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Unique Domination in Cross-Product Graphs

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Abstract

We present a method for approaching general domination and covering questions, tailored for cross-product graphs, and demonstrate it for questions concerning the existence of perfect dominating sets. Here, the definition of cross-product graphs can include families of cube-connected cycles, cube-connected cubes as well as families of hypercubes, tori, etc.

We introduce a condition, unique domination, which is closely related to many other domination properties. This condition survives many variations in the notion of domination and can be explicitly determined for any arbitrary finite graph. Considering the regularities which exist in many families of cross-product graphs, the existence of this condition can often be demonstrated for all the members with only simple methods. Our approach to questions of domination relies on combining proofs of the unique domination condition with other tools. Adding only a simple graph projection technique, we demonstrate short proofs of necessary conditions for the existence of perfect dominating sets in selected examples of cross-product graphs.

Keywords: Coverings, Cross-Product Graphs, Dominating Sets, Unique Domination, Grid Graphs, Perfect Dominating Sets, Tori, Cube-Connected Cycles.

1 Introduction

A multitude of complicated domination and covering questions arise from the applications of graph theory. For example, in the context of parallel network design, there may be replicable resources — such as code libraries — which are prohibitively expensive to place at all individual processor nodes; yet, having each node outside even some short, simple path distance of a resource would cause other problems. The most efficient arrangement of the libraries would cover the whole network with no overlap. We wish to know for which networks such an arrangement is possible. This translates to the question, “For which graphs do perfect dominating sets exist?”

There may be complications, however, as certain simple paths may be more desirable than shorter paths with many twists in some imposed sense of orientation; the edges themselves may be weighted; and some constraints may be implied by bandwidth rather than path distance. Thus we are led to consider the perfect domination problem with general dominating functions. In this context, the purpose of posing such questions is not to evaluate the suitability of a single underlying graph for a proposed network (even if computationally difficult, algorithmic solutions exist); rather, the purpose is to evaluate and compare whole families of related graphs.

In this area, as well as others, few generalizations apply to all the families of graphs under consideration. At best, they could usually be described as having some vague property of regular structure. Part of a shared methodology, however, is the intent to parameterize a large set of graphs with similar structural properties and, then, to reason from the parameters rather than from the graphs themselves — e.g., asymptotic analysis of computational power. For that reason, many interesting families of graphs are frequently specified as the cross-products of simple atomic graphs. Examples include meshes, tori, hypercubes, cube-connected-cycles, etc. Importantly, although each family is mathematically well-defined, the general class of families specified in this way is not. Generalizations of the graph cross-product operation are numerous and ever-increasing.

Yet, since regularities do exist in these structures, the possibility exists for parameterized solutions to the multitude of perfect domination questions. For example, in [6], the existence of perfect dominating sets, generalized to domination with distance d, is completely answered for the family of cube-connected cycles. Regrettably, this and many of the other known results in this area are deeply limited by specifics of the problem(s) under consideration. In fact, it is difficult to separate many of the techniques in [6] for reuse with different families of graphs.

Our intent in this paper is to provide a technique for demonstrating the nonexistence of perfect dominating sets on families of graphs, applicable to general dominating functions.

One of the two main tools we will use is graph projection. This reduces complexity in the cross-product graph in exchange for added complexity in the specification of the domination problem.

The other tool comes from linear algebra. We specify, relative to an arbitrary graph G and some general notion of domination, a domination matrix. We define the *unique domination* property as the invertibility of that matrix. The unique domination property implies that at most one perfect dominating set can exist. In some graphs, the existence of one perfect dominating set implies the existence of another, so for these graphs the unique domination property rules out the existence any perfect dominating set.

For arbitrary graphs, we can say little about whether or not the unique domination condition exists. Yet, exploiting the fact that many cross-product graphs have large automorphism groups, it is often possible to demonstrate this condition with modest algebraic and graph-theoretic tools.

2 Definitions

2.1 Graph Theory Notation

Implicitly, we restrict our discussion to finite, directed, simple graphs.

In the standard manner, when G represents a graph, let the expression $V(G)$ denote its vertex set, and $E(G)$ denote its edge set. An *automorphism* on the graph G is a bijective function $\sigma : V(G) \longrightarrow V(G)$ such that

$$(\sigma(v_1), \sigma(v_2)) \in E(G) \Longleftrightarrow (v_1, v_2) \in E(G).$$

A function f on the vertices of G is said to be *preserved* under an automorphism σ if $\forall v_1, v_2, \ldots \in V(G), f(v_1, v_2, \ldots) = f(\sigma(v_1), \sigma(v_2), \ldots)$. A set or group of automorphisms preserves f if every element preserves f. In addition, a group of automorphisms A is said to be *transitive* or *act transitively* on G if $\forall v_0, v_1 \in V(G), \exists \sigma \in A$ such that $\sigma(v_0) = v_1$. A single automorphism or a set of automorphisms which generates a transitive group is also said to *act transitively* on G.

In a less standard manner, many researchers prefer to separate the terms dominate and cover with the intent of specifying, indirectly, which graph elements are being considered in which context — a vertex *dominates* a set of vertices, but it *covers* a set of edges. Selecting a set of vertices so that each edge is covered exactly once is a different problem than selecting a set of vertices so that each vertex is dominated exactly once. A cycle of length three is sufficient to demonstrate this.

2.2 General Domination Questions

First, we examine vertex domination in the traditional sense:

Definition 2.1 *Given two vertices, $v_i, v_j \in V(G)$, v_i is said to* dominate v_j *if $v_i = v_j$ or there exists an edge from v_i to v_j.*

Definition 2.2 *A set of vertices $W \subseteq V(G)$ is said to be a* dominating set *for G if for each $v \in V(G)$, there exists a vertex $w \in W$ such that w dominates v.*

A set of vertices $W \subseteq V(G)$ is said to be a perfect dominating set *for G if for each $v \in V(G)$, there exists a unique vertex $w \in W$ such that w dominates v.*

An example of perfect domination is provided in Figure 1. We will distinguish this particular notion of domination from the generalizations which follow as the *most basic* notion of domination.

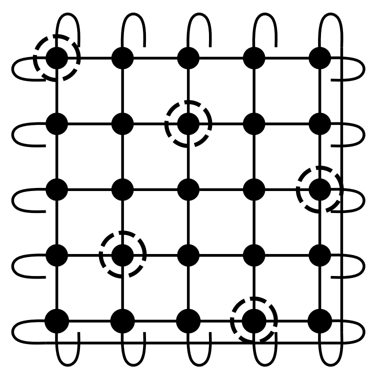

Figure 1: A Perfect Dominating Set for the 5 Torus

There are many simple graph quantities/properties/questions which can be expressed relative to this definition. The *domination number,* for example, is the minimal size of any dominating set. One of our particular motivating questions is "Do perfect dominating sets exist for a given graph G?" A variation which will also concern us is a *multi-domination* question, "Can we select a multi-set (that is, a set with possibly non-distinct elements) of vertices from G so that each vertex in G is dominated exactly m times?"

In addition to this generalization, it will be useful to alter the basic domination problem by replacing the definition of domination itself. This is not without precedent — there are attempts to capture abstractions where one vertex "partially" dominates another (*fractional-domination*) and, separately, attempts to capture irregular covering shapes by defining "is dominated by" as an abstract relation on the vertices of G.

We define a *dominating function* as *any* function f mapping pairs of vertices from a graph G into the real numbers. However, unless stated otherwise, we will assume the range to be non-negative integers. When the range f is further limited to the set $\{0, 1\}$, we will describe it as a *standard dominating function.* With standard dominating functions, we say that a vertex v_i *dominates* a vertex v_j iff $f(v_i, v_j) = 1$.

The domain of a dominating function f is properly a set of ordered triples $\{G, v_1, v_2\}$ where G is a graph and v_1 and v_2 are vertices in G. A dominating function f is *determined by path distance* if it can be defined as $f(G, v_i, v_j) = g \circ d(v_1, v_2)$ where $d(v_1, v_2)$ is the [directed] path distance from v_1 to v_2 in G and g is some well-defined function. We say a dominating function f is *limited by path distance* if there exists a constant c such that $f(G, v_i, v_j) = 0$ whenever $d(v_1, v_2) > c$. As with path distance itself, we will use the expression $f(v_1, v_2)$ whenever G is unambiguous.

For a typical example of a standard dominating function which is only slightly more general than that given in Definition 2.1, consider domination within distance d: $f(v_i, v_j)$ is defined as one when there exists a directed path from v_i to v_j of distance less than or equal to d; zero, otherwise. Relative to this notion, we may describe both dominating sets and perfect dominating sets exactly as in Definition 2.2; however, in order to formalize these sets for arbitrary dominating functions, we must proceed algebraically.

For simplicity, we will assume that G has n vertices indexed by the nonnegative integers; that is, $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$. [1] Let $M_{G,f}$ be the $n \times n$ matrix whose ith entry in the jth column is $f(v_i, v_j)$. We will refer to $M_{G,f}$ as the *domination matrix* for G and f.

Next, let W be any subset of $V(G)$ and let $\vec{W}$ be the n-length column vector whose ith entry is one if $v_i \in W$ and zero otherwise. Extending this notation to weighted subsets (and, hence, multi-sets with elements taken from $V(G)$), let $\vec{W}$ be the n-length column vector whose ith entry is zero if $v_i \notin W$ and the weight of v_i in W otherwise. Define 1_n as the n-length column with all entries set to one, and, for any two matrices/vectors X and Y having identical dimensions, let $X \sqsubseteq Y$ be the relation which is true iff each entry in X is less than or equal to the corresponding entry in Y.

Now, extending Definition 2.2:

Definition 2.3 *A set of vertices $W \subseteq V(G)$ is said to be a* dominating set *relative to f for G if $1_n \sqsubseteq M_{G,f} \cdot \vec{W}$*

A set of vertices $W \subseteq V(G)$ is said to be a perfect dominating set *relative to f for G if $1_n = M_{G,f} \cdot \vec{W}$*

With respect to multi-dominating problems, we obtain similar equations with 1_n replaced by $m \cdot 1_n$ and allowing W to be, as appropriate to the question, a multi-set or a weighted set.

2.3 Cross Product Graphs

For any graphs G and H with vertices $v_1, v_2 \in V(G)$, $w_1, w_2 \in V(H)$, let $\mathcal{P}(G, v_1, v_2, H, w_1, w_2)$ be the property that the vertices w_1 and w_2 are identical and the pair (v_1, v_2) represents an edge in G.

Define $S_{\mathcal{P}}(G, H)$ as

$$\{((v_1, w_1), (v_2, w_2)) : \mathcal{P}(G, v_1, v_2, H, w_1, w_2) \text{ is true}\}.$$

Given a graph G and a graph H, the standard cross-product operation $G \otimes H$ denotes the graph whose vertex set is $V(G) \times V(H)$ and whose edge set is $S_{\mathcal{P}}(G, H) \bigcup S_{\mathcal{P}}(H, G)$.

Using just the standard cross-product operation, one can build up significant classes of graphs. For example, when G and H are both cycles, $G \otimes H$ is a torus graph. When $G_i = L_2$ (the linear graph with two vertices), $1 \leq i \leq d$, the graph

$$G_1 \otimes G_2 \otimes \cdots \otimes G_d \tag{2.4}$$

is the d-dimensional hypercube (the notation Q_d is sometimes used for the hypercube of dimension d).

[1] In general, we will begin our indexing of entries in sets and vectors with 0.

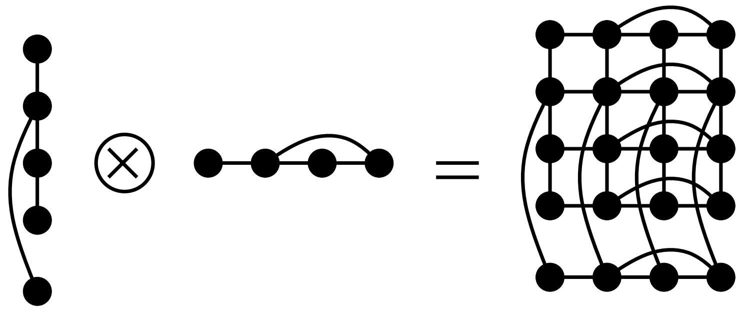

Figure 2: Standard Cross Product Operation

When $G_i = C_{\ell_i}$, the cycle on ℓ_i vertices, the graph indicated by expression 2.4 is a d-dimensional torus, $T_{\ell_1,\ell_2,\ldots,\ell_d}$ or, for brevity, $T_{(d)}$. When $\ell_i = \ell$ for $1 \leq i \leq d$, the graph indicated is a *regular* d-dimensional torus. The expression given for both hypercubes and d-dimensional tori is considered well-defined since, up to graph isomorphism, the standard cross-product operation is both commutative and associative.

Order is important, however. Using the implied order of operations with the natural indices of the atomic graphs, we find d-tuples which, very naturally, index the vertices of the cross-product graph. Moreover, we also note that, in a single cross-product operation, the edges contributed by $S_{\mathcal{P}}(G, H)$ are distinct from those contributed by $S_{\mathcal{P}}(H, G)$. Extending this observation in the obvious way, the order of operations in an expression for a cross-product graph allows us to give a well-defined label from $\{1, 2, \ldots, d\}$ to each edge. Informally, we say that the resulting edge sets lie along different dimensions, and the labeling indicates the dimension along which a given edge lies.

This labeling of the edges along dimensions is a significant step in generalizing the cross-product operation in useful ways. As an example, suppose that G is the hypercube of dimension d and that H is the cycle of length d. Let $\mathcal{D}_e$ be the intuitive mapping of edges in G onto the dimensions of G and let $\mathcal{D}_v$ be the function mapping the of vertices in H to their natural indices. Note that, for the given G and H, the range of $\mathcal{D}_e$ is exactly the same as that of $\mathcal{D}_v$, and let $\mathcal{Q}(G, v_1, v_2, H, w_1, w_2)$ be the property that both $\mathcal{P}(G, v_1, v_2, H, w_1, w_2)$ holds and that $\mathcal{D}_e((v_1, v_2)) = \mathcal{D}_v(w_1)$. With this notation, the *cube-connected cycle* of order d is the graph whose vertex set is $V(G) \times V(H)$ and whose edge set is $S_{\mathcal{Q}}(G, H) \bigcup S_{\mathcal{P}}(H, G)$.

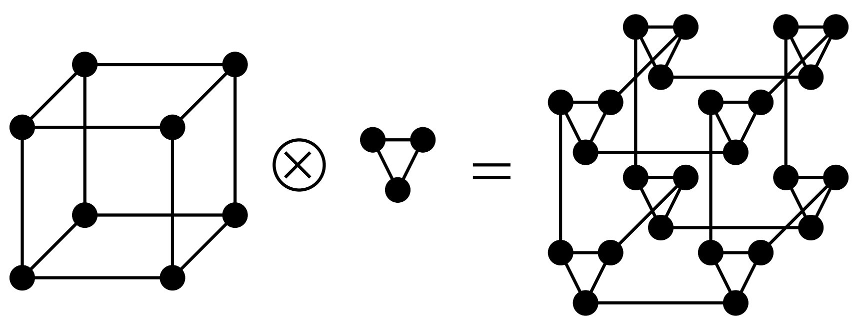

Figure 3: A Nonstandard Cross Product Operation

Continuing in this manner, we could develop further formalizations of interesting cross-product graphs: torus-connected cycles, cube-connected cubes, and, with a little thought, cube-connected cube-connected cycles. In fact, we could even attempt to standardize the *connection* operation. We will not do so here. The important point is that the significant kinds of graphs built from cross-product operations for general purposes — meaning those which are widely-studied and/or likely to come under study for applications (such as computer-related structures) — have common abstractable qualities. They are all constructed from extremely simple atomic graphs (linear graphs and cycles of arbitrary length), and each of these atomic graphs has at least one automorphism; with L_2 and cycles, in fact, a transitive automorphism exists. Moreover, reasonable cross-product operations on these atomic graphs result in graphs with related automorphisms.

3 Unique Domination

We make the following definition:

Definition 3.1 *A graph G has the* unique domination *condition relative to some dominating function f iff $M_{G,f}$ is invertible.*

Note that this definition is independent of the indexing of vertices in G.

The existence of the unique domination condition for a graph G relative to the dominating function f is equivalent to the existence of a unique solution for the real vector X in the following equation:

$$1_n \quad = \quad M_{G,f} \cdot X \tag{3.2}$$

3.1 General Observations

The existence of a perfect dominating set is equivalent to stating that there exists a set W such that $\vec{W} = X$ satisfies Equation 3.2. We observe:

Proposition 3.3 *If a graph G has the unique domination condition, then the number of distinct perfect dominating sets for G is at most one.*

Proposition 3.4 *If a graph G has the unique domination condition, then G has a perfect dominating set (in the simple sense of a subset of vertices) iff $1_n \cdot {M_{G,f}}^{-1}$ has entries from the set $\{0, 1\}$.*

Proposition 3.5 *If a graph G has the unique domination condition, and given a multi-domination problem where each vertex is required to dominated exactly m times and W is allowed to be a multi-set, this problem has a solution iff $m1_n \cdot {M_{G,f}}^{-1}$ has nonnegative integer entries.*

3.2 Proving Unique Domination

Proving the existence of the unique domination condition for a single graph with a single dominating function can be done immediately by calculating the determinant. A more interesting question is, “How does one prove invertibility over an entire family of graphs?” For the families of cross-product graphs, the automorphisms will be of central concern.

To begin, we examine a general method for deciding invertibility on a particular, well-studied class of matrices — circulant matrices. Recall that an $n \times n$ matrix M is *circulant* if $i_1 - i_2 \equiv j_1 - j_2 \pmod n$ implies that $M_{i_1,j_1} = M_{i_2,j_2}$.

As shown in figure 4, a circulant matrix is simply an arrangement of values within the matrix such that each successive row is the previous row “shifted” by one.

$$\begin{bmatrix} A & B & C & \cdots & X & Y & Z \\ Z & A & B & \cdots & W & X & Y \\ Y & Z & A & \cdots & V & W & X \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D & E & F & \cdots & A & B & C \\ C & D & E & \cdots & Z & A & B \\ B & C & D & \cdots & Y & Z & A \end{bmatrix}$$

Figure 4: Example of a Circulant Matrix

Demonstrating that a matrix is circulant is not difficult; however, in order to borrow the appropriate theorem from abstract algebra concerning its invertibility, we will require a bit of notation: For any circulant $n \times n$ matrix M, let R be the first row vector (given that a matrix M is circulant, the entire matrix can be uniquely specified by the entries from any given row); let R_i represent the value of the ith entry in R. Now, define p_R as the complex function

$$p_R(z) \quad = \quad \sum_{i=0}^{n-1} R_i z^i.$$

The function p_R is called the *representer* of the circulant matrix M. Also, let ω_n be $\exp(\frac{2\pi i}{n})$, a complex root of unity.

Our reference for the following theorem is page 75 of Davis [3]; within that source, it is suggested that problems concerning the invertibility of circulant matrices may have been originally posed and solved by Catalan:

Theorem 3.6 *If M is a circulant $n \times n$ matrix with representer p_R, then*

$$\textit{det}\ M \quad = \quad \prod_{j=0}^{n-1} p_R(\omega_n^j).$$

Corollary 3.7 *If M is a circulant $n \times n$ matrix, p_R the representer of M, then M is invertible iff $p_R(\omega_n^j) \neq 0$ for all $j \in \{0, 1, \ldots n-1\}$.*

Equivalently, M is invertible iff $p_R(z)$ has no solutions among the complex nth roots of unity.

The following proposition allows us to apply the results about circulant matrices to a broad class of graphs:

Proposition 3.8 *Let G be a graph with dominating function f. If there exists a transitive automorphism σ of G which preserves f, then there exists an indexing of the vertices such that $M_{G,f}$ is circulant.*

Proof: Let v_0 be any vertex. Index the remaining vertices by setting $v_i = \sigma^{(i)}(v_0)$. Suppose $i_1 - i_2 \equiv j_1 - j_2 \pmod{n}$, and set $k = i_2 - i_1 \bmod n$. Clearly, the order of σ is n. Then

$$\begin{aligned} f(v_{i_1}, v_{j_1}) &= f(\sigma^{(i_1)}(v_0), \sigma^{(j_1)}(v_0)) \\ &= f(\sigma^{(i_1+k)}(v_0), \sigma^{(j_1+k)}(v_0)) \\ &= f(\sigma^{(i_2)}(v_0), \sigma^{(j_2)}(v_0)) \\ &= f(v_{i_2}, v_{j_2}) \end{aligned}$$

The definition of the domination matrix completes the proof. □

Once we have that shown that a particular circulant matrix M is invertible, we know that, for a fixed column vector Y and variable column vector X, the equation $Y = M \cdot X$ has exactly one solution. Calculating M^{-1} in order to determine X is not necessary when Y has identical entries. The following lemma will apply. We omit the trivial proof.

Lemma 3.9 *Let M be a non-zero $n \times n$ matrix such that the sum of the all the entries in any row of M is a constant r. Then the equation $m1_n = M \cdot X$ has a solution for X where $X = \frac{m}{r} 1_n$.*

3.3 Examples for Cycles

For our first example, we examine domination within path distance d (as defined in Section 2.2) on simple cycles, and let f_d denote the assumed dominating function. To avoid a triviality, assume that C_n, the cycle of length n, is such that $n \geq 2d + 1$.

Note that the usual indexing of $V(C_n)$, with edges between v_i and v_j iff $|i - j| = 1 \bmod n$ suggests the transitive automorphism (C_n may have more than one), σ such that $v_i = \sigma^{(i)}(v_0)$. Hence, even without re-indexing the vertices, the the matrix M_{C_n, f_d} is circulant. We find that R, the first row of this domination matrix, has $2d + 1$ consecutive nonzero entries (with some allowance for wrap-around).

Without proof, we note the following:

Lemma 3.10 *The function $p_R(z) = \sum_{i=0}^{p-1} z^i$ has a zero among the nth roots of unity iff p divides n.*

Combining this with Corollary 3.7, we obtain the following corollary:

Corollary 3.11 *Let $n \geq 2d + 1$. Then C_n has the unique domination property relative to f_d iff $2d + 1$ does not divide n.*

The perfect domination problem is trivial on C_n; however, to illustrate how unique domination can be applied to perfect domination questions, we present a new proof:

If $n \geq 2d + 1$, and n not divisible by $2d + 1$, then $1_n = M_{C_n, f_d} \cdot X$ has exactly one solution for X. So by Lemma 3.9, it follows that every entry of X must be $\frac{1}{2d+1}$ — not an integer value. Thus, there exists no subset W of $V(C_n)$ such that $\vec{W} = X$; so there exists no perfect dominating set.

We proceed similarly with nonstandard dominating functions. Again we examine the n-length cycle C_n, and, this time, we define the dominating function

$$f_s(v_i, v_j) = \begin{cases} 3 & \text{if } v_i = v_j \\ 1 & \text{if } d(v_i, v_j) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

(This function will also be used in Section 4.)

The relevant polynomial $z^2 + 3z + 1$ has real roots, both of magnitude greater than one; it follows that the matrix M_{C_n,f_s} is invertible. Immediately, the unique domination condition exists, and reasoning in a similar manner as before, we find the only solution to $m1_n = M_{C_n,f_s} \cdot X$ in X has identical entries, $\frac{m}{5}$.

Lemma 3.13 *For the notion of domination provided by f_s, every cycle has the unique domination condition.*

4 The Projection Technique

By a *projection*, we mean only a surjective mapping between finite sets, particularly one related to graphs. Specifically, let G and H be arbitrary indexed graphs, a projection ρ of the vertices of G onto the vertices of H induces a projection ρ' of the indices of $V(G)$ onto the indices of $V(H)$ and vice-versa.

Definition 4.1 *A projection ρ from $V(G)$ to $V(H)$ is* balanced *if there exists a constant D_ρ such that, for all $w \in V(H)$, $D_\rho = |\rho^{-1}(w)|$. The constant D_ρ is the* depth *of the projection.*

Definition 4.2 *Given a projection ρ from $V(G)$ to $V(H)$ and a dominating function f for G, and suppose that for all $w_1, w_2 \in V(H)$,*

$$c_{w_1,w_2} = \sum_{v_2 \in \rho^{-1}(w_2)} f(v_1, v_2)$$

is constant for all $v_1 \in \rho^{-1}(w_1)$. Define the projected dominating function *$\rho \odot f$ for H by $\rho \odot f(w_1, w_2) = c_{w_1,w_2}$.*

The critical proposition for applying our technique to questions of perfect domination is the following:

Proposition 4.3 *Let G and H be indexed graphs, and let ρ be a balanced projection of G onto H with depth D_ρ. Let f be a dominating function for G and suppose that $\rho \odot f$ is well-defined.*

If the equation $1_n = M_{G,f} \cdot X$ has a solution for X with integer entries, then $D_\rho 1_{\frac{n}{D_\rho}} = M_{H,\rho\odot f} \cdot X$ also has a solution for X with integer entries.

Proof: Given a solution for $1_n = M_{G,f}X$ for X, a vector of length $|V(G)|$, we use ρ' to form a vector of length $|V(H)|$ solving $D_\rho 1_{\frac{n}{D_\rho}} = M_{H,\rho\odot f}X$. Explicitly, we construct the ith entry in the new vector by taking the sum over all $j \in {\rho'}^{-1}(i)$ of the jth entries in original. Showing that this new vector is a proper solution for desired equation is trivial. □

Corollary 4.4 *Let G and H be indexed graphs, ρ be a balanced projection of G onto H, f be a dominating function of G, and suppose that $\rho \odot f$ is well-defined.*

If H does not have a perfect dominating set with respect to the multi-domination problem implied by $D_\rho 1_{\frac{n}{D_\rho}} = M_{H,\rho\odot f}X$, then G does not have a perfect dominating set relative to the problem implied by $1_n = M_{G,f}X$.

We illustrate the application of Corollary 4.4 with an example: Consider the most basic notion of domination and let f_b represent the corresponding dominating function. Also, given the torus T_{ℓ_1,ℓ_2}, allow us to informally index the vertices in T_{ℓ_1,ℓ_2} by "row" and "column" positions (without loss of generality, let ℓ_1 be the number of columns). We note that, if $\ell_1 > 2$, $\ell_2 > 2$, a vertex dominates three vertices within the same column and one vertex in each of the adjacent columns.

The question of whether or not we can place some number of vertices appropriately in each column, without respect to the row position, so that each column is dominated precisely — i.e., three times the number placed in one column plus the number placed in each adjacent column will equal the number of row positions within a column — is, in some sense, an informal version of the projected domination problem. Formalizing, let ρ map vertices of the torus onto those in C_{ℓ_1} by sending $v_{i,j}$ to v_i. The depth of this projection is clearly ℓ_2, and we see that any row vector of $M_{T_{\ell_1,\ell_2},f_b}$, arranged in the manner of the torus has exactly this pattern of nonzero entries:

$$\begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array}$$

Summing along rows, we obtain this pattern: $1\ 3\ 1$, and noting the pattern is identical for every vertex in the T_{ℓ_1,ℓ_2} with the same column index, it follows that the projected dominating function is well-defined (and can to be shown to be the function f_s defined in Equation 3.12).

Proposition 4.5 *The torus T_{ℓ_1,ℓ_2} has a perfect dominating set (in the usual sense) iff both ℓ_1 and ℓ_2 are divisible by 5.*

Proof: When 5 does divide both ℓ_1 and ℓ_2, the set

$$\{v_{i,j} \in T_{\ell_1,\ell_2} : i + 2j \equiv 0 \pmod 5\}$$

can be demonstrated to be a perfect dominating set.

Now, project the torus onto either constituent cycle in the manner implied above. WLOG, the resulting multi-domination problem, stated in algebraic form is

$$\ell_1 Y_{C_{\ell_2}} = M_{C_{\ell_2},f_s} X.$$

By Lemma 3.13, the unique domination condition exists for the cycle C_{ℓ_2} with respect to f_s. By Lemma 3.9, the above multi-domination problem has a solution with integer entries iff $\frac{\ell_2}{5}$ is an integer. Corollary 4.4 completes the proof; ℓ_1 not divisible by 5 implies that T_{ℓ_1,ℓ_2} has no perfect dominating set. □

5 Application of the Techniques

Many of the specific results we demonstrate here may be strictly weaker than what is generally known and/or what the authors can show. Our intention is only to provide an example of a more general criterion for demonstrating the unique domination condition and give evidence of its value in conjunction with the the projection technique. Particularly, we will focus our demonstration on the d-dimensional torus graph, $T_{(d)} = T_{\ell_1,\ell_2,\ldots,\ell_d}$ (where the number of vertices is $n = \prod_{i=1}^{d} \ell_i$).

The first step in our approach to perfect domination questions for a given family of graphs is to tailor a result similar to Theorem 3.6 and Corollary 3.7 to provide criteria for unique domination. For tori, we illustrate this first step, finding conditions necessary and sufficient for unique domination, using the basic orthogonal automorphisms known to exist within that family of graphs.

For the second step, we use the principle of projection to take a domination question on an abstract cross-product graph H, exemplified by tori, (where H is constructed from *constituent* graphs $G_1, G_2, \ldots, G_d$; e.g., $H = G_1 \otimes G_2 \otimes G_3 \cdots$) and produce a different domination question on simpler graphs, typically combinations of the constituent graphs, where the simpler graphs are isomorphic to tori.

5.1 An Unique Domination Criteria for Tori

Projecting onto cycles allows us to use the results of Corollary 3.7. However, when projecting to tori, the domination matrix for the projected graph will not necessarily be circulant. Reducing only to cycles will preclude the most interesting results concerning the parameters.

For the general case, we must adapt Theorem 3.6 to apply to a wider notion of circulant properties; we define the class of matrices recursively:

Definition 5.1 *A matrix M is* meta-circulant[2] *if it can be decomposed into a circulant pattern of blocks strictly smaller than M such that each block is either circulant or meta-circulant.*

$$\begin{bmatrix} \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} & \begin{bmatrix} D & E & F \\ F & D & E \\ E & F & D \end{bmatrix} & \begin{bmatrix} G & H & I \\ I & G & H \\ H & I & G \end{bmatrix} & \begin{bmatrix} J & K & L \\ L & J & K \\ K & L & J \end{bmatrix} \\ \begin{bmatrix} J & K & L \\ L & J & K \\ K & L & J \end{bmatrix} & \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} & \begin{bmatrix} D & E & F \\ F & D & E \\ E & F & D \end{bmatrix} & \begin{bmatrix} G & H & I \\ I & G & H \\ H & I & G \end{bmatrix} \\ \begin{bmatrix} G & H & I \\ I & G & H \\ H & I & G \end{bmatrix} & \begin{bmatrix} J & K & L \\ L & J & K \\ K & L & J \end{bmatrix} & \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} & \begin{bmatrix} D & E & F \\ F & D & E \\ E & F & D \end{bmatrix} \\ \begin{bmatrix} D & E & F \\ F & D & E \\ E & F & D \end{bmatrix} & \begin{bmatrix} G & H & I \\ I & G & H \\ H & I & G \end{bmatrix} & \begin{bmatrix} J & K & L \\ L & J & K \\ K & L & J \end{bmatrix} & \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} \end{bmatrix}$$

Figure 5: Example of a Meta-Circulant Matrix

In order the establish the link between this class of matrices and tori, we will require some further notation relative to the a d-dimensional torus $T_{(d)} = T_{\ell_1, \ell_2, \ldots, \ell_d}$ where the number of vertices is $n = \prod_{i=1}^{d} \ell_i$.

Let μ be the bijection from $\{0, 1, 2, \ldots, n-1\}$ onto the natural indices of $T_{(d)}$ (d-tuples) in lexicographical order and let $\mu_k(i)$ be the k entry of $\mu(i)$; also, for the same torus, let ϕ_k be the circulant translation which sends the d-length index $\{i_1, i_2, \ldots, i_{k-1}, i_k, i_{k+1} \ldots i_d\}$ to $\{i_1, i_2, \ldots, i_{k-1}, i_k + 1 \bmod \ell_k, i_{k+1} \ldots i_d\}$.

For any $k \in \{1, 2, \ldots, d\}$, denote the automorphism of $T_{(d)}$ which maps the vertex with index $\vec{i}$ onto the vertex with index $\phi_k \vec{i}$ as the kth *basic orthogonal automorphism.* The set generates a group, and,

[2] The authors are unaware of any published reference to this class of matrices, but suspect the resulting theorem, or a variation of it, may be well-known.

without proof, we may index the elements of this group in the same manner as vertices from the torus. Define σ_j as the automorphism which sends the vertex at location $\{i_1, i_2, \ldots, i_d\}$ to the vertex at location $\{i_1 + \mu_1(j) \bmod n, i_2 + \mu_2(j) \bmod n, \ldots, \ldots i_d + \mu_d(j)\}$.

Definition 5.2 *For any $n \times n$ meta-circulant matrix M, a d-dimensional torus $T_{(d)} = T_{\ell_1,\ell_2,\ldots,\ell_d}$ (where the number of vertices is $n = \prod_{i=1}^{d} \ell_i$) is said to* describe the circulant properties *of M, if there exists a function g whose domain is the vertices of $T_{(d)}$ such that $M_{i,j} = g(\sigma_j(v_{\mu(i)}))$.*

Carefully following the subscripts and indices is sufficient to prove both of the following, and we omit the tedious proofs:

Lemma 5.3 *An $n \times n$ matrix M is meta-circulant iff there exists a torus $T_{(d)}$ which describes the circulant properties.*

Lemma 5.4 *Given $T_{(d)}$ together with a dominating function f preserved under the basic set of orthogonal automorphisms, the matrix $M_{T_{(d)},f}$ is meta-circulant and the torus $T_{(d)}$ itself describes the circulant properties.*

There many similarities between meta-circulant matrices and circulant matrices; for example, Lemma 3.9 trivially applies, and complex roots of unity ($\omega_n = \exp(\frac{2\pi i}{n})$) will be part of the expression for the determinant. To shorten the expression for the determinant, define, relative to a torus $T_{(d)}$, the following function:

$$q(i,j) \quad = \quad \prod_{k=1}^{d} \omega_{\ell_k}^{\mu_k(j)\mu_k(i)}.$$

For any meta-circulant $n \times n$ matrix M, let R be the first row vector; let $T_{(d)}$ be any torus which describes the circulant properties of M (by Lemma 5.3 at least one can be found). As before, let R_i represent the value of the ith entry in R.

Now, define p'_R relative to $T_{(d)}$ as the function

$$p'_R(j) \quad = \quad \sum_{i=0}^{n-1} R_i q(i,j)$$

where the function $q(i,j)$ is defined relative to $T_{(d)}$. Given a meta-circulant matrix M and a particular torus $T_{(n)}$ which describes the circulant properties of M, we denote the function, p'_R as the *representer of M relative to $T_{(d)}$*, and we describe the pair, $\{T_{(d)}, p'_R\}$ as a *representer* of M.

Theorem 5.5 *If M is a meta-circulant $n \times n$ matrix and $\{T_{(n)}, p'_R\}$ is any representer of M, then M is invertible iff $p'_R(j) \neq 0$ for all $j \in \{0, 1, \ldots n-1\}$ and*

$$\textit{det } M \quad = \quad \prod_{j=0}^{n-1} p'_R(j).$$

Proof: Let E_j be the column vector of length n whose ith entry is $q(i,j)$. Let E be the $n \times n$ matrix whose jth column is E_j.

The scalar product $p'_R(j)E_j$ is easily shown to equal the matrix product ME_j, and it follows that each $p'_R(j)$ must be an eigenvalue of M with E_j as a corresponding eigenvector. By induction on d, it can be shown that E has rank n, hence, the multi-set $\{p'_R(j) : j \in 0, 1, \ldots, n-1\}$ contains all the eigenvalues in their proper multiplicity. The Theorem follows since the determinant is the product of the eigenvalues. □

The condition that $p'_R(j) = 0$ may be equivalently stated purely in terms of the nth roots of unity (at the cost of brevity) with a polynomial in several variables, p_R, and the following corollary reflects this:

Corollary 5.6 *Let f a dominating function for $T_{(d)}$ which is preserved under the basic orthogonal automorphisms of $T_{(d)}$. Then $T_{(d)}$ has the unique domination condition iff the function*

$$p_R(z_1, z_2, \ldots, z_d) \quad = \quad \sum_{i=0}^{n-1} f(v_{\mu(0)}, v_{\mu(i)}) \prod_{k=1}^{d} z_k^{\mu_k(i)}$$

has no zeros in the set of d-tuples

$$\left\{ \{z_1, z_2, ..., z_d\} : \text{ each } z_i = \omega_{\ell_i}^{j_i} \text{ for some integer } j_i \right\}.$$

Proof: Relative to the given torus, we note that p_R as defined has the property that $p'_R(i) = p_R(\omega_{\ell_1}^{\mu_1(i)}, \omega_{\ell_2}^{\mu_2(i)}, \ldots, \omega_{\ell_d}^{\mu_d(i)})$.[3] The Corollary is then immediate. □

As a triviality, we note that each ℓ_i divides n; hence, if the function p_R in the above corollary has no solutions among d-tuples taken from the nth roots of unity, it will have no solutions in the more restricted set.

5.2 Supporting Material

In the particular case we examine here — the basic notion of domination on the d-dimensional torus — it is immediate that the domination matrix is meta-circulant. In fact, $M_{G,f}$ may be meta-circulant for a given G and f — even though the graph G is neither a torus or a cycle. Applications beyond tori are certainly possible and a proposition analogous to Proposition 3.8 may be useful. In a similar way, it is trivial to demonstrate, for our specific needs here, that all the involved projected dominating functions are well-defined. In more general settings, this is not a step to be overlooked. Thus, we explicitly provide the following supporting material:

Proposition 5.7 *Let G be a graph with dominating function f. If there exists a finite abelian group of automorphisms A which acts transitively on the vertices of G and preserves f, then there exists an indexing of the vertices such that $M_{G,f}$ is meta-circulant.*

Proof: Since A is finite and abelian, it can be written as the direct product of finitely many cyclic groups. The lemma follows from the repeating the steps in the proof of Proposition 3.8. □

[3] This function could be viewed as the *representer* of a meta-circulant matrix M.

Proposition 5.8 *Let G be a graph and let f be a dominating function, such that G has an abelian group of automorphisms A which both acts transitively on the vertices of G and preserves f.*

Suppose A can be written as the direct product of some group A' with cyclic groups $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_d$. where the order of each $\mathcal{C}_i$ is ℓ_i.

Then we can define a balanced projection ρ which sends G onto the torus $T_{(d)}$ such that the projected dominating function $\rho \odot f$ is well-defined.

Proof: We first map $V(G)$ onto the the equivalence classes determined by the orbits of A'. Looking at the action of the generator for each $\mathcal{C}_i$ on representatives of the equivalence classes will provide an indexing mapping the equivalence classes onto the torus. The composition will be ρ. Since path distance is preserved under any automorphism of G, the rest of the lemma follows by examining the preimages of ρ. □

Carefully examining the statement and proof of Proposition 5.8, an arbitrary projection can be shown to induce a well-defined projected dominating function if it is equivalent to the projection described and constructed in the proof.

Now, for d-dimensional tori, the group of automorphisms generated by basic orthogonal automorphisms is is an abelian, transitive automorphism group. Thus, Propositions 3.8 and 5.8 are more than sufficient for our purposes.

5.3 Perfect Domination in Tori

We examine only the most basic notion of domination on the d-dimensional torus graph, $T_{(d)} = T_{\ell_1,\ell_2,\ldots,\ell_d}$, attempting to find necessary and sufficient conditions on the parameters $\ell_1, \ell_2, \ldots, \ell_d$ for the existence of a perfect dominating set. Here, we presuppose that each of the constituent cycles is simple, i.e. $\ell_i \geq 3$, and that $d > 1$.

First, a useful lemma:

Proposition 5.9 *Let the finite graph G have a nonnegative dominating function f preserved under some abelian set of automorphisms A which acts transitively on vertices of G, and let v_0 be a vertex in G. Suppose there exists a vertex v_i such that*

$$f(v_0, v_i) \quad > \quad \sum \{f(v_0, v_j) : i \neq j, v_j \in V(g)\}$$

Then G has the unique domination condition.

Proof: Since A is finite, abelian, and acts transitively, by Proposition 5.7, the vertices of G can indexed so that that $M_{G,f}$ is meta-circulant. Now, if $j < k$, then the sum of any j roots of unity cannot equal, in magnitude, k. Examining the the function p_R relevant to $M_{G,f}$, we can immediately deduce that it has no solutions among any roots of unity. □

The following two lemmas illustrate part of the difference between projecting merely onto cycles and projecting onto tori of smaller dimension.

Lemma 5.10 *Given $T_{(d)}$ with $d \geq 2$, if, for some i in $\{1, 2, \ldots, d\}$, $\frac{n}{\ell_i}$ is not divisible by $2d+1$, then no perfect dominating set exists for $T_{(d)}$.*

Proof: If we project the torus $T_{(d)}$ orthogonally onto any of its constituent cycles, say C_{ℓ_1}, we have a projected dominating function satisfying the conditions of Lemma 5.9. Thus, as in the example in the previous section, since $\frac{n}{\ell_1}$ is the depth of the projection, and the sum of the entries in the first row of domination matrix is $2d+1$, an application of Lemma 3.9 completes the lemma. □

Lemma 5.11 *Given $T_{(d)}$, if, for some $I \subseteq \{1, 2, \ldots, d\}$ such that $d \leq 2|I|$, $\prod_{i \in I} \ell_i$ is not divisible by $2d+1$, then no perfect dominating set exists for $T_{(d)}$.*

Proof: Let $J = \{1, 2, \ldots, d\} \sim I$. We project the torus $T_{(d)}$ orthogonally onto the torus $\bigotimes_{j \in J} C_{\ell_j}$.

Of the $2d+1$ vertices dominated by a single vertex in $T_{(d)}$, only $2|J|$ are not mapped to the same vertex by the projection. Since $d \leq 2i|I|$, the projected dominating function satisfies the conditions of Lemma 5.9.

Proceeding as in the previous lemma, $\prod_{i \in I} \ell_i$ divisible by $2d+1$ is a necessary condition for a perfect dominating set to exist. □

Using the projection onto tori and the meta-circulant condition, we obtain a complete result for $T_{(d)}$ where $2d+1$ is prime.

Proposition 5.12 *When $p = 2d+1$ is prime, the torus $T_{(d)}$ has a perfect dominating set (in the usual sense) iff the length of each dimension is divisible by p.*

Proof: When p does divide the length of each dimension, the set

$$\left\{ v_{i_1, i_2, \ldots, i_d} \in T_{(d)} : \sum_{j=1}^{d} j i_j \equiv 0 \pmod{p} \right\}$$

can be demonstrated to be a perfect dominating set.

Either Lemma 5.10 or Lemma 5.11 is sufficient to show that at least two of lengths of the dimensions must be divisible by p in order for a perfect dominating to exist. Suppose $T_{(d)}$ has some set of dimensions whose length is divisible by p, but not all dimensions have this property.

WLOG, suppose only ℓ_1 and ℓ_2 are divisible by p. We project orthogonally onto T_{ℓ_1, ℓ_2}, and then, letting p^{k_i} be the greatest power of p which divides ℓ_i, we project T_{ℓ_1, ℓ_2} onto $T_{p^{k_1}, p^{k_2}}$ by the function which sends the vertex v with indices i, j to the vertex with indices $i \bmod p^{k_1}, j \bmod p^{k_1}$.

In the general case, the relevant function $p_R(z_1, z_2, \ldots)$ from Corollary 5.6 will have nonnegative integer coefficients summing to p, with one coefficient larger than one. Thus, the equation $p_f(z_1, z_2, \ldots) = 0$ does have for z_i in the roots of unity, but does not have solutions for z_i in the set $\{\omega_{p^i}^j : i, j \text{ integers}\}$. The complete factorization of $z^{p^k} - 1$ is sufficient to show this.

Hence, checking the depths of the involved projections and applying Lemma 3.9 is sufficient to reveal the contradiction. □

Remark 5.13 *The quality of Lemma 5.10 relative to that of Lemma 5.11 is apparent when considering primality conditions on the dimensions. Simply, a proof of Proposition 5.12 using only the unique domination criterion for the cycle (examining only circulant matrices) is not immediate, although it may be possible. By comparison, using the criterion for tori (looking at meta-circulant matrices) seems to provide a* much *shorter, clearer proof.*

Continuing in the manner demonstrated, we can apply the criterion for the unique domination condition to investigate $T_{(d)}$ where $2d+1$ is composite, extend the reasoning to derive results concerning the number of distinct perfect dominating sets, examine domination within distance d on both tori and hypercubes, etc.

5.4 Unique Domination in Tori

Tori often do not possess the unique domination property even when it is easily demonstrated that the perfect dominating sets do not exist. In spite of this, some conditions for the unique domination property with standard dominating functions may be found.

We begin with results for more general graphs than just tori:

Proposition 5.14 *Let G be a graph with n vertices, let f be a standard domination function, and let A be an abelian transitive automorphism group which preserves f. Also suppose that, under f, each vertex dominates exactly 5 vertices.*

If 5 does not divide n and 6 does not divide n, then G has the unique domination property.

Proof: From Proposition 5.7, the matrix $M_{G,f}$ is meta-circulant. The first row of this matrix has exactly five nonzero entries, each equal to one. From Corollary 5.6, the relevant function P_r will be the sum of five not-necessarily-distinct roots of unity.

From [5], there are exactly two ways in which five nth roots of unity can sum to zero: First, the five roots are evenly spaced on the unit circle, implying n is divisible by five. Second, when two of the roots are evenly spaced and the three remaining roots are evenly spaced, implying that n is divisible by six. The proposition follows. □

Proposition 5.15 *Let G be a graph with n vertices, let f be a standard domination function, and let A be an abelian transitive automorphism group which preserves f. Also suppose that, under f, each vertex dominates exactly 7 vertices.*

If 7 does not divide n, 10 does not divide n, and 12 does not divide n, then G has the unique domination property.

Proof: The proof is similar to that of Proposition 5.14, again we consider the possible combinations of the nth roots of unity which by [5] must be evenly spaced in order to sum to zero. □

Now, as corollaries, with the most basic notion of domination (that given in Definition 2.1, we obtain:

Corollary 5.16 *Let T be the two-dimensional torus on n vertices. If 5 does not divide n and 6 does not divide n, then T has the unique domination property.*

Corollary 5.17 *Let T be the three-dimensional torus on n vertices. If 5 does not divide n, 10 does not divide n, and 12 does not divide n, then T has the unique domination property.*

Remark 5.18 *From [5], there are suggestions that continuing to look at conditions such as "each vertex dominates exactly k vertices" for increasing k will result in weaker and weaker conditions.*

Moreover, examining the conditions closely in Corollaries 5.17 and 5.17, it is not a simple matter to strengthen them. An immediate example: For the most basic notion of domination, the two dimensional torus $T_{3,4}$ does have the unique domination property. Without some deeply extended reasoning, it would appear, at best, difficult to develop a result like Proposition 4.5 without relying on something equivalent to a projection and the corresponding alteration of the dominating function.

6 Extensions and Conclusions

The unique domination condition for a general cross-product graph G and dominating function f, particularly the criteria which demonstrate its existence, appear to be valuable in a variety of domination-related problems. Ongoing work by the authors includes developing the criteria for unique domination in various nonstandard cross-product graphs as well as considering applications to deeper questions concerning domination.

To a very limited extent, we provide a number of observations for extending our approach into related questions. While we cannot expect to adequately represent every area of shared concern, we do hope to make a few connections between our basic methods and the ongoing research of others.

6.1 Reliance on Transitive Automorphisms

Even the simple technique used for proving the existence of the unique domination property on certain cycles made strong use of an indexing of the vertices through a transitive automorphism. The critical problem for a common cross-product graph, the mesh, is that the atomic graphs (linear graphs) have no transitive automorphism.

However, linear graphs, here denoted L_n, are "almost" cycles. (We assume the vertices are indexed in usual manner with edges between v_i and v_j iff $|i-j|=1$.) Looking at dominating functions f which are both determined by and limited by path distance, the matrix $M_{L_n,f}$ has the property that it is a sub-matrix of $M_{C_{n+m},f}$ for some constant m and variable n.

Properly, the matrix $M_{L_n,f}$ is *Toeplitz*. Recall that a matrix M is Toeplitz if $\forall i,j \in \{0,1,\ldots,n-2\}, M_{i,j} = M_{i+1,j+1}$. Showing that the domination matrix for C_{n+m} is invertible first may be easier than directly approaching the dominating matrix for L_n. As an example, with the dominating function f_s defined in Equation 3.12, the invertibility of M_{C_{n+1},f_s} can be used to derive the invertibility of the matrix M_{L_n,f_s}. The proof is not difficult.

Unfortunately, when extending these results toward results concerning the existence of perfect dominating sets, the usefulness of Lemma 3.9 is lost.

6.2 Reliance on Commutative Automorphisms

For some families of graphs produced by nonstandard cross-product operations — such as the cube-connected cycle or torus-connected cycle — there is an obvious group of transitive automorphisms; but this group is not abelian.

Developing an unique domination criterion similar to that which relied on meta-circulant matrices is not as difficult as it may appear. With the above two examples, we find the appropriate matrix is decomposable into blocks where each block is meta-circulant (and blocks within each diagonal are related to each other by a rotation property).

The class of graphs implied by these matrices is vast; however, it does not include many of the more esoteric graphs proposed for computer networks (e.g. twisted cube-connected cubes), nor does it suggest simple, appropriate projections. As noted earlier, this is part of ongoing research by the authors.

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