Diversity Gain Region for MIMO Fading Broadcast Channels

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Abstract — **In this work, we introduce the notion of the *diversity gain region* for a multi-user channel. This region specifies the set of diversity-gain vectors that are simultaneously achievable by all users in the multi-user channel. This is done by associating different probabilities of error for different users, contrary to the traditional approach where a single *probability of system error* is considered. We derive an inner bound (achievable region) and an outer bound for the diversity gain region of a MIMO fading broadcast channel.**

I. Introduction

It is well-known that the error exponent for a single-user channel provides the rate of exponential decay of the average probability of error as a function of the block length of the codebooks [1]. The concept of the error exponent was extended to a Gaussian multiple access channel (MAC) in [2], where an upper bound on the *probability of system error* (i.e., the probability that any user is in error) was derived for random codes. Recently, Zheng *et al.* considered error exponents in high signal-to-noise ratio (SNR) approximation, called diversity gains, for multi-input-multi-output (MIMO) fading single-user channels [3].

In many applications of multi-user networks, different users might have different reliability requirements. For instance, in an uplink (or downlink) of a cellular system, a user running an FTP application might have more stringent reliability requirements than a user running a multimedia application which is designed for graceful degradation. Based on the traditional approaches [2] which consider a single probability of system error, a network can only be designed to satisfy the most stringent reliability requirement. This might result in a mismatch of resources allocation, and thus, it is inherently suboptimal.

To address this issue, we introduced the notion of error exponent region (EER) for a general multi-user channel in [4]. For a given operating point, i.e., a rate-pair (R_1, R_2), the error exponent region consists of all achievable error exponents when the channel is operated at that point. An EER depends on the channel operating point, and for a given channel, there are numerous EERs depending on which operating point we consider. In this paper, we focus on the EER at high SNR for a MIMO fading broadcast channel.

The rest of the paper is structured as follows. In Section II, we introduce the notion of multiplexing gain region (MGR) and diversity gain region (DGR). The MGR and DGR are the channel capacity region (CCR) and the error exponent region at high SNR respectively. In Section III, we derive the DGR inner bound using two strategies - channel-splitting and superposition. In Section IV, we propose a unified encoding scheme, generalized-superposition, which includes both channel-splitting and superposition as special cases. In Section V, we derive the DGR outer bound. Two important results are also shown in Section V. First, either one of the two users in the broadcast channel can achieve the single-user diversity gain if the data rates are low. Second, the DGR inner bound and DGR outer bound are tight for equal diversity gains if the broadcast channel is symmetric. We conclude our work in Section VI.

II. Multiplexing Gain Region and Diversity Gain Region

Consider a MIMO fading broadcast channel with m transmit antennas and n_1 and n_2 receive antennas for user 1 and user 2. The channel model is

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{H}_1\mathbf{X} + \mathbf{Z}_1 \tag{1}$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}}\mathbf{H}_2\mathbf{X} + \mathbf{Z}_2. \tag{2}$$

The channel fading matrices between the transmitter and the receiver 1 and the receiver 2 are represented by an $n_1 \times m$ matrix $\mathbf{H}_1$ and an $n_2 \times m$ matrix $\mathbf{H}_2$. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block length l, and change to a new independent realization in the next block length l. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry has a complex Gaussian distribution $\mathcal{CN}(0,1)$. We assume that fading matrices are known by the receivers but unknown by the transmitter. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average transmit power at each antenna is one. The noise $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $n_1 \times l$ and $n_2 \times l$ matrices with i.i.d. entries $\mathcal{CN}(0,1)$.

In [3], an encoding scheme $\mathcal{C}(SNR)$ (a family of codes) in a MIMO fading single-user channel is said to achieve multiplexing gain r and diversity gain d if

$$\lim_{SNR\to\infty} \frac{R(SNR)}{\log SNR} = r, \lim_{SNR\to\infty} \frac{\log P_e(SNR)}{\log SNR} = -d, \tag{3}$$

where $R(SNR)$ and $P_e(SNR)$ are the rate and the average probability of error of the code $C(SNR)$ respectively. Define $R(SNR) \cong r \log SNR$ and $P_e(SNR) \doteq SNR^{-d}$ if equalities hold in the limit, and $\tilde{\geq}$, $\tilde{\leq}$, $\dot{\geq}$, $\dot{\leq}$ are defined similarly. Following the same notations in [3], we define an encoding scheme

$\mathcal{C}(SNR)$ to achieve multiplexing gain pair (r_1, r_2) and diversity gain pair (d_1, d_2) in a MIMO fading broadcast channel if

$$\lim_{SNR\to\infty} \frac{R_1(SNR)}{\log SNR} = r_1, \lim_{SNR\to\infty} \frac{\log P_{e1}(SNR)}{\log SNR} = -d_1, \tag{4}$$

$$\lim_{SNR\to\infty} \frac{R_2(SNR)}{\log SNR} = r_2, \lim_{SNR\to\infty} \frac{\log P_{e2}(SNR)}{\log SNR} = -d_2, \tag{5}$$

where $R_1(SNR)$, $R_2(SNR)$, $P_{e1}(SNR)$, $P_{e2}(SNR)$ are the rates and the probabilities of codeword error for user 1 and user 2. Multiplexing gain region (MGR) is thus defined as the set of all achievable multiplexing gain pair (r_1, r_2) for all encoding schemes. The MGR is the CCR at high SNR.

As mentioned earlier, an EER depends on the operating point (R_1, R_2). Similarly, given a multiplexing gain pair (r_1, r_2), we define the diversity gain region (DGR) as the set of all achievable diversity gain pair (d_1, d_2). The diversity gain region is the EER at high SNR. Before continuing, we derive a MGR inner bound and a MGR outer bound for a MIMO fading broadcast channel. We summarize the result in the following theorem.

Theorem 1 *For a MIMO fading broadcast channel with m transmit antennas and n_1, n_2 receive antennas, an MGR inner bound is*

$$MGR_{in} = \{(r_1, r_2) : \frac{r_1}{\min(m, n_1)} + \frac{r_2}{\min(m, n_2)} \leq 1\}, \tag{6}$$

and an MGR outer bound is

$$\begin{aligned} MGR_{out} = \big\{(r_1, r_2) :\ & 0 \leq \alpha \leq 1, \\ & r_1 \leq (\min(m, n_1) - \min(m, n_2))^+ \\ & \quad + \alpha \min\{\min(m, n_1), \min(m, n_2)\}, \\ & r_2 \leq (\min(m, n_2) - \min(m, n_1))^+ \\ & \quad + (1-\alpha) \min\{\min(m, n_1), \min(m, n_2)\}\big\}, \end{aligned} \tag{7}$$

where $(x)^+ = \max(x, 0)$. □

Before deriving the DGR inner and outer bounds, we summarize all the definitions of the diversity gains used in this paper. More detail explanations and exact formulas of these diversity gains are given in later sections.

(1) $d_{m,n,l}(r)$: random coding diversity gain with m transmit antennas, n receive antennas, and block length l.
(2) $d^{ex}_{m,n,l}(r)$: expurgated diversity gain with m transmit antennas, n receive antennas, and block length l.
(3) $d^{out}_{m,n}(r)$: outage diversity gain with m transmit antennas and n receive antennas.
(4) $d^{ns}_{m,n,l,p}(r)$: naive single-user diversity gain with m transmit antennas, n receive antennas, block length l, and normalized side-interference power $SNR^{-(1-p)}$.
(5) $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$: non-uniform power random coding diversity gain with m transmit antennas, n receive antennas, block length l, normalized power $SNR^{-(1-p_1)}$ for length βl, and normalized power $SNR^{-(1-p_2)}$ for length $(1-\beta)l$.

III. Achievable Diversity Gain Region

Before continuing, let's review the diversity-multiplexing tradeoff for a MIMO fading single-user channel derived in [3]. It was shown in [3] that both the random coding diversity gain $d_{m,n,l}(r)$ and the expurgated diversity gain $d^{ex}_{m,n,l}(r)$ are achievable in a MIMO fading single-user channel with m transmit antennas, n receive antennas, and block length l. In addition, the diversity gain was shown to be upper bounded by the outage diversity gain $d^{out}_{m,n}(r)$, where $d^{out}_{m,n}(r)$ is the piecewise linear function connecting the points $(k, d^{out}_{m,n}(k)) = (k, (m-k)(n-k)), k \in \mathbb{Z}^+$. Finally, it was also shown that $d_{m,n,l}(r)$ and $d^{out}_{m,n}(r)$ coincide for $l \geq m+n-1$.

For the MIMO fading broadcast channel considered in this paper, we propose two encoding strategies - channel-splitting and superposition. In channel-splitting, we allocate βl symbols to user 1 and $(1-\beta)l$ symbols to user 2 inside each block length l, where $\beta = \frac{k}{l}$ and $1 \leq k \leq l-1$, $k \in \mathbb{Z}$ (see Fig. 1). Thus, the achievable diversity gains are

$$d_1^{cs} = \max\{d_{m,n_1,\beta l}(\frac{r_1}{\beta}), d^{ex}_{m,n_1,\beta l}(\frac{r_1}{\beta})\} \tag{8}$$

$$d_2^{cs} = \max\{d_{m,n_2,(1-\beta)l}(\frac{r_2}{1-\beta}), d^{ex}_{m,n_2,(1-\beta)l}(\frac{r_2}{1-\beta})\}, \tag{9}$$

where the superscript "cs" denotes channel-splitting.

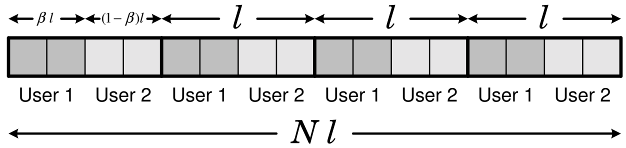


Figure 1: Channel-splitting

For superposition encoding, the channel input is $X = X_1 + X_2$, where X_1 and X_2 have i.i.d. entries $\mathcal{CN}(0,1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ $(0 < p < 1)$ respectively. We use two decoding strategies - joint maximum-likelihood (ML) decoding and naive single-user decoding. In joint ML decoding, user 1 decodes his own message $\hat{i}$ based on the pair (i,j) maximizing $P(Y_1|X_1(i), X_2(j))$ and user 2 decodes his own message $\hat{j}$ based on the pair (i,j) maximizing $P(Y_2|X_1(i), X_2(j))$, where $X_1(i)$ and $X_2(j)$ are the i^{th} and j^{th} codewords for user 1 and user 2 respectively. We can derive the achievable diversity gains using joint ML decoding as

$$d_1^s = \min\{d^{out}_{m,n_1}(r_1), d^{out}_{m,n_1}(r_1+r_2)\} = d^{out}_{m,n_1}(r_1+r_2) \tag{10}$$

$$d_2^s = \min\{p\, d^{out}_{m,n_2}(\frac{r_2}{p}), d^{out}_{m,n_2}(r_1+r_2)\} \tag{11}$$

when the block length $l \geq m + \max(n_1, n_2) - 1$, where the superscript "s" denotes superposition. For the block length $l < m + \max(n_1, n_2) - 1$, the outage diversity gain in (10), (11) is replaced by the random coding diversity gain. In (10), $d^{out}_{m,n_1}(r_1)$ accounts for the error event when user 1 decodes $X_1(i)$ as a wrong codeword, but decodes $X_2(j)$ correctly, which is referred as the type 1 error in [2], and $d^{out}_{m,n_1}(r_1+r_2)$ accounts for the error event when user 1 decodes both messages $X_1(i)$, $X_2(j)$ as wrong codewords, which is referred as the type 3 error in [2]. In (11), $p\, d^{out}_{m,n_2}(\frac{r_2}{p})$ accounts for the type 2 error when user 2 decodes $X_2(j)$ as a wrong codeword, but decodes $X_1(i)$ correctly, and $d^{out}_{m,n_2}(r_1 + r_2)$ accounts for the type 3 error when user 2 decodes both messages $X_1(i)$, $X_2(j)$ as wrong codewords.

In naive single-user decoding, user 1 simply regards user 2's interference as noise. In this case, user 1 can achieve diversity gain $d^{ns}_{m,n_1,l,p}(r_1)$, where $d^{ns}_{m,n_1,l,p}(r_1)$ is called the naive single-user diversity gain and its derivation and exact formula are given in the following subsection. Since user 1 can choose either joint ML decoding or naive single-user decoding, the following diversity gains are achievable

$$d_1^{'s} = \max\{d^{out}_{m,n_1}(r_1 + r_2), d^{ns}_{m,n_1,l,p}(r_1)\} \qquad (12)$$

$$d_2^{'s} = \min\{p\, d^{out}_{m,n_2}(\frac{r_2}{p}), d^{out}_{m,n_2}(r_1 + r_2)\}, \qquad (13)$$

where we assume $l \geq m + \max(n_1, n_2) - 1$.

Similarly, we can exchange the role of user 1 and user 2 in superposition encoding, i.e. we may assume X_2 and X_1 have i.i.d. entries $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ $(0 < p < 1)$ respectively. Thus, the following diversity gains are also achievable

$$d_1^{''s} = \min\{p\, d^{out}_{m,n_1}(\frac{r_1}{p}), d^{out}_{m,n_1}(r_1 + r_2)\} \qquad (14)$$

$$d_2^{''s} = \max\{d^{out}_{m,n_2}(r_1 + r_2), d^{ns}_{m,n_2,l,p}(r_2)\}. \qquad (15)$$

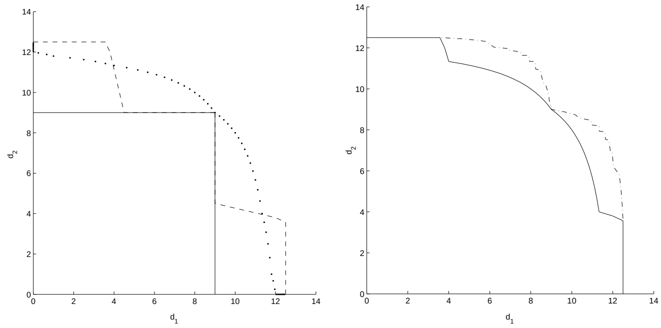
In Fig. 2(a), the solid curve is the boundary of the achievable DGR by superposition using joint ML decoding, and the dashed curve is the boundary of the achievable DGR by superposition using joint ML decoding and naive single-user decoding, which merges with the solid curve at $(d_1, d_2) = (4.5, 9)$ and $(d_1, d_2) = (9, 4.5)$. The dotted curve in Fig. 2(a) is the boundary of the achievable DGR by channel-splitting.

A. Naive Single-user Decoding

We now derive an achievable diversity gain using naive single-user decoding, thus obtaining an explicit expression for $d^{ns}_{m,n,l,p}(r)$. If we use superposition encoding for the MIMO fading broadcast channel, i.e., the channel input $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$, we can write the channel output for user 1 as

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{H}_1(\mathbf{X}_1 + \mathbf{X}_2) + \mathbf{Z}_1, \qquad (16)$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ have i.i.d entries $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ $(0 < p < 1)$ respectively. If we decode user 1's message using naive single-user decoding, i.e.,



(a) (b)

Figure 2: Diversity gain region for $m = n_1 = n_2 = 4, l = 60, r_1 = r_2 = 0.5$ (a) channel-splitting(dotted) and superposition(solid and dashed), (b) union of channel-splitting and superposition (solid), and generalized-superposition (dash-dotted).

user 1 simply treats user 2 as noise, we can derive an achievable diversity gain for user 1. This is equivalent to considering the following MIMO fading side-interference single-user channel

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}}\mathbf{H}(\mathbf{X} + \mathbf{S}) + \mathbf{Z}, \qquad (17)$$

where $\mathbf{H}$ is an $n \times m$ matrix with i.i.d. entries $\mathcal{CN}(0, 1)$, and $\mathbf{Z}$ and $\mathbf{S}$ are $n \times l$ noise and side-interference matrices with i.i.d. entries $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ respectively. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average transmit power at each antenna is one. Define $(x)^+ = \max(x, 0)$ and $\mathbb{R}^n_+$ as the set of real n-vectors with non-negative elements. We summarize the result of the achievable diversity gain of the side-interference channel in the following theorem.

Theorem 2 *For a MIMO fading side-interference channel operated at a multiplexing gain r with m transmit antennas, n receive antennas, the optimal probability of detection error is upper-bounded by*

$$P_e(SNR) \dot{\leq} SNR^{-d^{ns}_{m,n,l,p}(r)}, \qquad (18)$$

where

$$d^{ns}_{m,n,l,p}(r) = \min_{\underline{\alpha} \in \mathcal{B}^c} \Big\{ \sum_{i=1}^{\min(m,n)} (2i - 1 + |m - n|)\alpha_i + l\Big[\sum_{i=1}^{\min(m,n)} (1 - \alpha_i - (p - \alpha_i)^+)^+ - r\Big]\Big\}, \qquad (19)$$

and

$$\mathcal{B} = \Big\{\underline{\alpha} \in \mathbb{R}^{\min(m,n)}_+ \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1 - \alpha_i - (p - \alpha_i)^+)^+ \leq r\Big\} \qquad (20)$$

$$\mathcal{B}^c = \mathbb{R}^{\min(m,n)}_+ - \mathcal{B}. \qquad (21)$$

Proof : See Appendix B. $\square$

IV. Improved Achievable Diversity Gain Region by Generalized-Superposition Encoding

In this section, we introduce a new encoding scheme, generalized-superposition, which includes channel-splitting and superposition as special cases. Before introducing generalized-superposition, we first derive the non-uniform power random coding diversity gain $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$. Consider a random codebook CB with M codewords (see Fig. 3 (a)). Denote C_i the i^{th} codeword with block length l in the codebook CB. Denote $C_i(k)$ the k^{th} element in the codeword C_i. Each random variable $C_i(k)$ is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $1 \le k \le \beta l$, and $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $\beta l+1 \le k \le l$, where $\beta = \frac{q}{l}$ and $0 \le q \le l$, $q \in \mathbb{Z}$. If we plot $E\{|C_i(k)|^2\}$ versus k, we have a function with value $SNR^{-(1-p_1)}$ for $1 \le k \le \beta l$ and with value $SNR^{-(1-p_2)}$ for $\beta l + 1 \le k \le l$ (see Fig. 3(a)).

Extending the derivation of the random coding diversity gain $d_{m,n,l}(r)$ in [3], we can derive the non-uniform power random coding diversity gain $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ for a non-uniform power random codebook CB as

$$d^{np}_{m,n,l,p_1,p_2,\beta}(r) = \min_{\underline{\alpha}\in\mathcal{B}^c} \Big\{ \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i \; + \\ l\big[\beta \sum_{i=1}^{\min(m,n)} (p_1-\alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2-\alpha_i)^+ - r\big]\Big\}, \tag{22}$$

and

$$\mathcal{B} = \Big\{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{\min(m,n)} \ge 0; \\ \beta \sum_{i=1}^{\min(m,n)} (p_1-\alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2-\alpha_i)^+ \le r\Big\} \tag{23}$$

$$\mathcal{B}^c = \mathbb{R}_+^{\min(m,n)} - \mathcal{B}. \tag{24}$$

The derivation for $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ is a straight forward extension of the derivation for $d_{m,n,l}(r)$ and is omitted.

We are now ready to introduce generalized-superposition. In generalized-superposition, we construct two independent random codebooks CB_1 and CB_2. Denote $C_{1,i}$ and $C_{2,j}$ the i^{th} and j^{th} codewords with block length l in codebooks CB_1 and CB_2 respectively. Denote $C_{1,i}(k)$ the k^{th} element in the codeword $C_{1,i}$ and $C_{2,j}(k)$ the k^{th} element in the codeword $C_{2,j}$. Each random variable $C_{1,i}(k)$ is i.i.d. with $\mathcal{CN}(0,1)$ for $1 \le k \le \beta l$, and $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $\beta l + 1 \le k \le l$. Similarly, each random variable $C_{2,j}(k)$ is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $1 \le k \le \beta l$ and $\mathcal{CN}(0,1)$ for $\beta l+1 \le k \le l$ (see Fig. 3(b)). If we plot $E\{|C_{1,i}(k)|^2\}$ versus k, we have a function with value 1 for $1 \le k \le \beta l$ and with value $SNR^{-(1-p_1)}$ for $\beta l + 1 \le k \le l$. Similarly, $E\{|C_{2,j}(k)|^2\}$ is a function with value $SNR^{-(1-p_2)}$ for $1 \le k \le \beta l$ and with

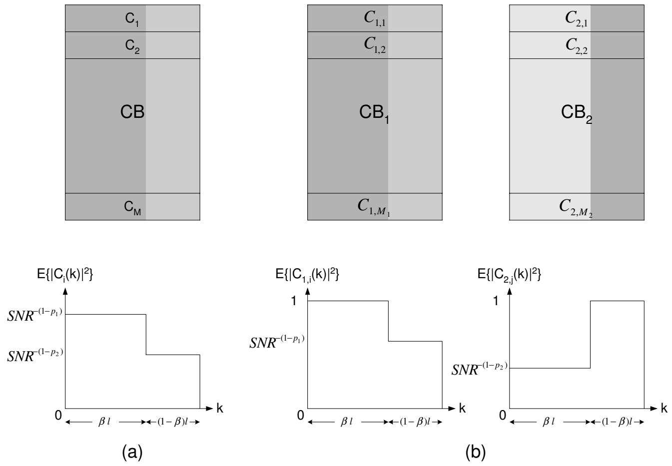


Figure 3: (a) Non-uniform power random codebook, (b) Generalized-superposition.

value 1 for $\beta l+1 \le k \le l$. It is clear that both superposition and channel-splitting are special cases of generalized-superposition.

In the receivers, we use a mixture of joint ML and naive single-user decoding. We summarize the result in the following theorem.

Theorem 3 *For a MIMO fading broadcast channel with m transmit antennas, n_1, n_2 receive antennas for user 1, user 2, and block length $l \ge m + \max(n_1, n_2) - 1$, an achievable DGR_{gs} is given by*

$$DGR_{gs}(r_1, r_2) = \Big\{(d_1, d_2) : \beta = \frac{k}{l}, 0 \le k \le l, k \in \mathbb{Z}, \\ 0 \le p_1 \le 1, 0 \le p_2 \le 1, \\ d_1 \le \max\Big\{ \min\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d^{out}_{m,n_1}(r_1+r_2)\}, \\ d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta})\Big\} \tag{25}$$

$$d_2 \le \max\Big\{ \min\{d^{np}_{m,n_2,l,p_2,1,\beta}(r_2), d^{out}_{m,n_2}(r_1+r_2)\}, \\ d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta})\Big\}\Big\}. \tag{26}$$

$\square$

Note that $d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta})$ should be interpreted as 0 for $\beta = 0$ and $d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta})$ should be interpreted as 0 for $\beta = 1$ in the above theorem, since the diversity gain is zero for any scheme with encoding block length 0.

In Fig. 2(b), the solid curve is the boundary of the union of the channel-splitting and superposition achievable DGRs, and the dash-dotted curve is the boundary of the achievable DGR by generalized-superposition, which merges with the solid curve at $(d_1, d_2) = (3.6, 12.5)$ and $(d_1, d_2) = (12.5, 3.6)$.

V. Outer Bound for Diversity Gain Region

For a MIMO fading broadcast channel, the probability of decoding error for user i can always be lower bounded by the probability of decoding error for user i operating over a point-to-point channel defined by the marginal distribution $P(Y_i|X)$, for $i = 1, 2$. Further, we use the fact that the performance of a broadcast channel depends only on the marginal distributions $P(Y_1|X)$ and $P(Y_2|X)$, not on the joint distribution $P(Y_1, Y_2|X)$. To be specific, consider another broadcast channel with marginal distributions the same as those in the original broadcast channel, i.e., $P'(Y_1|X) = P(Y_1|X)$ and $P'(Y_2|X) = P(Y_2|X)$, but with $P'(Y_1, Y_2|X) \neq P(Y_1, Y_2|X)$ in general. The DGR of this new broadcast channel is the same as the DGR of the original broadcast channel, since the probability of error of each user depends only on the corresponding marginal distributions [5]. If we now allow the two receivers in the new broadcast channel to cooperate, we have a single-user channel, whose probability of error (using an optimal receiver), P'_e, should be less than or equal to the probability of system error P_e in the original broadcast channel. Using the union bound, it is also easy to show that $P_e \leq 2\max\{P_{e1}, P_{e2}\}$, where P_{ei} denotes the probability of error for user i in the original broadcast channel. Collecting all these ideas, we have the following outer bound for the DGR

$$d_1 \leq d^{out}_{m,n_1}(r_1) \tag{27}$$

$$d_2 \leq d^{out}_{m,n_2}(r_2) \tag{28}$$

$$\min\{d_1, d_2\} \leq \max\{d^{out}_{m,n_1}(r_1 + r_2), d^{out}_{m,n_2}(r_1 + r_2)\}. \tag{29}$$

In Fig. 4, the solid curve is the boundary of the DGR inner bound and the dash-dotted curve is the boundary of the DGR outer bound. Two important results are observed in Fig. 4: (i) the DGR inner bound and the DGR outer bound are tight at the right, lower corner and at the left, upper corner; (ii) the DGR inner bound and the DGR outer bound are tight at $d_1 = d_2$. Result (i) implies that the appearance of the second (first) user does not affect the first (second) user since the first (second) user achieves the single-user diversity gain. We summarize these two results in the following theorem.

Theorem 4 *Consider a MIMO fading broadcast channel with block length $l > m + \min(n_1, n_2) - 1$. Either user 1, with a multiplexing gain $r_1 < 1$, can achieve his maximum single-user diversity gain if user 2's multiplexing gain $r_2 < (1 - r_1)(1 - \frac{m+n_1-1}{l})\min(m, n_2)$, or user 2, with a multiplexing gain $r_2 < 1$, can achieve his maximum single-user diversity gain if user 1's multiplexing gain $r_1 < (1 - r_2)(1 - \frac{m+n_2-1}{l})\min(m, n_1)$. If the MIMO fading broadcast channel is symmetric, i.e., $n_1 = n_2$, the DGR inner bound and the DGR outer bound are tight at $d_1 = d_2$.*

Proof:

The result that the DGR inner bound and the DGR outer bound are tight at $d_1 = d_2$ for a symmetric MIMO fading broadcast channel is a direct consequence of the DGR upper bound (29).

Consider user 1 with a multiplexing gain $r_1 < 1$ and define $p_1 = (1 - r_1)(1 - \frac{m+n_1-1}{l})$. For superposition encoding with channel input $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ have i.i.d. entries $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, SNR^{-(1-p_1)})$ respectively, the achievable diversity gain by naive single-user decoding for user 1 is $d^{ns}_{m,n_1,l,p_1}(r_1) = d^{out}_{m,n_1}(r_1)$ (note that p_1 is the largest value this equation $d^{ns}_{m,n_1,l,p_1}(r_1) = d^{out}_{m,n_1}(r_1)$ holds, i.e., $d^{ns}_{m,n_1,l,p}(r_1) < d^{out}_{m,n_1}(r_1)$ for $p_1 < p$). The achievable diversity gain for user 2 is $\min\{p_1\, d^{out}_{m,n_2}(\frac{r_2}{p_1}), d^{out}_{m,n_2}(r_1 + r_2)\}$, so we have a positive diversity gain for user 2 if $r_2 < p_1 \min(m, n_2) = (1 - r_1)(1 - \frac{m+n_1-1}{l})\min(m, n_2)$. The proof for user 2, with a multiplexing gain $r_2 < 1$, achieving his maximum single-user diversity gain if $r_1 < (1 - r_2)(1 - \frac{m+n_2-1}{l})\min(m, n_1)$ is similar. $\square$

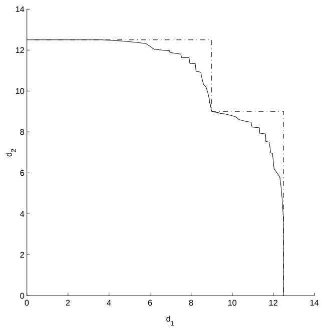


Figure 4: Diversity Gain Region for $m = n_1 = n_2 = 4, l = 60, r_1 = r_2 = 0.5$: inner (solid) and outer (dash-dotted) bounds.

VI. Conclusion

In this work, we introduce the notion of multiplexing gain region and diversity gain region, which are the channel capacity region and the error exponent region at high SNR. We derive the DGR inner and outer bounds for a MIMO fading broadcast channel. We prove that when the data rates are low, either user 1 or user 2 can achieved the single-user diversity gain. For a symmetric MIMO fading broadcast channel, the DGR inner bound and the DGR outer bound are tight at equal diversity gains ($d_1 = d_2$) . The concept of the MGR and the DGR is very general and can be applied to other multi-user channels, such as a MIMO fading multiple access channel.

VII. Appendix

A. Proof of Theorem 1

It is sufficient to prove that (7) is an outer bound for the MGR, since (6) is achieved by superposition encoding (see (12), (13)). Without loss of generality, we may assume that all the nonzero eigenvalues of $\mathbf{H}_1'\mathbf{H}_1$ and $\mathbf{H}_2'\mathbf{H}_2$ are one, where $\mathbf{H}_1'$ and $\mathbf{H}_2'$ are the conjugate transpose of $\mathbf{H}_1$ and $\mathbf{H}_2$. The probability of the event $\mathcal{A} = \{\underline{\lambda} \in [SNR^{-\epsilon} \quad SNR^{\epsilon}]^{\min\{m,n\}}\}$ goes to one as $SNR \to \infty$, where ϵ is any positive constant, $\underline{\lambda}$ is a vector of the nonzero eigenvalues of $\mathbf{H}_1'\mathbf{H}_1$ (or $\mathbf{H}_2'\mathbf{H}_2$), and $n = n_1$ (or

n_2). The integral of multiplexing gains over the range $\mathcal{A}^c$ can be shown to be negligible, so we may assume $SNR^{-\epsilon} \leq \lambda_i \leq SNR^{\epsilon}$, where λ_i is any entry of $\underline{\lambda}$ and $i = 1, 2, \cdots, \min(m, n)$. Since ϵ is any positive constant, we can make ϵ arbitrarily small and assume that $\underline{\lambda}$ is a vector with every entry equal to one.

Based on the assumption that all nonzero eigenvalues are in the value of one, we can consider the following equivalent broadcast channel (after singular value decomposition of the fading matrices)

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{V}_1 I_1 \mathbf{W}_1' \mathbf{X} + \mathbf{Z}_1 \quad (30)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{V}_2 I_2 \mathbf{W}_2' \mathbf{X} + \mathbf{Z}_2. \quad (31)$$

$I_1 = [I_{1,ij}]$ is an $n_1 \times m$ matrix with $I_{1,ii} = 1$ for $1 \leq i \leq \min(m, n_1)$ and $I_{1,ij} = 0$ otherwise, where $I_{1,ij}$ is the element on the i^{th} row and the j^{th} column of I_1. $I_2 = [I_{2,ij}]$ is an $n_2 \times m$ matrix with $I_{2,ii} = 1$ for $1 \leq i \leq \min(m, n_2)$ and $I_{2,ij} = 0$ otherwise, where $I_{2,ij}$ is the element on the i^{th} row and the j^{th} column of I_2. $\mathbf{V}_1$ and $\mathbf{V}_2$ are $n_1 \times n_1$ and $n_2 \times n_2$ unitary random matrices respectively. $\mathbf{W}_1$ and $\mathbf{W}_2$ are $m \times m$ unitary random matrices. $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $\min(m, n_1) \times l$ and $\min(m, n_2) \times l$ matrices with i.i.d. entries $\mathcal{CN}(0, 1)$.

It is well-known that the capacity region of a broadcast channel depends only on the marginal distributions, so we can consider the capacity region of the following broadcast channel

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}} \mathbf{V}_1 I_1 \mathbf{W}_1' \mathbf{X} + \mathbf{Z}_1 \quad (32)$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}} \mathbf{V}_2 I_2 \mathbf{W}_1' \mathbf{X} + \mathbf{Z}_2. \quad (33)$$

If we now assume that the channel matrices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$ are known both at the transmitter and the receivers in the broadcast channel defined in (32), (33), we can consider the following equivalent broadcast channel

$$\tilde{\mathbf{Y}}_1 = \sqrt{\frac{SNR}{m}} I_1 \tilde{\mathbf{X}} + \tilde{\mathbf{Z}}_1 \quad (34)$$

$$\tilde{\mathbf{Y}}_2 = \sqrt{\frac{SNR}{m}} I_2 \tilde{\mathbf{X}} + \tilde{\mathbf{Z}}_2, \quad (35)$$

where $\tilde{\mathbf{X}} = \mathbf{W}_1' \mathbf{X}$, $\tilde{\mathbf{Y}}_1 = \mathbf{V}_1' \mathbf{Y}_1$, $\tilde{\mathbf{Y}}_2 = \mathbf{V}_2' \mathbf{Y}_2$, $\tilde{\mathbf{Z}}_1 = \mathbf{V}_1' \mathbf{Z}_1$, and $\tilde{\mathbf{Z}}_2 = \mathbf{V}_2' \mathbf{Z}_2$. The MGR of the broadcast channel defined in (34), (35) is an outer bound of the MGR of the original fading broadcast channel defined in (1), (2). However, it is easy to see that the MGR of the broadcast channel defined in (34), (35) is exactly (7). This completes the proof.

B. Proof of Theorem 2

At high SNR, we can ignore the integral of the probability of error over the range $\mathbf{H} \notin \mathbb{R}_+^{\min(m,n)}$, so

$$P_e(SNR) \dot{\leq} P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \in \mathcal{B}^c). \quad (36)$$

We prove that $P(\text{error}, \mathbf{H} \in \mathcal{B}^c)$ is upper bounded by $SNR^{-d^{ns}_{m,n,l,p}(r)}$. The proof that $P(\mathbf{H} \in \mathcal{B})$ is also upper bounded by $SNR^{-d^{ns}_{m,n,l,p}(r)}$ is trivial.

Assume $X(0)$, $X(1)$ are two possible transmitted codewords and $\Delta X = X(1) - X(0)$. Suppose $X(0)$ is transmitted, then the probability that a receiver will make a detection error in favor of $X(1)$, conditioned on a certain realization of H, is

$$\begin{aligned} &P(X(0) \to X(1)|H) \\ &= P\left(\frac{SNR}{m} \left\| \frac{1}{2} \left(\frac{SNR^p}{m} HH' + I \right)^{-\frac{1}{2}} H \Delta X \right\|_F^2 \leq ||w||^2 \right) \\ &\leq \exp\left\{ -\frac{SNR}{4m} \left\| \left(\frac{SNR^p}{m} HH' + I \right)^{-\frac{1}{2}} H \Delta X \right\|_F^2 \right\}, \end{aligned} \quad (37)$$

where w is the additive noise with variance 1/2, I is an identity matrix, and $\| \cdot \|_F$ is the Frobenius norm. Average over the ensemble of random codes, we have the average pairwise error probability (PEP) given the channel realization H

$$\begin{aligned} &PEP(H) \leq \\ &|I + \frac{SNR}{2m} (\frac{SNR^p}{m} HH' + I)^{-\frac{1}{2}} HH' (\frac{SNR^p}{m} HH' + I)^{-\frac{1}{2}} |^{-l} \\ &\doteq \prod_{i=1}^{\min(m,n)} SNR^{-l(1-\alpha_i-(p-\alpha_i)^+)^+}, \end{aligned} \quad (38)$$

where $\lambda_i = SNR^{-\alpha_i}$ ($\alpha_1 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0$) and λ_i's are the nonzero eigenvalues of HH'. Apply the union bound and integrate over $\mathcal{B}^c$, we have

$$P(\text{error}, \underline{\alpha} \in \mathcal{B}^c) \dot{\leq} SNR^{-d^{ns}_{m,n,l,p}(r)}. \quad (39)$$

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