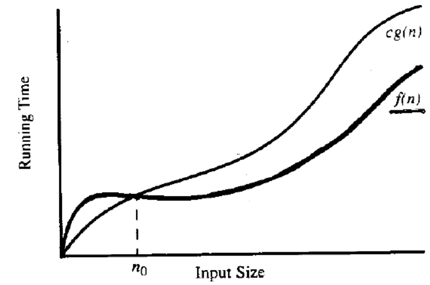
eecs281 DATA STRUCTURES AND ALGORITHMS

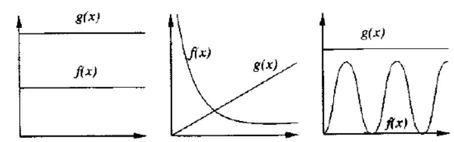
Lecture 3: Algorithm Analysis
Foundational Data Structures
(Review of Some 280 Material)

Asymptotic Algorithm Analysis

An algorithm with complexity $f(n)$ is said to be not slower than another algorithm with complexity $g(n)$ if $f(n)$ is bounded by $g(n)$ for large n



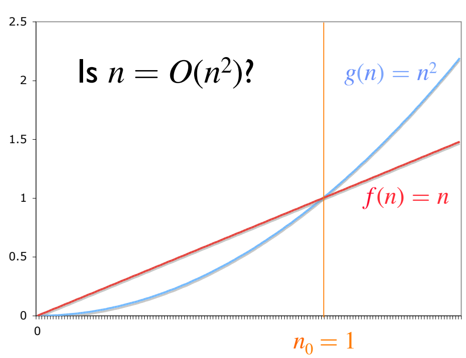
Commonly written as
$f(n) = O(g(n))$
(read: $f(n)$ is big-Oh $g(n)$),
a.k.a. the asymptotic (or big-Oh) notation



Big-Oh – Definition

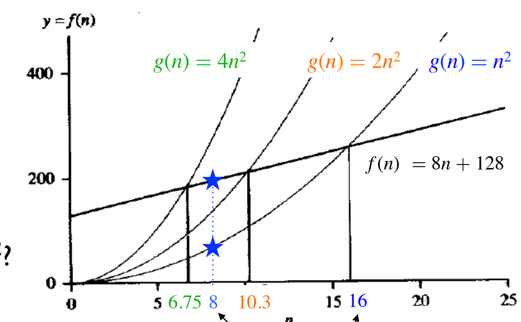
$f(n) = O(g(n))$ if and only if there are constants

$$\left.\begin{array}{l} c > 0 \\ n_0 \geq 0 \end{array}\right\} \text{such that } f(n) \leq c\ g(n) \text{ whenever } n \geq n_0$$



Big-Oh – Example

Let
$f(n) = 8n + 128$
$g(n) = n^2$
Is $f(n) = O(g(n))$?
Is $8n + 128 \leq c\ n^2$?



Let $c = 1$, clearly, for $n = 8$, $f(n) > g(n)$
At what value of n_0 is $g(n) > f(n), \forall n \geq n_0$?

How about for $c = 2$ and $c = 4$?

Big-Oh – Definition

As long as there is a $c > 0$, and $n_0 \geq 0$ such that $c \cdot g(n) \geq f(n)$ for all $n \geq n_0$, we say that $f(n) = O(g(n))$

In this example, $8n + 128 = O(n^2)$

Mathematically:

$f(n) = O(g(n))$ iff $\exists\, c > 0, n_0 \geq 0 \mid \forall n, n \geq n_0, f(n) \leq c\, g(n)$

$O(g(n)) = \{ f(n) \colon \exists\, c > 0, n_0 \geq 0 \mid \forall n, n \geq n_0, 0 \leq f(n) \leq c\, g(n) \}$

So more accurately, $f(n) \in O(g(n))$

but conveniently people write $f(n) = O(g(n))$,

though NOT $f(n) \leq O(g(n))$

Big-Oh – Definition

In other words, we only care about LARGE n, it doesn't matter what c is

- obviously, c cannot be 10^{100} (one googol, the conjectured upper bound on the number of atoms in the observable universe)!

Also, asymptotically, $n^2 + k = O(n^2)$, k constant (Why?)

Big-Oh: Sufficient (but not necessary) Condition

If $\left[\lim_{n\to\infty} \left(\frac{f(n)}{g(n)} \right) = c < \infty \right]$ then $f(n)$ is $O(g(n))$

$\boxed{\log_2 n = O(2n)?}$

$f(n) = \log_2 n$

$g(n) = 2n$

$\lim_{n\to\infty} \left(\frac{\log n}{2n} \right)$

$= \lim_{n\to\infty} \left(\frac{1}{2n} \right)$

$= 0 = c < \infty$

∞ / ∞

Use L'Hôpital's Rule

$\boxed{\Rightarrow \log_2 n = O(2n)}$

$\boxed{\sin\left(\frac{n}{100}\right) = O(100)?}$

$f(n) = \sin\left(\frac{n}{100}\right)$

$g(n) = 100$

$\lim_{n\to\infty} \left(\frac{\sin\left(\frac{n}{100}\right)}{100} \right)$

Condition does not hold but nevertheless it is true that $f(n) = O(g(n))$

L'Hôpital's Rule

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$

and $\lim_{x\to c} f'(x)/g'(x)$ exists then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Also useful, derivative of log:

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln(b)}$$

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

wikipedia

Log Identities

Identity	Example
$\log_a(xy) = \log_a x + \log_a y$	$\log_2(12) =$
$\log_a(x/y) = \log_a x - \log_a y$	$\log_2(4/3) =$
$\log_a(x^r) = r \log_a x$	$\log_2 8 =$
$\log_a(1/x) = -\log_a x$	$\log_2 1/4 =$
$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$	$\log_7 9 =$
$k = \log_2 n$ iff $2^k = n$	$\log_a a = ?$ $\log_a 1 = ?$

Power Identities

Identity	Example
$a^{(n+m)} = a^n a^m$	$2^5 =$
$a^{(n-m)} = a^n / a^m$	$2^{3-2} =$
$(a^{(n)})^m = a^{nm}$	$(2^2)^3 =$
$a^{-n} = \frac{1}{a^n}$	$2^{-4} =$
$a^{-1} = ?$ $a^0 = ?$ $a^1 = ?$	

Big-Oh: We Can Drop Constants

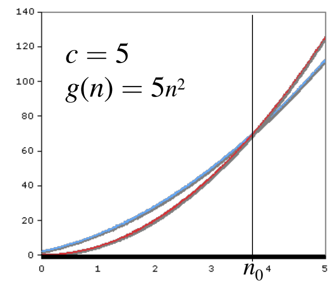
$$3n^2 + 7n + 42 = O(n^2)?$$

$$\boxed{\begin{aligned} f(n) &= 3n^2 + 7n + 42 \\ g(n) &= n^2 \end{aligned}}$$

Definition

$c > 0, n_0 \geq 0$ such that

$f(n) \leq c \cdot g(n), \forall n \geq n_0$



Sufficient Condition

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$$

$$= \lim_{n\to\infty}\left(\frac{3n^2 + 7n + 42}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{6n+7}{2n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{6}{2}\right)$$

Big-Oh – Common Mistakes

Mistake #0: $f(n) = O(g(n)) \Rightarrow f(n) = g(n)$ (NOT)

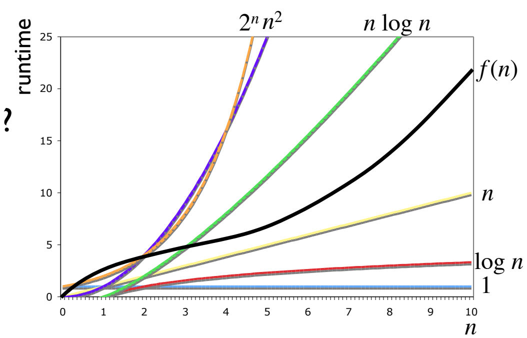
Mistake #1: If $f_1(n) = h(n)$ and $f_2(n) = h(n)$ then $f_1(n) = f_2(n)$; it follows that if $f_1(n) = O(g(n))$ and $f_2(n) = O(g(n))$ means $f_1(n) = f_2(n)$ (NOT)

Mistake #2: $f(n) = O(g(n)) \Rightarrow g(n) = O^{-1}(f(n))$ (NOT)

(There's no $O^{-1}()$!)

How Fast is Your Code?

Is $f(n) = O(2^n)$?
Is $f(n) = O(n^2)$?
Is $f(n) = O(n \log n)$?
Is $f(n) = O(n)$?
Is $f(n) = O(\log n)$?



Let $f(n)$ be the complexity of your code, how fast would you advertise it as?

While $f(n) = O(g(n)) \not\Rightarrow f(n) = g(n)$, you want to pick a $g(n)$ that is as close to $f(n)$ as possible (a "tight" bound)

About Big-Oh

Asymptotic analysis deals with the performance of algorithms for LARGE input sizes

Big-Oh provides a short-hand to express upper bound, it is not an exact notation

- be careful how big c is
- be careful how big n_0 must be

Big-Oh asymptotic analysis is language independent

Big-Oh – Rules

Rule 1: For $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$
$\Rightarrow f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n))$

Example: $f_1(n) = n^3 \in O(n^3), f_2(n) = n^2 \in O(n^2)$
$\Rightarrow f_1(n) + f_2(n) = O(?)$

Rule 2: For $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$
$\Rightarrow f_1(n) * f_2(n) = O(g_1(n)*g_2(n))$

- If your code calls a function within a loop, the complexity of your code is the complexity of the function you call times the loop's complexity

Rule 3: If $f(n) = O(g(n))$ and $g(n) = O(h(n))$
then $f(n) = O(h(n))$

Big-Oh – More Common Mistakes

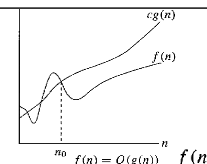
Mistake #3: Let $f(n) = g_1(n)*g_2(n)$
If $f(n) \le cg_1(n)$ where $c = g_2(n)$,
then $f(n) = O(g_1(n))$ (NOT)

Mistake #4: Let $f_1(n) = O(g_1(n)), f_2(n) = O(g_2(n))$,
and $g_1(n) < g_2(n) \Rightarrow f_1(n) < f_2(n)$ (NOT)

Counter-example:

$f_1(n) = ?$
$g_1(n) = ?$
$f_2(n) = ?$
$g_2(n) = ?$

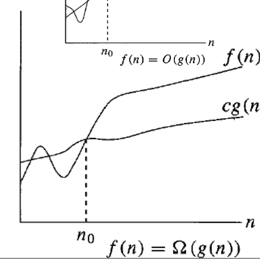
Relatives of Big-Oh



Big-Omega ($\Omega()$): asymptotic lower bound

For $f(n) > 0, \forall n \ge 0, f(n) = \Omega(g(n))$ if
$\exists\, c > 0, n_0 > 0 \mid \forall n, n \ge n_0, f(n) \ge c\, g(n)$

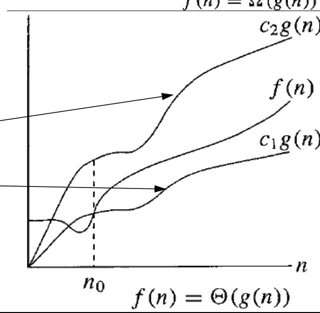
$h_1(n) = O(h_2(n)) \Leftrightarrow h_2(n) = \Omega(h_1(n))$



Big-Theta ($\Theta()$):

$f(n) = \Theta(g(n))$ iff
$f(n) = O(g(n))$ and
$f(n) = \Omega(g(n))$

$f(n)$ grows as fast as $g(n)$



Big-Theta

Does $f(n) = \Theta(g(n)) \Rightarrow g(n) = \Theta(f(n))$?

Does $f(n) = \Theta(g(n)) \Rightarrow f(n) = g(n)$?

Does $f(n) = \Theta(g(n)) \Rightarrow f(n)$ is the same order as $g(n)$?

Relatives of Big-Oh

little-oh ($o()$):

$f(n) = o(g(n))$ if $f(n) = O(g(n))$ but $f(n) \neq \Theta(g(n))$

$f(n) = o(g(n))$ if $\exists\, n_0 > 0 \mid \forall c > 0, \forall n, n \geq n_{0,}\, f(n) \leq c\, g(n)$

In contrast to $O()$, $o()$ is forall $c > 0$, whereas $O()$ only requires there exists $c > 0$; so $O()$ is sloppier than $o()$, which is why we use it more often!

Example: $2n^2 = O(n^2)$ is asymptotically tight, but $2n = O(n^2)$ is not

little-omega ($\omega()$):

$f(n) = \omega(g(n))$ iff $g(n) = o(f(n))$

In the Limit

$O(): f(n) = O(g(n)) \Leftrightarrow f(n) \leq c_1 g(n)$ and $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c_1$

$\Omega(): f(n) = \Omega(g(n)) \Leftrightarrow f(n) \geq c_2 g(n)$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)} \leq c_2$

$\Theta(): f(n) = \Theta(g(n)) \Leftrightarrow$ both $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c_1$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)} \leq c_2$

$o(): f(n) = o(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$\omega(): f(n) = \omega(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

The Common Case: Empirical Performance Evaluation

If $n_0 >$ the common case n, the asymptotic analysis result is not very useful . . . .

To determine the common case performance, given known workload, run empirical performance measurement/evaluation

Note that common case performance is not necessarily the average case performance (Why not?)

Empirical evaluation is also useful for evaluating complex algorithm or large software systems

Experiment Setup

Factors that affect the accuracy of your empirical performance evaluation:

- system speed
- system load
- compiler optimization

What you need:

- workload generator: must generate realistic common cases
- reduce system variability:
  - use the same compiler
  - use the same machine
  - minimize concurrent/background tasks
    - for shared systems, run experiment around the same time of day

Measuring Time

```
struct timeval{
   //--- seconds
   unsigned int tv_sec;

   //--- microseconds
   unsigned int tv_usec;
}
```

```
struct rusage{
   //--- user time used
   struct timeval ru_utime;

   //--- system time used
   struct timeval ru_stime;
   …
}
```

```
#include <iostream>
#include <sys/resource.h>
#include <sys/time.h>

void main(){
    struct rusage startu;
    struct rusage endu;

    getrusage(RUSAGE_SELF, &startu);
    //---- Do computations here
    getrusage(RUSAGE_SELF, &endu);

    double start_sec = start.ru_utime.tv_sec + startu.ru_utime.tv_usec/1000000.0);
    double end_sec  = endu.ru_utime.tv_sec + endu.ru_utime.tv_usec/1000000.0);
    double duration = end_sec - start_sec;
}
```

initialize `rusage` variables

set data of `rusage` variables at start/end of computation

duration = end – start

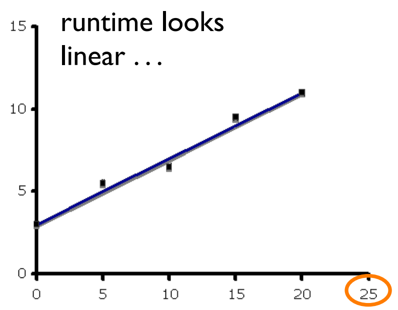
`struct rusage` contains other useful information, e.g., memory usage

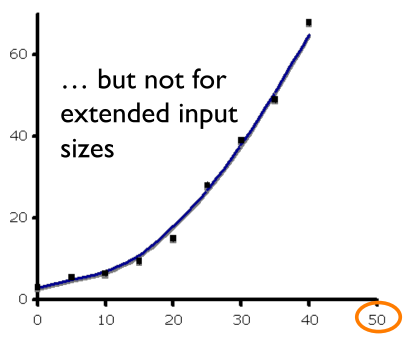
Empirical Results

Repeat experiment several times with the same input and take the average or minimum

Plot algorithm runtimes for varying input sizes

Include a large range to accurately display trend





Analysis vs. Evaluation

When experimental results differ from analysis . . .

- check for correctness in complexity analysis
- check for error in coding
  - extra loop
  - algorithm implemented is different from the one analyzed!
- if no error, experiment may simply have not covered worst case scenario
- external factors, e.g., hardware/software system (performance) bug?

Self-Study Questions

1. Which of these are true? Why?
   $10^{100} = O(1)$
   $3n^4 + 45n^3 = O(n^4)$
   $3^n = O(2^n)$
   $2^n = O(3^n)$
   $45\log(n) + 45n = O(\log(n))$
   $\log(n^2) = O(\log(n))$
   $[\log(n)]^2 = O(\log(n))$
2. Let $f(n) = (n-1)n/2$, $c = n/2$, is $f(n) = O(n)$?
   If so, why? If not, what's the big-Oh of $f(n)$?
3. Is $\log n = O(n)$? Is $\log n = O(n^2)$? Which is a tighter bound?
4. Given 4 consecutive statements: S_1; S_2; S_3; S_4; Let $S_1 = O(\log n)$, $S_2 = O(\log n^3)$, $S_3 = O(n)$, $S_4 = O(3n)$. What is the big-Oh time complexity of the four consecutive statements together? Prove it mathematically using the definition of big-Oh.
5. You ran two programs to completion. Both have running time of 10 ms. Can you say that the two programs have the same big-Oh time complexity? Why or why not?
6. Find $f(n)$ and $g(n)$, such that $f(n)$ is not $O(g(n))$ and $g(n)$ is not $O(f(n))$

Foundational Data Structures

Data structures from which we build abstract data types (ADTs):
- arrays
- linked lists

Example ADTs?

Since they are so foundational to all the more complicated data structures, it is of upmost importance that you thoroughly understand how to work with them

Arrays Review

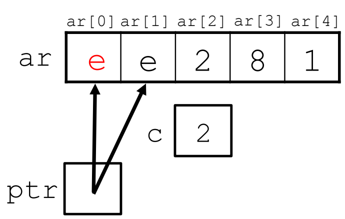
What is an array?

```
char ar[] = {'m', 'e', '2', '8', '1'};
ar[0] = 'e';

char c = ar[2];
// now we have c=='2'

char *ptr = ar;
// now ptr points to "ee281"

ptr = &ar[1];
// now ptr points to "e281"
// same as ptr = ar+1;
```



Copying with Pointers

How can we copy data from `src_ar` to `dest_ar`?

```
double size = 4;
double src_ar[] = {3, 5, 6, 1};
double dest_ar[size];
```

Without pointer

```
for (int i = 0; i< size; i++){
    dest_ar[i] = src_ar[i];
}
```

With pointer?

In which cases would you want to use pointers?

Arrays: Common Bugs

Two most common bugs (in various guises):

1. out-of-bound access
   - index variable not initialized
   - null-termination error
   - off-by-one errors
   - bounds not checked

2. dangling pointers into/out of array elements
   - pointers in array not de-allocated → memory leak
   - when moved (or realloc-ed), pointers to array elements not moved

Index Variable Not Initialized

```
int i;
printf( “%c\n” ,y[i]);
```

What’s the bug?

Correct programs always
run correctly on correct input

Buggy programs sometimes
run correctly on correct input
- sometimes they crash
  even when input doesn’t change!

Off-by-One Errors

```
const int size = 5;
int x[size];

// set values to 0-4
for(int j=0; j<=size; j++){
    x[j] = j;
}
// copy values from above
for(int k=0; k<=(size-1); k++) {
    x[k] = x[k+1];
}
// set values to 1-5
for(int m=1; m<size; m++){
    x[m-1] = m;
}
```

NULL-termination Errors

```
int i;
char x[10];
strcpy(x, “0123456789”);

// allocate memory
char* y =
  (char*)malloc(strlen(x));

for(i = 1; i < 11; i++) {
    y[i] = x[i];
}
y[i] = ‘\0’;
printf(“%s\n”,y);
```

Lookup/confirm the behavior of various libraries by reading the manual pages (under Linux or Mac OS X) or http://www.cplusplus.com/reference/clibrary/

Bounds Not Checked

```
int main(int argc, const char* argv[]) {
    char name[20];
    strcpy(name, argv[1]);
}
```

What errors may occur when running the code?

How can the code be made safer?

Container Classes

Wrapper for objects
- allows for control/protection over editing of objects
- e.g., adding bounds checking to arrays

Container class operations:
- Constructor
- Destructor
- `addElement()`
- `removeElement()`
- `getElement()`
- `getSize()`
- `copy()`
- `assign()`

Example of a Container Class: Adding Bounds Checking to Arrays

```
class Array{
  int* data;              // array data
  unsigned int length;    // array size
  // Why aren't data and length public?

public:
  // Constructor:
  Array(unsigned len=0):length(len) {
    data = (len ? new char[len] : NULL);
  }
  // other methods to follow in next slides…
};
```

Array Class: Inserting an Element

```
bool insert(int index, double val){
 if (index >= size || index < 0)
  return false;
 for(int i=size-1; i > index; i--) {
  data[i] = data[i-1];
 }
 data[index] = val;
 return true;
}
```

why is `i` decremented instead of incremented?

ar	1.6	3.1	4.2	5.9			Original array

`ar.insert(1, 3.4);` Call insert

Are arrays desirable when many insertions are needed?

Array Class: Complexity of Insertion

```
bool insert(int index, double val){
 if (index >= size || index < 0)
  return false;
 for(int i=size-1; i > index; i--) {
  data[i] = data[i-1];
 }
 data[index] = val;
 return true;
}
```

← at most n times

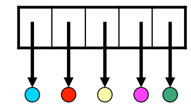
Best case: $O(1)$

Worst case: $O(n)$

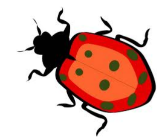
Average case: $O(n)$

Memory Leak

If `ar` is deleted/freed using either: `free(ar);` or `delete ar;` objects it points to become inaccessible, causing memory leak

ar


How to delete `ar` correctly?

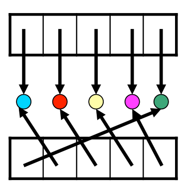


Memory Ownership

array A =

Both arrays contain pointers to objects

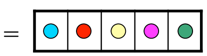
array B =



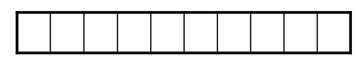
When should the objects be freed?

How to implement this?

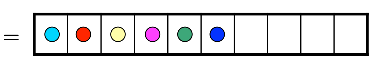
Array Class: Append Example

Original `ar` =


How can we append one more element ● ?

Create a new `temp_ar` =


Copy existing elements into new array and add new element:

New `ar` =


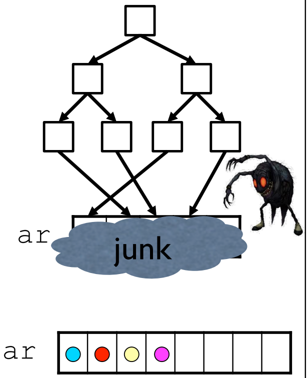
Delete old array so that memory can be reused (but be careful of dangling pointers!)

Why do we have to make a new array?
How big shall we make the new array?

Dangling Pointers

Say we have a binary search tree (BST) pointing to elements in an unsorted array `ar` (the BST acts as an "index" to speed up search)

Now if we need a larger array, we'd need to reallocate a larger chunk of memory and copy each element of the old array to the new array



Leak if the BST is not updated and continues to point to the old space

How to fix this?

Amortized Complexity

A type of worst-case complexity analysis spread out over a given input size

Considers the average cost over a sequence of operations
- in contrast: best/worst/average-case only considers a single operation

Justifies the cost of expandable arrays

Array Class: Complexity of Append

Appending n additional elements to an already full array of size n

On first append
- double array size from n to $2n$ (1 step)
- copy n items from original array to new array (n steps)

On remaining n–1 appends
- place element in appropriate location (n–1 times 1 step)

Total: $1+n+(n-1) = 2n$ steps

Amortized complexity of appending additional n elements: $2n/n = 2$ steps per append $= O(1)$

Pros and Cons of Arrays

Name 2 advantages of using an array:

Name 3 disadvantages of using an array:

2D Arrays in C/C++

```
int arr[3][3];
int val =0;

// For each row
for (int r=0; r < 3; r++){
 // For each column
 for (int c=0; c < 3; c++){
  arr[r][c] = val++;
 }
}
```

		column 0	column 1	column 2
row	0	0	1	2
	1	3	4	5
	2	6	7	8

Limitations:

2D Arrays in 1D

1D array

0	1	2	3	4	5	6	7	8

```
num_columns = 3
row =
column =
```

Want:

3x3 2D array

		column 0	column 1	column 2
row	0	0	1	2
	1	3	4	5
	2	6	7	8

```
index =
```

Limitation:

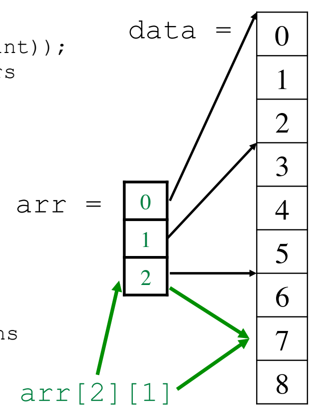
- indexing cumbersome, makes code hard to read
  - prefer to address elements as `array[][]`

2D Arrays with Row Pointers

```
int data = (int *) malloc(9*sizeof(int));
int *arr[3]; // array of row pointers


// assign row pointers
for (int r=0, nc=3; r < 3; r++) {
 arr[r] = &array[r*nc];
}

int val=0;
for(int r=0; r < 3; r++){ //rows
 for(int c=0; c < 3; c++){ // columns
  arr[r][c] = val++;
 }
}
```



Use this method in your programming assignments

Self-Study Questions

1. Who owns the memory in a container class?
2. What are the disadvantages of arrays?
3. Why do you need a `const` and a non-`const` version of some operators? What should a non-`const op[]` return?
4. How many destructor calls (min, max) can be invoked by: `operator delete` and `operator delete[]`
5. Why would you use a pointer-based copying algorithm ?
6. Are C++ strings null-terminated?
7. Give two examples of off-by-one bugs.
8. How do I set up a 2D array class?
9. Perform an amortized complexity analysis of an automatically-resizable container with doubling policy.
10. Discuss the pros and cons of pointers and references when implementing container classes.