B} = \left\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n_1)} \mid \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\min(m,n_1)} \geq 0; \sum_{i=1}^{\min(m,n_1)} (1 - \alpha_i - (p_2 - \alpha_i)^+)^+ < r_1 \right\} \quad (28)$$

Proof: The proof of this lemma, together with an explicit expression for $d_{m,n_1,l,p_2}^{ns}(r_1)$ in the form of a piecewise linear function, is presented in Appendix B. ■

We are now ready to derive the DGR inner bound using the superposition encoding strategy shown in Fig. 3. Denote P_{e11} , type 11 error probability, the probability that user 1 decodes (i, j) as $(\hat{i}, \hat{j})$, and denote P_{e13} , type 13 error probability, the probability that user 1 decodes (i, j) as $(\hat{i}, \hat{j})$, where $i \neq \hat{i}$ and $j \neq \hat{j}$. Similarly, denote P_{e22} , type 22 error probability, the probability that user 2 decodes (i, j) as $(\hat{i}, \hat{j})$, and denote P_{e23} , type 23 error probability, the probability that user 2 decodes (i, j) as $(\hat{i}, \hat{j})$. Applying the random coding argument, it can be shown that there exist codebooks for user 1 and user 2 using joint ML decoding such that

$$P_{e11} \leq SNR^{-d_{m,n_1,l,p_1,\beta}^{np}(r_1)} \quad (29a)$$

$$P_{e22} \leq SNR^{-d_{m,n_2,l,p_2,1,\beta}^{np}(r_2)} \quad (29b)$$

$$P_{e13} \leq SNR^{-d_{m,n_1,l}(r_1+r_2)} \quad (29c)$$

$$P_{e23} \leq SNR^{-d_{m,n_2,l}(r_1+r_2)}. \quad (29d)$$

The probabilities of error for user 1 and user 2 using joint ML decoding can be upper bounded by

$$\begin{aligned} P_{e1} = P_{e11} + P_{e13} &\leq SNR^{-d_{m,n_1,l,1,p_1,\beta}^{np}(r_1)} + SNR^{-d_{m,n_1,l}(r_1+r_2)} \\ &\leq 2SNR^{-\min\{d_{m,n_1,l,1,p_1,\beta}^{np}(r_1), d_{m,n_1,l}(r_1+r_2)\}} \end{aligned} \quad (30a)$$

$$\begin{aligned} P_{e2} = P_{e22} + P_{e23} &\leq SNR^{-d_{m,n_2,l,p_2,1,\beta}^{np}(r_2)} + SNR^{-d_{m,n_2,l}(r_1+r_2)} \\ &\leq 2SNR^{-\min\{d_{m,n_2,l,p_2,1,\beta}^{np}(r_2), d_{m,n_2,l}(r_1+r_2)\}}. \end{aligned} \quad (30b)$$

Thus, the achievable diversity gains using joint ML decoding are

$$d_1 = \min \{d_{m,n_1,l,1,p_1,\beta}^{np}(r_1), d_{m,n_1,l}(r_1 + r_2)\} \quad (31a)$$

$$d_2 = \min \{d_{m,n_2,l,p_2,1,\beta}^{np}(r_2), d_{m,n_2,l}(r_1 + r_2)\}. \quad (31b)$$

In the previous discussion, we first showed that for any (d_1, d_2) satisfying (31), there exists a pair of *random* codebooks with $E\{P_{e1}\} \leq SNR^{-d_1}$ and $E\{P_{e2}\} \leq SNR^{-d_2}$, where the expectation is over the probability space of the random codebooks. Note that this does not imply that there exists a pair of *deterministic* codebooks satisfying $P_{e1} \leq SNR^{-d_1}$ and $P_{e2} \leq SNR^{-d_2}$ *simultaneously*. The proof of the existence of such a pair is long and technical in nature and thus is omitted. In [22], a proof of the existence of a pair of deterministic codebooks satisfying $P_{e1} \leq E\{P_{e1}\}$ and $P_{e2} \leq E\{P_{e2}\}$ simultaneously was given for the case of a Gaussian broadcast channel, so a similar argument can be applied here.

When naive single-user decoding is utilized, the probability of error for user 1 can be upper bounded by

$$P_{e1} \leq SNR^{-d_{m,n_1,\beta l,p_2}^{ns}(\frac{r_1}{\beta})}. \quad (32)$$

Similarly, the probability of error for user 2 (using naive single-user decoding) can be upper bounded by

$$P_{e2} \leq SNR^{-d_{m,n_2,(1-\beta)l,p_1}^{ns}(\frac{r_2}{1-\beta})}. \quad (33)$$

Thus, the achievable diversity gains using naive single-user decoding are

$$d_1 = d_{m,n_1,\beta l,p_2}^{ns}(\frac{r_1}{\beta}) \quad (34a)$$

$$d_2 = d_{m,n_2,(1-\beta)l,p_1}^{ns}(\frac{r_2}{1-\beta}). \quad (34b)$$

Since both users can choose either joint ML decoding or naive single-user decoding, the maximum of the corresponding diversity gains are achievable. We summarize all the above results in the following theorem.

Theorem 1 *For the MIMO fading broadcast channel (20), an achievable DGR is given by*

$$\begin{aligned}
 DGR_{in}(r_1, r_2) = & \left\{ (d_1, d_2) : \beta = \frac{a}{l}, a \in \{0, 1, \dots, l\}, 0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1, \right. \\
 & d_1 \leq \max \left\{ \min \left\{ d_{m,n_1,l,1,p_1,\beta}^{np}(r_1), d_{m,n_1,l}(r_1 + r_2) \right\}, d_{m,n_1,\beta l,p_2}^{ns}\left(\frac{r_1}{\beta}\right) \right\} \\
 & \left. d_2 \leq \max \left\{ \min \left\{ d_{m,n_2,l,p_2,1,\beta}^{np}(r_2), d_{m,n_2,l}(r_1 + r_2) \right\}, d_{m,n_2,(1-\beta)l,p_1}^{ns}\left(\frac{r_2}{1-\beta}\right) \right\} \right\}, \quad (35)
 \end{aligned}$$

where $d_{m,n_1,\beta l,p_2}^{ns}(\frac{r_1}{\beta})$ should be interpreted as 0 for $\beta = 0$ and $d_{m,n_2,(1-\beta)l,p_1}^{ns}(\frac{r_2}{1-\beta})$ should be interpreted as 0 for $\beta = 1$.

Proof: Follows from the previous discussion. ■

Several comments are in order at this point.

- One might question the efficiency of superposition encoding when coding over a single block (or a finite number of blocks). This scheme is motivated by the fact that superposition encoding is a capacity achieving strategy for degraded broadcast channels [23], [24] (by coding over sufficiently large N blocks). It should be clear that in the present setting we directly evaluate the corresponding error probabilities without using the property that the broadcast channel considered in (20) is degraded.
- It is possible to improve the DGR inner bound for small block length l by expurgating codebooks and deriving a *nonuniform-power expurgated diversity gain*. The derivation of such improved bound is a straightforward extension of Theorem 1 and thus is omitted.
- Other decoding strategies can also be applied to provide upper bounds on the error probabilities of the two users. For instance, a decoding technique commonly used in the literature is successive cancellation. Diggavi *et al.* derived an achievable diversity gain pair for a single-user channel with two different messages (high- and low-reliability messages) using successive cancellation [14]. Our numerical results indicated that this decoding strategy does not improve the DGR inner bound given in Theorem 1.

- It may seem surprising that we use two different probability distributions $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p_1)})$ to construct the random codebook CB_1 (and similarly for CB_2). This requires some explanation. Consider two special cases of superposition encoding - “uniform” superposition and “on-off” superposition. In uniform superposition, the parameter β in Fig. 3 is chosen to be zero or one, so the random codebook CB_1 (CB_2) has i.i.d. (uniform) entries. In on-off superposition, the parameters p_1 and p_2 in Fig. 3 are chosen to be zero, so the transmitter switches between user 1 and user 2 (on-off) during the transmission. In Fig. 5, the achievable DGRs obtained by these two special cases are illustrated. In this figure, the solid square is the boundary of the achievable DGR using uniform superposition and joint ML decoding, and the dotted curve is the boundary of the achievable DGR using on-off superposition. It seems that for asymmetric diversity gain pair, on-off superposition is superior to uniform superposition. There are two possible explanations for this, though we can not verify which one is the main reason: either joint ML decoding is significantly inferior to individual ML decoding, or the bound derived for joint ML decoding is loose. Based on these two encoding schemes, it is now clear that superposition encoding includes the above two cases (uniform and on-off) and also serves as a smooth transition between these two encoding schemes. One may ask if it is possible to improve the DGR by using three, four, or even more probability distributions to construct each random codebook. Our numerical results indicated that going beyond two distributions provides only marginal improvements. However, multiple distributions might be beneficial for a broadcast channel with more than two users.
- The result that the performance bound based on joint ML decoding can be improved by using naive single-user decoding might not have been anticipated. The dashed curve in Fig. 5 is the boundary of the achievable DGR using uniform superposition and a mixture of joint ML and naive single-user decoding (the dashed curve merging with the solid square at $(d_1, d_2) = (9, 4.5)$ and $(d_1, d_2) = (4.5, 9)$). The improvement (comparing to using only joint ML decoding) is in the lower-right and upper-left corner regions. It appears that joint ML decoding is close to optimum individual ML decoding when the broadcast channel is nearly symmetric ($n_1 \approx n_2$) and the requirements (diversity gains) for user 1 and user 2 are similar, and is much worse than individual ML decoding when the broadcast channel is extremely asymmetric ($n_1 \gg n_2$) or the requirements for user 1 and user 2 are quite

different ($d_1 \gg d_2$). To illustrate this point, consider an extremely asymmetric channel with $m = 1$, $n_1 = 1$, $n_2 = 1000$ operated at $r_1 = 0.02$ and $r_2 = 1$. Construct two independent random codebooks CB_1 and CB_2 with i.i.d elements $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-0.99})$, respectively. It is easy to check from Theorem 1 that the achievable diversity of user 2 using joint ML decoding is positive. For joint ML decoding at user 1, however, type 13 error consists of roughly $M_1 M_2 = SNR^{l(r_1+r_2)}$ error events, each one consisting of a pair of codewords from the codebooks. The maximum achievable multiplexing gain for user 1 is $m \times n_1 = 1$, which is less than $r_1 + r_2 = 1.02$. If we simply apply the union bound for P_{e13} , we have *zero* diversity gain for user 1 due to type 13 error. This suggests that the joint ML decoding might work poorly for user 1 in this case. On the other hand, the allocated power for user 2 ($SNR^{-0.99}$) is much smaller than the allocated power for user 1 ($SNR^0 = 1$), so we expect that the diversity gain for user 1 derived using individual ML decoding should be close to the single-user diversity gain $d_{m,n_1,l}(r_1)$. This is because even if user 1 simply regards the signal X_2 as some interfering Gaussian noise, the performance degradation (compared to the point-to-point case without side interference from user 2) should be small. This suggests that in this case naive single-user decoding might provide a superior performance compared to joint ML decoding.

In Fig. 5, the dash-dotted curve is the boundary of the achievable DGR using superposition encoding, which, as mentioned above, provides a smooth transition between the achievable region using uniform superposition and the achievable region using on-off superposition.

D. Outer Bound for Diversity Gain Region

In this subsection, we derive an DGR outer bound valid for any encoding scheme over sufficiently large N blocks ($N \rightarrow \infty$), so this DGR outer bound is also valid for any encoding scheme over a finite blocks.

It is true that for any broadcast channel the probability of decoding error for user i can always be lower bounded by the probability of decoding error for user i operating over a point-to-point channel defined by the marginal distribution $P(Y_i|X)$. This implies that $d_i \leq d_{m,n_i}^{out}(r_i)$, for $i = 1, 2$.

Given an encoding and decoding scheme, it is true that

$$P_{e,sys} \leq P_{e1} + P_{e2} \leq 2 \max\{P_{e1}, P_{e2}\}, \quad (36)$$

Fig. 5. DGR inner bound for $m = n_1 = n_2 = 4, l = 60, r_1 = r_2 = 0.5$: uniform superposition with joint ML decoding (solid), uniform superposition with a mixture of joint ML and naive single-user decoding (dashed), on-off superposition (dotted), and superposition (dash-dotted).

where the first inequality follows from the union bound. The broadcast channel considered in (20) is a *stochastically* degraded broadcast channel [12, Chap. 14]. Since the performance of a broadcast channel depends only on the conditional marginals, we may further assume that the broadcast channel considered in (20) is a *physically* degraded broadcast channel, i.e., $P(Y_1, Y_2|X) = P(Y_1|X)P(Y_2|Y_1)$ if $n_1 \geq n_2$. If we now allow the two receivers to cooperate, we have a single-user channel with $n_1 + n_2$ receive antennas, whose probability of error, P'_e , should be less than or equal to the probability of system error $P_{e,sys}$ in the original broadcast channel [25]. The probability of error P'_e of the new single-user channel can be lower bounded by

$$SNR^{-\max\{d_{m,n_1}^{out}(r_1+r_2), d_{m,n_2}^{out}(r_1+r_2)\}} \leq P'_e, \quad (37)$$

since the broadcast channel considered here is physically degraded. Combining (36), (37), we have

$$SNR^{-\max\{d_{m,n_1}^{out}(r_1+r_2), d_{m,n_2}^{out}(r_1+r_2)\}} \leq P'_e \leq P_{e,sys} \leq 2 \max\{P_{e1}, P_{e2}\}, \quad (38)$$

which implies that $\min\{d_1, d_2\} \leq \max\{d_{m,n_1}^{out}(r_1 + r_2), d_{m,n_2}^{out}(r_1 + r_2)\}$.

We summarize all of the above results in the following theorem.

Theorem 2 *For the MIMO fading broadcast channel (20), an outer bound for DGR is given by*

$$d_1 \leq d_{m,n_1}^{\text{out}}(r_1) \quad (39a)$$

$$d_2 \leq d_{m,n_2}^{\text{out}}(r_2) \quad (39b)$$

$$\min\{d_1, d_2\} \leq \max\{d_{m,n_1}^{\text{out}}(r_1 + r_2), d_{m,n_2}^{\text{out}}(r_1 + r_2)\}. \quad (39c)$$

Proof: Follows from the above arguments. ■

Note that the single-user diversity gain $d(N, r)$ is upper bounded by $d_{m,n}^{\text{out}}(r)$ for any N , so the DGR outer bound given in the above theorem is valid for any N .

In Fig. 6, the derived inner and outer DGR bounds are shown for two channel scenarios. In this figure, the solid curve is the boundary of the DGR inner bound and the dash-dotted curve is the boundary of the DGR outer bound. Two important results are observed in Fig. 6: (i) the DGR inner and outer bounds are tight at the lower-right and the upper-left corners; (ii) for a symmetric MIMO fading broadcast channel, the DGR inner and outer bounds are tight at $d_1 = d_2$ (Fig. 6(a)). Result (i) implies that the appearance of the second user does not affect the first user (for a certain range of diversity gains for the second user) since the first user achieves the optimal single-user diversity gain $d_{m,n_1}^{\text{out}}(r_1)$ (and similarly for the second user). These results are formally expressed in the following theorem.

Theorem 3 *For the MIMO fading broadcast channel (20), the following are true:*

- (a) *If $r_1 < 1$ and $r_2 < (1 - r_1)(1 - \frac{m+n_1-1}{l}) \min(m, n_2)$, then user 1 achieves the single-user diversity gain $d_{m,n_1,l}(r_1)$, and simultaneously user 2 achieves a diversity gain $d_2 > 0$. A similar result holds for user 2.*
- (b) *If the MIMO fading broadcast channel is symmetric, i.e., $n_1 = n_2$, the DGR inner and outer bounds are tight at $d_1 = d_2$.*

Proof:

- (a) Consider user 1 with a multiplexing gain $r_1 < 1$. The tightness of the DGR inner and outer bounds at the lower-right corner is a direct consequence applying naive single-user decoding to user 1. From (68a), (68b), user 1 using superposition encoding with $\beta = 1$ can achieve the single-user diversity gain $d_{m,n_1}^{\text{out}}(r_1)$ using naive single-user decoding as

long as

$$p_2 < 1 - \frac{m + n_1 - 1}{l} \quad (40a)$$

$$r_1 \leq \frac{(1 - p_2)l - (m + n_1 - 1)}{l - (m + n_1 - 1)}. \quad (40b)$$

Choose $p_2^* = (1 - r_1)(1 - \frac{m+n_1-1}{l})$, then both conditions in (40) are satisfied. The achievable diversity gains for user 1 using naive single-user decoding and for user 2 using joint ML decoding are

$$d_1 = d_{m,n_1,l,p_2^*}^{ns}(r_1) = d_{m,n_1}^{out}(r_1) \quad (41a)$$

$$d_2 = \min \left\{ d_{m,n_2,l,p_2^*,1,\beta}^{np}(r_2), d_{m,n_2,l}(r_1 + r_2) \right\} = \min \left\{ p_2^* d_{m,n_2}^{out}\left(\frac{r_2}{p_2^*}\right), d_{m,n_2}^{out}(r_1 + r_2) \right\}. \quad (41b)$$

Note that the chosen p_2^* is the largest value for which the equation $d_{m,n_1,l,p_2}^{ns}(r_1) = d_{m,n_1}^{out}(r_1)$ still holds, i.e., $d_{m,n_1,l,p_2}^{ns}(r_1) < d_{m,n_1}^{out}(r_1)$ for $p_2 > p_2^*$. It is clear that if $r_2 < p_2^* \min(m, n_2) = (1 - r_1)(1 - \frac{m+n_1-1}{l}) \min(m, n_2)$, both $\frac{r_2}{p_2^*}$ and $r_1 + r_2$ in (41b) are smaller than $\min(m, n_2)$, and thus $d_2 > 0$. The proof of this argument for user 2, with a multiplexing gain $r_2 < 1$, achieving the optimal single-user diversity gain if $r_1 < (1 - r_2)(1 - \frac{m+n_2-1}{l}) \min(m, n_1)$ is similar.

- (b) For a symmetric MIMO fading broadcast channel with $n_1 = n_2 = n$, the achievable diversity gains using uniform superposition with $\beta = 1$, $p_2 = 1$ and joint ML decoding are

$$\begin{aligned} d_1 &= \min \left\{ d_{m,n_1,l,1,p_1,\beta}^{np}(r_1), d_{m,n_1,l}(r_1 + r_2) \right\} = \min \left\{ d_{m,n,l}(r_1), d_{m,n,l}(r_1 + r_2) \right\} \\ &= d_{m,n,l}(r_1 + r_2) = d_{m,n}^{out}(r_1 + r_2) \end{aligned} \quad (42a)$$

$$\begin{aligned} d_2 &= \min \left\{ d_{m,n_2,l,p_2,1,\beta}^{np}(r_2), d_{m,n_2,l}(r_1 + r_2) \right\} = \min \left\{ d_{m,n,l}(r_2), d_{m,n,l}(r_1 + r_2) \right\} \\ &= d_{m,n,l}(r_1 + r_2) = d_{m,n}^{out}(r_1 + r_2). \end{aligned} \quad (42b)$$

Comparing these with the DGR upper bound (39c), it is clear that the inner bound and bounds are tight at $d_1 = d_2$ for a symmetric broadcast channel. ■

Before leaving this section, we define the CCR counterpart in a MIMO fading broadcast channel, namely the multiplexing gain region (MGR).

Fig. 6. Diversity gain region inner bound (solid) and outer bound (dash-dotted) for (a) $m = n_1 = n_2 = 4, l = 60, r_1 = r_2 = 0.5$, (b) $m = n_1 = 4, n_2 = 3, l = 55, r_1 = r_2 = 0.5$.

Definition 2 Consider the MIMO fading broadcast channel (20) with coding over N blocks. The MGR is defined as the closure of the set of all multiplexing gain pairs (r_1, r_2) with $d_1(N, r_1, r_2) > 0$ and $d_2(N, r_1, r_2) > 0$ for all encoding schemes.

Note that in the above definition, the MGR is a function of N . We now derive a MGR inner bound and a MGR outer bound and summarize the result in the following theorem. Examples of typical MGR inner and outer bounds are shown in Fig. 7.

Theorem 4 For the MIMO fading broadcast channel (20), an MGR inner bound MGR_{in} and an MGR outer bound MGR_{out} for any encoding scheme over N blocks are

$$MGR_{in} = \left\{ (r_1, r_2) : \frac{r_1}{\min(m, n_1)} + \frac{r_2}{\min(m, n_2)} \leq 1 \right\} \quad (43)$$

$$\begin{aligned}
MGR_{out} = & \left\{ (r_1, r_2) : 0 \leq \alpha \leq 1, \right. \\
& r_1 \leq (\min(m, n_1) - \min(m, n_2))^+ + \alpha \min \{ \min(m, n_1), \min(m, n_2) \}, \\
& \left. r_2 \leq (\min(m, n_2) - \min(m, n_1))^+ + (1 - \alpha) \min \{ \min(m, n_1), \min(m, n_2) \} \right\}.
\end{aligned} \tag{44}$$

Proof: The proof is given in Appendix C. The inner bound (43) is derived using uniform superposition. It can also be shown that the achievable MGR derived using (nonuniform) superposition is the same region as given in (43), but it's a little more involved and is omitted. ■

Note that although the MGR might be a function of N , the inner bound (derived for $N = 1$) and the outer bound (derived for $N \rightarrow \infty$) given in Theorem 4 are valid for any N .

Fig. 7. Multiplexing gain region inner bound (solid) and outer bound (dash-dotted) for (a) $m = 3, n_1 = 4, n_2 = 4$ (solid=dash-dotted), (b) $m = 3, n_1 = 3, n_2 = 2$.

V. DIVERSITY GAIN REGIONS FOR MIMO FADING MULTIPLE ACCESS CHANNELS

Consider a MIMO fading multiple access channel with m_1 , m_2 transmit antennas for user 1 and user 2, respectively and n receive antennas. The channel model is given by

$$\mathbf{Y} = \sqrt{\frac{SNR}{m_1}} \mathbf{H}_1 \mathbf{X}_1 + \sqrt{\frac{SNR}{m_2}} \mathbf{H}_2 \mathbf{X}_2 + \mathbf{Z}. \quad (45)$$

The channel fading matrices between transmitter 1, transmitter 2 and the receiver are represented by an $n \times m_1$ matrix $\mathbf{H}_1$ and an $n \times m_2$ matrix $\mathbf{H}_2$, respectively. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block with length l , and change to a new independent realization in the next block. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry is distributed as $\mathcal{CN}(0, 1)$. We assume that the fading matrices are known by the receiver but not known by the transmitters. The channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ are $m_1 \times l$ and $m_2 \times l$ matrices, respectively, and are normalized such that the average power at each transmit antenna is 1, which means that the average power of each user's signal at each receive antenna is SNR . The noise $\mathbf{Z}$ is an $n \times l$ matrix with i.i.d. entries distributed as $\mathcal{CN}(0, 1)$. The channel output $\mathbf{Y}$ is an $n \times l$ matrix.

Similar to a MIMO fading broadcast channel, an encoding scheme $\mathcal{C}(SNR)$ for a MIMO fading MAC over N blocks is said to achieve a multiplexing gain pair (r_1, r_2) and a diversity gain pair (d_1, d_2) if

$$\lim_{SNR \rightarrow \infty} \frac{R_1(SNR)}{\log SNR} = r_1 \quad (46a)$$

$$\lim_{SNR \rightarrow \infty} \frac{R_2(SNR)}{\log SNR} = r_2, \quad (46b)$$

and

$$d_1(N, r_1, r_2) \triangleq \lim_{SNR \rightarrow \infty} - \frac{\log P_{e1}(N, R_1(SNR), R_2(SNR))}{N \log SNR} \quad (47a)$$

$$d_2(N, r_1, r_2) \triangleq \lim_{SNR \rightarrow \infty} - \frac{\log P_{e2}(N, R_1(SNR), R_2(SNR))}{N \log SNR}, \quad (47b)$$

where $R_1(SNR)$, $R_2(SNR)$, $P_{e1}(N, R_1(SNR), R_2(SNR))$, $P_{e2}(N, R_1(SNR), R_2(SNR))$ are the rates and the probabilities of error for user 1 and user 2, respectively. We use $d_1(r_1, r_2)$ and $d_2(r_1, r_2)$ as shorthand notations for $d_1(1, r_1, r_2)$ and $d_2(1, r_1, r_2)$, respectively. Given a multiplexing gain pair (r_1, r_2) , the DGR is defined as the set of all achievable diversity gain pairs $(d_1(N, r_1, r_2), d_2(N, r_1, r_2))$ for all encoding schemes. Similarly, the MGR is defined as the closure of the set of all multiplexing gain pairs (r_1, r_2) with $d_1(N, r_1, r_2) > 0$ and $d_2(N, r_1, r_2) > 0$.

0 for all encoding schemes. The MGR for a MIMO fading MAC with m_1, m_2 transmit antennas and n receive antennas is

$$r_1 \leq \min(m_1, n) \quad (48a)$$

$$r_2 \leq \min(m_2, n) \quad (48b)$$

$$r_1 + r_2 \leq \min(m_1 + m_2, n). \quad (48c)$$

The achievability of (48) for $N = 1$ was given in [18]. The proof that (48) is also an outer bound for sufficiently large N ($N \rightarrow \infty$) can be derived easily from the channel capacity for a general MAC [12, Chap. 14], and the channel capacity for a MIMO fading single-user channel [19].

A. Inner Bound for Diversity Gain Region (Achievability)

As in a broadcast channel, an inner bound is developed for $N = 1$ in this subsection, and thus is valid for any value of N .

Motivated by (nonuniform) superposition encoding in a MIMO fading broadcast channel, it is reasonable to consider a similar technique for a MIMO fading MAC. The technique that enables an efficient bound using superposition strategy in the MIMO fading broadcast channel is based on a combination of joint ML and naive single-user decoding. However, this bounding technique does not provide a tight bound in the case of MACs. To see this, recall that in a MIMO fading broadcast channel, when the intended signal of one of the two users suffers from the channel fading, the side interference (from the other user) also suffers from *the same* channel fading. In a MIMO fading MAC, on the other hand, the intended signal and the side interference go through *different* and *independent* fading channels. This implies that sometimes the intended user's signal might suffer deep fading but the side interference is “unfaded” at the channel output. Thus naive single-user decoding might work poorly in a MIMO fading MAC.

Using our bounding techniques, it was observed that (nonuniform) superposition using joint ML decoding does not provide an advantage over uniform and on-off superposition. Therefore, due to lack of an effective decoding strategy to close the gap between individual ML decoding and joint ML decoding, we only derive an inner bound using two special cases of superposition encoding - uniform superposition and on-off superposition. Recall that in uniform superposition encoding, the channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ have i.i.d. entries $\mathcal{CN}(0, 1)$. The bounds on the users' error probabilities are obtained using joint ML decoding. Following [10], we define three types of

error events, type 1, type 2, and type 3, as in (5). Assuming the block length $l \geq m_1 + m_2 + n - 1$, it can be shown by using random coding arguments that there exist codebooks for user 1 and user 2, respectively, such that

$$P_{e,t1} \leq SNR^{-d_{m_1,n}^{out}(r_1)} \quad (49a)$$

$$P_{e,t2} \leq SNR^{-d_{m_2,n}^{out}(r_2)} \quad (49b)$$

$$P_{e,t3} \leq SNR^{-d_{m_1+m_2,n}^{out}(r_1+r_2)}. \quad (49c)$$

The probabilities of error for user 1 and user 2 can be upper bounded by

$$\begin{aligned} P_{e1} = P_{e,t1} + P_{e,t3} &\leq SNR^{-d_{m_1,n}^{out}(r_1)} + SNR^{-d_{m_1+m_2,n}^{out}(r_1+r_2)} \\ &\leq 2SNR^{-\min\{d_{m_1,n}^{out}(r_1), d_{m_1+m_2,n}^{out}(r_1+r_2)\}} \end{aligned} \quad (50a)$$

$$\begin{aligned} P_{e2} = P_{e,t2} + P_{e,t3} &\leq SNR^{-d_{m_2,n}^{out}(r_2)} + SNR^{-d_{m_1+m_2,n}^{out}(r_1+r_2)} \\ &\leq 2SNR^{-\min\{d_{m_2,n}^{out}(r_2), d_{m_1+m_2,n}^{out}(r_1+r_2)\}}. \end{aligned} \quad (50b)$$

Thus, the achievable diversity gains derived from joint ML decoding are

$$d_1^{us} = \min \{d_{m_1,n}^{out}(r_1), d_{m_1+m_2,n}^{out}(r_1+r_2)\} \quad (51a)$$

$$d_2^{us} = \min \{d_{m_2,n}^{out}(r_2), d_{m_1+m_2,n}^{out}(r_1+r_2)\}, \quad (51b)$$

where the superscript “us” denotes uniform superposition.

Recall that in on-off superposition, the channel input $\mathbf{X}_1$ has i.i.d. entries $\mathcal{CN}(0, 1)$ for the first βl transmissions (on) and is zero for the remaining $(1 - \beta)l$ transmissions (off), where $\beta = \frac{a}{l}$ for some $a \in \{1, 2, \dots, l - 1\}$. Similarly, The channel input $\mathbf{X}_2$ is zero for the first βl transmissions (off) and has i.i.d. entries $\mathcal{CN}(0, 1)$ for the remaining $(1 - \beta)l$ transmissions (on).

We can derive achievable diversity gains for this strategy as

$$d_1^{os} = \max \left\{ d_{m_1,n,\beta l} \left(\frac{r_1}{\beta} \right), d_{m_1,n,\beta l}^{ex} \left(\frac{r_1}{\beta} \right) \right\} \quad (52a)$$

$$d_2^{os} = \max \left\{ d_{m_2,n,(1-\beta)l} \left(\frac{r_2}{1-\beta} \right), d_{m_2,n,(1-\beta)l}^{ex} \left(\frac{r_2}{1-\beta} \right) \right\}, \quad (52b)$$

where the superscript “os” denotes on-off superposition.

The union of achievable DGRs obtained using uniform and on-off superposition is a DGR inner bound. We summarize all the above results in the following theorem.

Theorem 5 For the MIMO fading multiple access channel (45) with block length $l \geq m_1 + m_2 + n - 1$, an achievable DGR is $DGR_{in}(r_1, r_2) = DGR_{us}(r_1, r_2) \cup DGR_{os}(r_1, r_2)$, where $DGR_{us}(r_1, r_2)$ and $DGR_{os}(r_1, r_2)$ are given by

$$DGR_{us}(r_1, r_2) = \left\{ (d_1, d_2) : \right. \\ \left. d_1 \leq \min \{d_{m_1, n}^{out}(r_1), d_{m_1+m_2, n}^{out}(r_1 + r_2)\} \right. \\ \left. d_2 \leq \min \{d_{m_2, n}^{out}(r_2), d_{m_1+m_2, n}^{out}(r_1 + r_2)\} \right\} \quad (53a)$$

$$DGR_{os}(r_1, r_2) = \left\{ (d_1, d_2) : \beta = \frac{a}{l}, a \in \{1, 2, \dots, l-1\}, \right. \\ \left. d_1 \leq \max \left\{ d_{m_1, n, \beta l}^{ex} \left(\frac{r_1}{\beta} \right), d_{m_1, n, \beta l}^{ex} \left(\frac{r_1}{\beta} \right) \right\} \right. \\ \left. d_2 \leq \max \left\{ d_{m_2, n, (1-\beta)l}^{ex} \left(\frac{r_2}{1-\beta} \right), d_{m_2, n, (1-\beta)l}^{ex} \left(\frac{r_2}{1-\beta} \right) \right\} \right\}. \quad (53b)$$

Proof: Follows from the above discussion. ■

In Fig. 8(a) we depict the achievable DGR using on-off superposition. In this figure, each point is a diversity gain pair corresponding to one particular $\beta = \frac{a}{l}$ in (52), and the achievable DGR using on-superposition is shown in Fig. 8(b). Note that the boundary in Fig. 8(b) is not simply a smooth curve connecting the corresponding achievable diversity gain pairs in Fig. 8(a). This is because the value of βl assigned to user 1 must be an integer and each consecutive point in Fig. 8(a) corresponds to one consecutive integer.

From Theorem 5, the MGR of a MIMO fading MAC can be divided into four regions r_{12} , r_{13} , r_{23} , and r_3

$$r_{12} \triangleq \{(r_1, r_2) : d_{m_1, n}^{out}(r_1) \leq d_{m_1+m_2, n}^{out}(r_1 + r_2), d_{m_2, n}^{out}(r_2) \leq d_{m_1+m_2, n}^{out}(r_1 + r_2)\} \quad (54a)$$

$$r_{13} \triangleq \{(r_1, r_2) : d_{m_1, n}^{out}(r_1) \leq d_{m_1+m_2, n}^{out}(r_1 + r_2) \leq d_{m_2, n}^{out}(r_2)\} \quad (54b)$$

$$r_{23} \triangleq \{(r_1, r_2) : d_{m_2, n}^{out}(r_2) \leq d_{m_1+m_2, n}^{out}(r_1 + r_2) \leq d_{m_1, n}^{out}(r_1)\} \quad (54c)$$

$$r_3 \triangleq \{(r_1, r_2) : d_{m_1+m_2, n}^{out}(r_1 + r_2) \leq d_{m_1, n}^{out}(r_1), d_{m_1+m_2, n}^{out}(r_1 + r_2) \leq d_{m_2, n}^{out}(r_2)\}, \quad (54d)$$

depending on whether the bound of type 1 error, type 2 error, or type 3 error dominates when using uniform superposition (see Fig. 9). In region r_{12} , the achievable diversity gains using uniform superposition are $d_1^{us} = d_{m_1, n}^{out}(r_1)$ and $d_2^{us} = d_{m_2, n}^{out}(r_2)$. In this region, each user attains the optimal single-user diversity gain, and thus there is no diversity gain tradeoff between the users. In region r_{13} , the achievable diversity gains using uniform superposition

Fig. 8. Diversity gains using on-off superposition for $m_1 = m_2 = n = 4, l = 12, r_1 = r_2 = 0.5$ (a) achievable diversity gain pairs, (b) achievable diversity gain region.

are $d_1^{us} = d_{m_1, n}^{out}(r_1)$ and $d_2^{us} = d_{m_1+m_2, n}^{out}(r_1 + r_2)$. In this region, the first user achieves the optimal single-user diversity gain, while the second user's error probability is dominated by type 3 error. Thus it is possible to provide a tradeoff of diversity gains between the users using on-off superposition. In Fig. 10 we show the DGRs for two channel scenarios. Observe that in Fig. 10(a), user 2's diversity gain d_2 can go beyond $d_2^{us} = 5$ by reducing user 1's diversity gain, where the solid curve in Fig. 10(a) is the boundary of the achievable DGR using uniform superposition encoding, and the dotted curve in Fig. 10(a) is the boundary of the achievable DGR using on-off superposition. A similar result also holds for region r_{23} . In region r_3 , the achievable diversity gains using uniform superposition are $d_1^{us} = d_2^{us} = d_{m_1+m_2, n}^{out}(r_1 + r_2)$. In this region, type 3 error is dominant over both type 1 and type 2 errors, so it is possible to provide a tradeoff of diversity gains between the two users. This is illustrated in Fig. 10(b), where the solid curve is the boundary of the achievable DGR by uniform superposition, and the dotted curve is the boundary of the achievable DGR by on-off superposition.

Fig. 9. Multiplexing gain region for $m_1 = m_2 = n = 4$.

Fig. 10. Diversity gain region for (a) $m_1 = m_2 = n = 4, l = 60, r_1 = 2.5, r_2 = 0.5$ (region r_{13}): uniform superposition (solid) and on-off superposition (dotted), (b) $m_1 = m_2 = n = 4, l = 240, r_1 = 3.4, r_2 = 0.5$ (region r_3): uniform superposition (solid) and on-off superposition (dotted).

B. Outer Bound for Diversity Gain Region

As in a broadcast channel, an outer bound is developed for sufficiently large N ($N \rightarrow \infty$) in this subsection, and thus is valid for any finite value of N .

For a MIMO fading MAC, the probabilities of decoding error for user 1 and user 2 can always be lower bounded by the probabilities of decoding error for user 1 and user 2 operating over the point-to-point channels defined by $\mathbf{Y} = \sqrt{\frac{SNR}{m_i}} \mathbf{H}_i \mathbf{X}_i + \mathbf{Z}$, for $i = 1, 2$. Furthermore, if we allow the two transmitters in the MIMO fading MAC to cooperate, we have a MIMO fading single-user channel with $m_1 + m_2$ transmit antennas and n receive antennas, whose probability of error (using an optimal receiver) should be less than or equal to the probability of system error in the original MAC. Note that the diversity gain corresponding to the probability of system error is equal to $\min\{d_1, d_2\}$. Collecting all these ideas, we have the following outer bound for the DGR

$$d_1 \leq d_{m_1, n}^{\text{out}}(r_1) \quad (55a)$$

$$d_2 \leq d_{m_2, n}^{\text{out}}(r_2) \quad (55b)$$

$$\min\{d_1, d_2\} \leq d_{m_1+m_2, n}^{\text{out}}(r_1 + r_2). \quad (55c)$$

Assuming the block length $l \geq m_1 + m_2 + n - 1$, we can also conclude from (51a), (51b), (55a), (55b) that all the operating points inside the region r_{12} in Fig. 9 have tight DGR inner and outer bounds.

In Fig. 11, two examples are considered for illustrating the derived inner and outer bounds. The solid curve is the boundary of the achievable DGR, and the dashed-dotted curve is the DGR outer bound. It is noted that in these particular examples, it is not clear whether a tradeoff between the optimal diversity gains of the two users exists. Furthermore, it is noted that in Fig. 11(b), the inner bound achieved by uniform superposition completely contains that achieved by on-off superposition.

VI. CONCLUSION

We have introduced the notion of DGR for a MIMO fading multi-user channel. This region specifies the set of diversity gain vectors that are simultaneously achievable by all users. This is done by associating different probabilities of error for different users, contrary to the traditional

Fig. 11. Diversity gain region inner bound (solid) and outer bound (dash-dotted) for (a) $m_1 = m_2 = n = 4, l = 60, r_1 = 2.5, r_2 = 0.5$, (b) $m_1 = 5, m_2 = 2, n = 4, l = 55, r_1 = 2.5, r_2 = 1$.

approach where a single *probability of system error* is considered. Therefore, there are multiple diversity gains, one for each user, for a given multi-user channel. We have derived an inner bound (achievable region) and an outer bound for the DGRs of MIMO fading broadcast channels and MIMO fading multiple access channels. The concept of DGR, or more generally EER, can be applied to any multi-terminal communication system. Our future work involves a characterization of the EER for cooperative MIMO fading communication systems, as well as the construction of practical coding schemes to achieve the derived performance limits for these channels.

APPENDIX

A. Nonuniform-power Random Coding Diversity Gain

It is easy to verify that $d_{m,n,l,p_1,p_2,\beta}^{np}(r) = p_2 d_{m,n,l}(\frac{r}{p_2})$ for $\beta = 0$, and $d_{m,n,l,p_1,p_2,\beta}^{np}(r) = p_1 d_{m,n,l}(\frac{r}{p_1})$ for $\beta = 1$. For $0 < \beta < 1$, we may assume $p_2 \leq p_1$ without loss of generality. $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ is a piecewise linear function and it is easy to see that $d_{m,n,l,p_1,p_2,\beta}^{np}(r) = 0$ for $r \geq \min(m,n)(\beta p_1 + (1-\beta)p_2)$. For $(k-1)(\beta p_1 + (1-\beta)p_2) < r < k(\beta p_1 + (1-\beta)p_2)$ and $k \in \{1, 2, \dots, \min(m,n)\}$, $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ consists of two line segments with slopes

$\min(\frac{m+n-2k+1}{\beta}, l)$ and $\min(m+n-2k+1, l)$, respectively, i.e. $d_{m,n,l,p_1,p_2,\beta}^{np}(r)$ is a function connecting the following three points $(r, d) =$

$$\begin{aligned} & \left((k-1)(\beta p_1 + (1-\beta)p_2), \sum_{q=k}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2 [\min(s_q, l) - \min(s_q, \beta l)]\} \right), \\ & \left(k(\beta p_1 + (1-\beta)p_2) - p_2, \sum_{q=k+1}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2 [\min(s_q, l) - \min(s_q, \beta l)]\} + p_2 \min(s_k, l) \right), \\ & \left(k(\beta p_1 + (1-\beta)p_2), \sum_{q=k+1}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2 [\min(s_q, l) - \min(s_q, \beta l)]\} \right), \end{aligned}$$

where $s_q = m + n - 2q + 1$ (see Fig. 12).

B. Naive Single-User Decoding Diversity Gain

Proof of Lemma 2: If we use superposition encoding with $\beta = 1$, the channel output for user 1 is

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{H}_1 (\mathbf{X}_1 + \mathbf{X}_2) + \mathbf{Z}_1, \quad (56)$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and have i.i.d elements $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p_2)})$, respectively. If we decode user 1's message using naive single-user decoding, i.e., user 1 simply regards user 2 as noise, we can consider the following MIMO fading side-interference single-user channel

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}} \mathbf{H} (\mathbf{X} + \mathbf{S}) + \mathbf{Z}, \quad (57)$$

where $\mathbf{H}$ is an $n \times m$ matrix with i.i.d. entries $\mathcal{CN}(0, 1)$, and $\mathbf{Z}$ and $\mathbf{S}$ are $n \times l$ noise and side-interference matrices with i.i.d. entries $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ respectively. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average power at each transmit antenna is 1. We want to show that the probability of detection error in this side-interference channel is upper bounded by $SNR^{-d_{m,n,l,p}^{ns}(r)}$.

At high SNR, we can ignore the integral of the probability of error over the range $\mathbf{H} \notin \mathbb{R}_+^{\min(m,n)}$. The probability of error can be upper bounded by

$$P_e \leq P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \notin \mathcal{B}), \quad (58)$$

Fig. 12. Nonuniform-power random coding diversity gain ($0 < \beta < 1$, $p_2 \leq p_1$).

where

$$\mathcal{B} = \left\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1 - \alpha_i - (p - \alpha_i)^+)^+ < r \right\}. \quad (59)$$

Since

$$\begin{aligned}
P(\mathbf{H} \in \mathcal{B}) &\doteq SNR^{-\inf_{\underline{\alpha} \in \mathcal{B}} \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i} \\
&\leq SNR^{-\min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} \left\{ \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i + l \left[\sum_{i=1}^{\min(m,n)} (1-\alpha_i - (p-\alpha_i)^+) + -r \right] \right\}} \\
&= SNR^{-d_{m,n,l,p}^{ns}(r)},
\end{aligned} \tag{60}$$

it remains to prove that $P(\text{error}, \mathbf{H} \notin \mathcal{B})$ is also upper bounded by $SNR^{-d_{m,n,l,p}^{ns}(r)}$.

Conditioned on a channel realization $\mathbf{H} = H$, we can write the channel as

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}} H \mathbf{X} + \left(\sqrt{\frac{SNR}{m}} H \mathbf{S} + \mathbf{Z} \right). \tag{61}$$

Since H is known at the receiver, we can whiten the noise $\sqrt{\frac{SNR}{m}} H \mathbf{S} + \mathbf{Z}$ by multiplying $\left(\frac{SNR^p}{m} H H' + I \right)^{-\frac{1}{2}}$ at the channel output Y , where H' is the conjugate transpose of H . Thus, we have the following equivalent channel

$$\tilde{\mathbf{Y}} = \sqrt{\frac{SNR}{m}} \left(\frac{SNR^p}{m} H H' + I \right)^{-\frac{1}{2}} H \mathbf{X} + \tilde{\mathbf{Z}}, \tag{62}$$

where $\tilde{\mathbf{Z}}$ is an $n \times l$ matrix with i.i.d. entries $\mathcal{CN}(0, 1)$

Assume $X(0), X(1)$ are two possible transmitted codewords and $\Delta X = X(1) - X(0)$. Suppose $X(0)$ is transmitted, then the probability that a receiver will make a detection error in favor of $X(1)$ is

$$\begin{aligned}
P(\mathbf{X}(0) = X(0) \rightarrow \mathbf{X}(1) = X(1) \mid \mathbf{H} = H) = \\
P \left(\frac{SNR}{m} \left\| \frac{1}{2} \left(\frac{SNR^p}{m} H H' + I \right)^{-\frac{1}{2}} H \Delta X \right\|_F^2 \leq \|w\|^2 \right),
\end{aligned} \tag{63}$$

where w is the additive noise with variance 1/2 on the direction of $\left(\frac{SNR^p}{m} H H' + I \right)^{-\frac{1}{2}} H \Delta X$, I is an identity matrix, and $\|\cdot\|_F$ is the Frobenius norm. With the standard approximation of the Gaussian tail function: $Q(t) \leq 1/2 \exp(-t^2/2)$, we have

$$\begin{aligned}
P(\mathbf{X}(0) = X(0) \rightarrow \mathbf{X}(1) = X(1) \mid \mathbf{H} = H) \leq \\
\exp \left\{ -\frac{SNR}{4m} \left\| \left(\frac{SNR^p}{m} H H' + I \right)^{-\frac{1}{2}} H \Delta X \right\|_F^2 \right\}.
\end{aligned} \tag{64}$$

Averaging over the ensemble of random codes, we have the average pairwise error probability (PEP) given the channel realization H

$$\begin{aligned}
P(\mathbf{X}(0) \rightarrow \mathbf{X}(1) \mid \mathbf{H} = H) &= \left| I + \frac{SNR}{2m} \left(\frac{SNR^p}{m} HH' + I \right)^{-\frac{1}{2}} HH' \left(\frac{SNR^p}{m} HH' + I \right)^{-\frac{1}{2}} \right|^{-l} \\
&= \left\{ \prod_{i=1}^{\min(m,n)} \left(1 + \frac{\frac{SNR}{2m} \lambda_i}{1 + \frac{SNR^p}{m} \lambda_i} \right) \right\}^{-l} \\
&\doteq \prod_{i=1}^{\min(m,n)} SNR^{-l(1-\alpha_i-(p-\alpha_i)^+)^+}, \tag{65}
\end{aligned}$$

where $\lambda_i = SNR^{-\alpha_i}$ ($\alpha_1 \geq \dots \geq \alpha_{\min(m,n)} \geq 0$) and λ_i 's are the nonzero eigenvalues of HH' .

Applying the union bound, we have

$$\begin{aligned}
P(\text{error} \mid \underline{\alpha}) &\leq SNR^{lr} \prod_{i=1}^{\min(m,n)} SNR^{-l(1-\alpha_i-(p-\alpha_i)^+)^+} \\
&= SNR^{-l \left[\left(\sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ \right) - r \right]}. \tag{66}
\end{aligned}$$

In high SNR, $P(\underline{\alpha})$ can be approximated by $\prod_{i=1}^{\min(m,n)} SNR^{-(2i-1+|m-n|)\alpha_i}$ [17], so we have

$$\begin{aligned}
P(\text{error}, \mathbf{H} \notin \mathcal{B}) &\doteq \int_{\mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} P(\underline{\alpha}) P(\text{error} \mid \underline{\alpha}) d\underline{\alpha} \\
&\leq \int_{\mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} SNR^{-\sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i} SNR^{-l \left[\sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ - r \right]} d\underline{\alpha} \\
&\doteq SNR^{-d_{m,n,l,p}^{ns}(r)} \tag{67}
\end{aligned}$$

This completes the proof. ■

$d_{m,n,l,p}^{ns}(r)$ is a piecewise linear function and it is easy to see that $d_{m,n,l,p}^{ns}(r) = 0$ for $r \geq \min(m,n)(1-p)$. For $r \leq \min(m,n)(1-p)$, $d_{m,n,l,p}^{ns}(r)$ can be classified into the following three cases.

Case 1: $(1-p)l \geq m+n-1$

$d_{m,n,l,p}^{ns}(k(1-p)) = (m-k)(n-k)$ for $k \in \{0, \dots, \min(m,n)\}$. For $(k-1)(1-p) < r < k(1-p)$, $d_{m,n,l,p}^{ns}(r)$ consists of two line segments with slopes $m+n-2k+1$ and l , i.e. $d_{m,n,l,p}^{ns}(r)$ is a function connecting the following three points $(r, d) = ((k-1)(1-p), (m-k+1)(n-k+1))$, $(k(1-p) - \frac{sp}{1-s}, (m-k)(n-k) + \frac{sp}{1-s}l)$, and $(k(1-p), (m-k)(n-k))$, where $s = \frac{m+n-2k+1}{l}$.

(see Fig. 13).

Case 2: $|m - n| + 1 < (1 - p)l < m + n - 1$

Define $k' = \max\{a \in \mathbb{Z} \mid (1 - p)l \leq m + n - 2a + 1\}$. For $r \leq k'(1 - p)$, $d_{m,n,l,p}^{ns}(r) = (m - k')(n - k') + (k'(1 - p) - r)l$. For $r > k'(1 - p)$ and $(k - 1)(1 - p) < r < k(1 - p)$, $d_{m,n,l,p}^{ns}(r)$ consists of two line segments with slopes $m + n - 2k + 1$ and l , i.e. $d_{m,n,l,p}^{ns}(r)$ is a function connecting the following three points $(r, d) = ((k - 1)(1 - p), (m - k + 1)(n - k + 1))$, $(k(1 - p) - \frac{sp}{1-s}, (m - k)(n - k) + \frac{sp}{1-s}l)$, and $(k(1 - p), (m - k)(n - k))$, where $s = \frac{m+n-2k+1}{l}$ (see Fig. 14).

Case 3: $(1 - p)l \leq |m - n| + 1$

$d_{m,n,l,p}^{ns}(r)$ is a linear function with slope l , i.e. $d_{m,n,l,p}^{ns}(r) = (\min(m, n)(1 - p) - r)l$ (see Fig. 15).

Naive single-user diversity gains for case 1, 2, and 3 are shown in Fig. 16.

Fig. 13. Naive single-user diversity gain: $(1 - p)l \geq m + n - 1$

In case 1, $d_{m,n,l,p}^{ns}(r) = d_{m,n,l}(r)$ for $r \leq \frac{(1-p)l - (m+n-1)}{l - (m+n-1)}$. This means that one can still achieve the single-user diversity gain $d_{m,n,l}(r)$ even with side interference. This is a quite interesting

Fig. 14. Naive single-user diversity gain: $|m - n| + 1 < (1 - p)l < m + n - 1$

and important phenomenon, so we give some intuition behind this. Consider the following two conditions

$$p < 1 - \frac{m + n - 1}{l} \quad (68a)$$

$$r \leq \frac{(1-p)l - (m + n - 1)}{l - (m + n - 1)}. \quad (68b)$$

The side-interference power at the receiver is $SNR \cdot SNR^{-(1-p)} = SNR^p$. Roughly speaking, the first condition $p < 1 - \frac{m+n-1}{l}$ says that the side-interference power is small and the second condition $r \leq \frac{(1-p)l - (m+n-1)}{l - (m+n-1)}$ says that the channel is lightly used (small multiplexing gain). Note that $\frac{(1-p)l - (m+n-1)}{l - (m+n-1)}$ is less than $1 - p$, so we can never achieve full single-user diversity gain for $r \geq 1 - p$. To understand the implications of these two conditions in more detail, we can consider a single-input-single-output (SISO) fading side-interference single-user channel ($m = n = 1$). The optimum probability of detection error for a SISO fading single-user channel can be upper

Fig. 15. Naive single-user diversity gain: $(1-p)l \leq |m-n| + 1$

Fig. 16. Naive single-user diversity gain for $m = n = 4, l = 30$ (a) $p = 0.5$, (b) $p = 0.85$, (c) $p = 0.97$.

bounded by

$$P_e(SNR) \leq P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \notin \mathcal{B}), \quad (69)$$

where

$$\mathcal{B} = \left\{ \alpha \geq 0 \mid (1 - \alpha - (p - \alpha)^+)^+ < r \right\} \quad (70)$$

is the “bad” channel event. We can upper bound $P(\mathbf{H} \in \mathcal{B})$ by

$$P(\mathbf{H} \in \mathcal{B}) \leq SNR^{-\inf_{\alpha \in \mathcal{B}} \alpha}, \quad (71)$$

and upper bound $P(\text{error}, \mathbf{H} \notin \mathcal{B})$ by

$$P(\text{error}, \mathbf{H} \notin \mathcal{B}) \leq SNR^{-d_{1,1,l,p}^{ns}(r)} = SNR^{-\min_{\alpha \in \mathbb{R}_+ \setminus \mathcal{B}} \{\alpha + l[(1 - \alpha - (p - \alpha)^+)^+ - r]\}}. \quad (72)$$

Under conditions (68a), (68b), the minimization in (71), (72) is achieved by $\alpha = 1 - r$. This implies that the dominant error event happens when the side-interference power at the channel output

$$SNR^p \cdot SNR^{-\alpha} = SNR^{-(1-r-p)} < 1 \quad (73)$$

is less than the power of the Gaussian noise Z . Small multiplexing gain (condition (68b)) implies that the dominant error event, e.g. outage event, happens for large α . This together with small side interference (condition (68a)) suggests that the side interference might be negligible compared to additive Gaussian noise. This explains why one can still achieve the single-user diversity gain even with (small) side interference. In general, the minimization values in (71), (72) might be different and there is a gap between the naive single-user diversity gain $d_{m,n,l,p}^{ns}(r)$ and the random coding diversity gain $d_{m,n,l}(r)$.

C. Proof of Theorem 4

The inner bound is proved for $N = 1$ and the outer bound is proved for $N \rightarrow \infty$, so both bounds are valid for any N .

We first show that (43) is an inner bound for the MGR. Without loss of generality, we may assume $n_1 \geq n_2$. Applying Theorem 1 with $\beta = 0$, we can achieve a diversity gain pair (d_1, d_2)

$$\begin{aligned} d_1 &= \min \left\{ d_{m,n_1,l,1,p_1,\beta}^{mp}(r_1), d_{m,n_1,l}(r_1 + r_2) \right\} = \min \left\{ p_1 d_{m,n_1,l}\left(\frac{r_1}{p_1}\right), d_{m,n_1,l}(r_1 + r_2) \right\} \\ d_2 &= d_{m,n_2,(1-\beta)l,p_1}^{ms}\left(\frac{r_2}{1-\beta}\right) = d_{m,n_2,l,p_1}^{ms}(r_2) \end{aligned}$$

Note that under the assumption $n_1 \geq n_2$, $d_{m,n_1,l}(r_1 + r_2) > 0$ for any interior point (r_1, r_2) of MGR_{in} , so $d_1 > 0$ as long as $d_{m,n_1,l}\left(\frac{r_1}{p_1}\right) > 0$. Since $d_{m,n_1,l}\left(\frac{r_1}{p_1}\right) > 0$ for $r_1 < \min(m, n_1)p_1$,

and $d_2 = d_{m,n_2,l,p_1}^{ns}(r_2) > 0$ for $r_2 < \min(m, n_2)(1 - p_1)$, it is clear that by choosing p_1 a value between 0 and 1 we have $d_1 > 0$ and $d_2 > 0$ for any interior point of MGR_{in} .

It remains to show that (44) is an outer bound for the MGR. Denote $\underline{\lambda}_1 = (\lambda_{1,1}, \lambda_{1,2}, \dots, \lambda_{1,\min\{m,n_1\}})$ and $\underline{\lambda}_2 = (\lambda_{2,1}, \lambda_{2,2}, \dots, \lambda_{2,\min\{m,n_2\}})$ the vectors of the nonzero eigenvalues of $\mathbf{H}'_1\mathbf{H}_1$ and $\mathbf{H}'_2\mathbf{H}_2$, where $\mathbf{H}'_1$ and $\mathbf{H}'_2$ are the conjugate transposes of $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. Applying singular value decomposition to $\mathbf{H}_1$ and $\mathbf{H}_2$, we can re-write the broadcast channel as

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{V}_1 \mathbf{\Sigma}_1 \mathbf{W}_1 \mathbf{X} + \mathbf{Z}_1 \quad (74a)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{V}_2 \mathbf{\Sigma}_2 \mathbf{W}_2 \mathbf{X} + \mathbf{Z}_2, \quad (74b)$$

where $\mathbf{\Sigma}_1 = [\sigma_{1,ij}]$ is an $n_1 \times m$ matrix with $\sigma_{1,ii} = \sqrt{\lambda_{1,i}}$ for $1 \leq i \leq \min(m, n_1)$ and $\sigma_{1,ij} = 0$ otherwise, and $\mathbf{\Sigma}_2 = [\sigma_{2,ij}]$ is an $n_2 \times m$ matrix with $\sigma_{2,ii} = \sqrt{\lambda_{2,i}}$ for $1 \leq i \leq \min(m, n_2)$ and $\sigma_{2,ij} = 0$. $\mathbf{V}_1$ and $\mathbf{V}_2$ are $n_1 \times n_1$ and $n_2 \times n_2$ unitary random matrices, and $\mathbf{W}_1$ and $\mathbf{W}_2$ are $m \times m$ unitary random matrices.

It is well-known that the capacity region of a broadcast channel depends only on the marginal distributions, so we can consider the capacity region of the following broadcast channel by replacing $\mathbf{W}_2$ with $\mathbf{W}_1$ in (74b)

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{V}_1 \mathbf{\Sigma}_1 \mathbf{W}_1 \mathbf{X} + \mathbf{Z}_1 \quad (75a)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{V}_2 \mathbf{\Sigma}_2 \mathbf{W}_1 \mathbf{X} + \mathbf{Z}_2. \quad (75b)$$

If we now assume that the channel matrices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$ are known both at the transmitter and the receivers in the broadcast channel (75), we can consider the following equivalent broadcast channel

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{\Sigma}_1 \mathbf{X} + \mathbf{Z}_1 \quad (76a)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{\Sigma}_2 \mathbf{X} + \mathbf{Z}_2, \quad (76b)$$

whose MGR is an outer bound of the original broadcast channel defined in (20).

For any encoding scheme over N blocks, define the following notations.

$\mathbf{U}_1, \mathbf{U}_2$: input messages for user 1 and user 2.

$\mathbf{X}^N$: $m \times lN$ channel input matrix. $\mathbf{X}^N = f(\mathbf{U}_1, \mathbf{U}_2)$ for some function depending on the encoding scheme.

$\mathbf{Y}_1^N$: $n_1 \times lN$ channel output matrix of user 1.

$\mathbf{H}_1^N$: $n_1 \times mN$ channel fading matrix for user 1.

$\mathbf{H}_1^{(k)}$: $n_1 \times m$ channel fading matrix of user 1 for the k^{th} block, i.e., $\mathbf{H}_1^N = [\mathbf{H}_1^{(1)} \mathbf{H}_1^{(2)} \dots \mathbf{H}_1^{(N)}]$.

$\Sigma_1^{(k)}$: $n_1 \times m$ channel singular value matrix of user 1 for the k^{th} block $\mathbf{H}_1^{(k)}$.

Σ_1^N : $\Sigma_1^N = [\Sigma_1^{(1)} \Sigma_1^{(2)} \dots \Sigma_1^{(N)}]$.

$\underline{\lambda}_1^{(k)}$: $1 \times \min\{m, n_1\}$ vector of the nonzero eigenvalues of $\mathbf{H}_1^{(k)'} \mathbf{H}_1^{(k)}$, where $\mathbf{H}_1^{(k)'}$ is the conjugate transpose of $\mathbf{H}_1^{(k)}$.

$\underline{\lambda}_1^N$: $\underline{\lambda}_1^N = [\underline{\lambda}_1^{(1)} \underline{\lambda}_1^{(2)} \dots \underline{\lambda}_1^{(N)}]$.

$\mathcal{A} \triangleq [SNR^{-\epsilon} \ SNR^{\epsilon}]^{\min\{m, n_1\}}$: $\min\{m, n_1\}^{th}$ -fold Cartesian product of the interval $[SNR^{-\epsilon} \ SNR^{\epsilon}]$.

$\mathcal{A}^N$: N^{th} -fold Cartesian product of $\mathcal{A}$, i.e., $\mathcal{A}^N = [SNR^{-\epsilon} \ SNR^{\epsilon}]^{\min\{m, n_1\}N}$.

Since $\mathbf{U}_1$ and Σ_1^N are independent, the achievable rate R_1 for user 1 can be written as

$$\begin{aligned} R_1 &= \frac{1}{N} I(\mathbf{U}_1; \mathbf{Y}_1^N, \Sigma_1^N) = \frac{1}{N} I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N) \\ &= \frac{1}{N} \int_{\underline{\lambda}_1^N \in \mathcal{A}^N} I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N + \\ &\quad \frac{1}{N} \int_{\underline{\lambda}_1^N \in (\mathcal{A}^N)^c} I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N. \end{aligned} \quad (77)$$

The second term in (77) can be upper bounded by

$$\begin{aligned} &\frac{1}{N} \int_{\underline{\lambda}_1^N \in (\mathcal{A}^N)^c} I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N \\ &\leq \frac{1}{N} \int_{\underline{\lambda}_1^N \in (\mathcal{A}^N)^c} I(\mathbf{X}^N; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N \\ &\leq \int_{\underline{\lambda}_1 \in \mathcal{A}^c} \log \left| I + \frac{SNR}{m} \Sigma_1' \Sigma_1 \right| P(\Sigma_1) d\Sigma_1 \\ &= \int_{\underline{\lambda}_1 \in \mathcal{A}^c} \sum_{i=1}^{\min\{m, n_1\}} \log \left(1 + \frac{SNR}{m} \lambda_{1,i} \right) P(\underline{\lambda}_1) d\underline{\lambda}_1, \end{aligned} \quad (78)$$

where the first inequality in (78) is due to $\mathbf{U}_1 - \mathbf{X}^N - \mathbf{Y}_1^N$ forming a Markov chain and the second inequality in (78) is based on the fact that $I(\mathbf{X}^N; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N)$ is maximized when $\mathbf{X}^N$ has i.i.d. complex Gaussian entries. Note that $P(\underline{\lambda}_1)$ goes to zero exponentially for large

$\underline{\lambda}_1$ but the integrand is in the order of $\log(SNR)$, so the last term $\int_{\underline{\lambda}_1 \in \mathcal{A}^c} \sum_{i=1}^{\min\{m, n_1\}} \log(1 + \frac{SNR}{m} \lambda_{1,i}) P(\underline{\lambda}_1) d\underline{\lambda}_1$ in (78) goes to zero as $SNR \rightarrow \infty$. From (77), (78), we have $R_1 \approx \frac{1}{N} \int_{\underline{\lambda}_1^N \in \mathcal{A}^N} I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N$ at high SNR, i.e., regarding the data rate for user 1, we may assume that $SNR^{-\epsilon} \leq \lambda_{1,i} \leq SNR^\epsilon$. Denote $I_1 = [I_{1,ij}]$ an $n_1 \times m$ matrix with $I_{1,ii} = 1$ for $1 \leq i \leq \min(m, n_1)$ and $I_{1,ij} = 0$ otherwise, and $I_2 = [I_{2,ij}]$ an $n_2 \times m$ matrix with $I_{2,ii} = 1$ for $1 \leq i \leq \min(m, n_2)$ and $I_{2,ij} = 0$ otherwise. Since $I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = SNR^{-\epsilon} I_1) \leq I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = \Sigma_1^N) \leq I(\mathbf{U}_1; \mathbf{Y}_1^N | \Sigma_1^N = SNR^\epsilon I_1)$ for $\underline{\lambda}_1^N \in \mathcal{A}^N$ and ϵ is any positive constant, we can take ϵ arbitrarily small and on the scale of our interest (multiplexing gain) we may assume that $\underline{\lambda}_1$ is a vector with every entry equal to one. Similar argument also applies to user 2. Therefore, regarding multiplexing gains, we can consider the following broadcast channel

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} I_1 \mathbf{X} + \mathbf{Z}_1 \quad (79a)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} I_2 \mathbf{X} + \mathbf{Z}_2. \quad (79b)$$

The MGR of the broadcast channel defined in (79) is an outer bound of the MGR of the original fading broadcast channel defined in (20). However, it is easy to see that the MGR of the broadcast channel defined in (79) is exactly (44). This completes the proof.

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