eecs 281 Data Structures and Algorithms

Lecture 6: Trees
Binary Search Trees (BST)

Trees

A tree is a "natural" way to represent hierarchical structure and organization

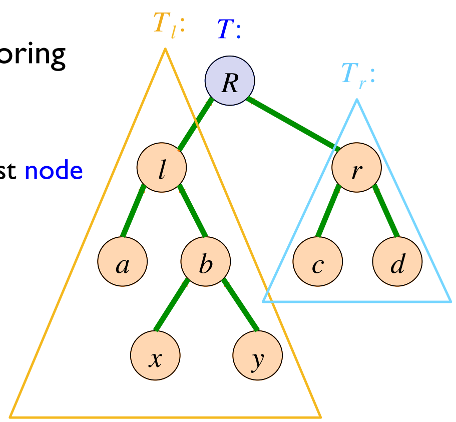
A lot of problems in computer systems can be solved by breaking it down into smaller pieces and arranging the pieces in some form of hierarchical structure

For example: binary search

Parent-Child

Tree (T): a set of nodes storing elements in a parent-child relationship such that:

- there is one root, the topmost node
- the root node has no parent
- all other nodes have exactly one parent
- parent-child relationship is denoted by direct link in tree
- subtrees:
  - T_l: left subtree of root
  - T_r: right subtree of root
  - we will use T_l, T_r to denote left and right subtrees of a node in general, not just of the root node



Extended Family

R: root node of tree T

l is a child of root

l is a parent of b

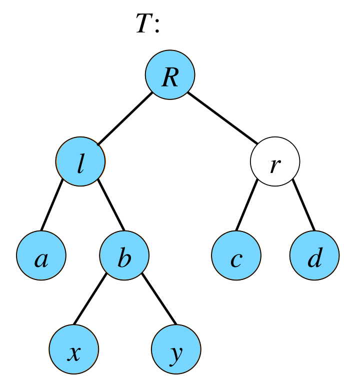
b is a grandchild of root

a, c, and d are siblings of b

a, c, d, x, and y are leaf nodes

degree of a tree: maximum number of children each node can have

The example T is a binary tree



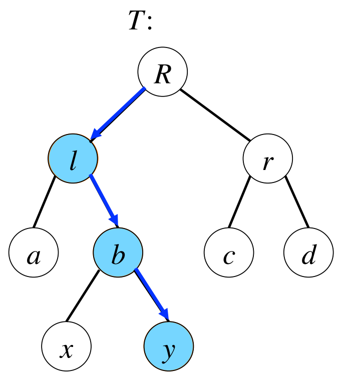
Node Path

A path: the set of nodes visited to get from a node higher up on the tree to a node lower down, not including the originating, higher up node

There is a unique path from one node to another, e.g.,

- path from root to y is $\{l, b, y\}$
- the path length of root to y is 3 (hops)

Path length may be 0, e.g., l going to itself is a path



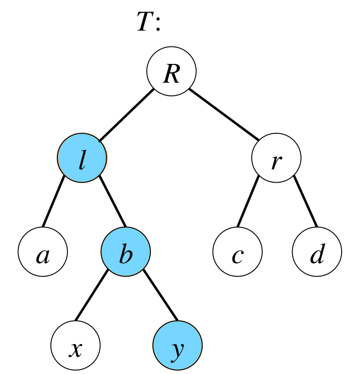
Ancestors and Descendants

Ancestor: l and b are ancestors of y: there is a path from l to y and b to y

- each node is its own ancestor
- node i is a proper ancestor of node j if the path length from i to j is not 0

Descendant: b and y are descendants of l: there is a path from l to b and l to y

- node j is a proper descendant of node i if the path length from i to j is not 0



Depth and Height

The depth of node i is the length of the path from the root node to i

- $\text{depth}(R) = 0, \text{depth}(x) = 3$

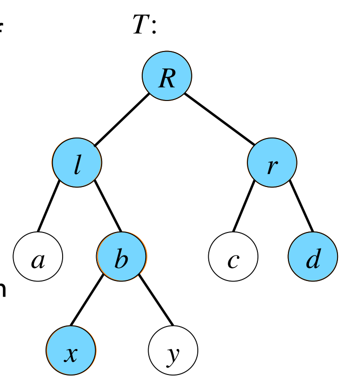
All nodes on a level of the tree have the same depth

- the root is at level 0

The depth of a tree is the maximum depths of all nodes, T is of depth 3

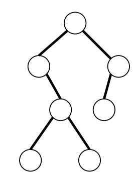
The height of node i is the longest path from i to a leaf node

- $\text{height}(x) = \text{height}(d) = 0,$
- $\text{height}(b) = \text{height}(r) = 1,$
- $\text{height}(l) = 2, \text{height}(R) = 3$

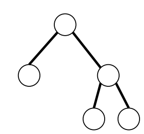


Binary Tree Characteristics

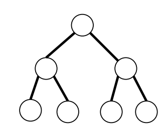
Every node in a binary tree has $0, 1,$ or 2 children



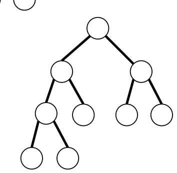
Every node in a proper binary tree has 0 or 2 children



Every level in a perfect binary tree is fully populated



Every level except the lowest in a complete binary tree is fully populated; the lowest level is populated left to right



Binary Tree Representation

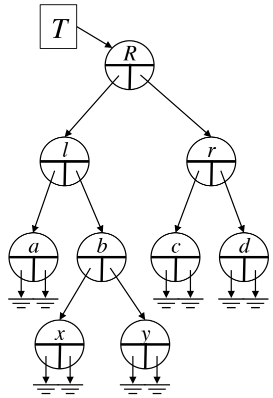
A binary tree can be represented as a linked structure:

```
struct Node {
   Item item;
   Node *left, *right;
};
```

Efficient for moving down a tree from parent to child

How to move up the tree?

How to remove a node from, and add a node to, a binary tree?



Binary Tree Representation

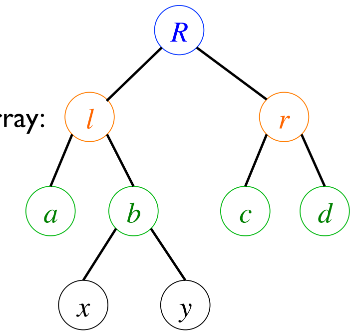
A binary tree can also be represented as an ordered set:

T: $\{R, \{l, \{a\}, \{b, \{x\}, \{y\}\}\}, \{r, \{c\}, \{d\}\}\}$

which can be implemented using an array:

R	l	r	a	b	c	d	–	–	x	y

T:



For a binary tree:

- a node at index i has its children at which indices?
- a node at index i has its parent at which index?

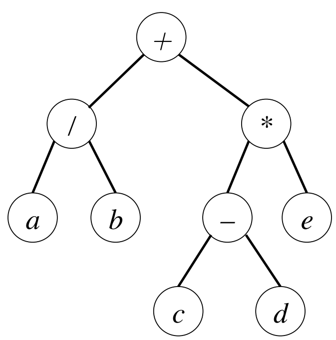
Tree Traversal

The expression $a/b + (c - d)e$ has been encoded as T

How would you traverse T to re-create (print out) the expression?

- to ensure correct evaluation precedence, enclose the printout of each subtree in parentheses, e.g., $(((a)/(b)) + (((c) - (d)) * (e)))$

T:



Write a pseudo-code recursive function to print T

```
void rtraverse(Node *root)
```

Tree Traversal

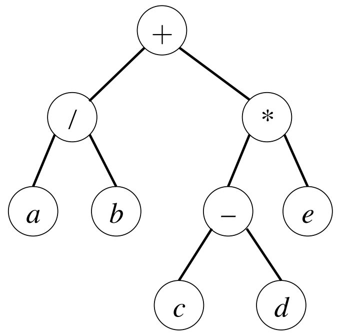
In-order depth-first traversal with stack

```
void
itraverse(Node *root)
{
  Stack stack;
  // node with '(' or ')' as Item
  Node lparen, rparen;
  Node node = root;

  print(lparen);
  do {
    if (!node->right &&
        !node->left) {
      print(node);
    } else {
      if (node->right) {
        stack.push(rparen);
        stack.push(node->right);
        stack.push(lparen);
        node->right = NULL;
      }
      push(node);
      if (node->left) {
        stack.push(rparen);
        stack.push(node->left);
        stack.push(lparen);
        node->left = NULL;
      }
    }
  } while (node = pop());
  print(rparen);
}
```

Tree Traversal

T:



Aside from in-order depth-first traversal, we could also traverse the tree depth-first pre-order or post-order

- in-order: visit T_l, visit node, visit T_r
- pre-order: visit node, visit T_l, visit T_r
- post-order: visit T_l, visit T_r, visit node
- which traversal order will give you Reverse Polish Notation (RPN)? $ab/cd–e*+$
- and the Polish Notation? $+/ab*–cde$

Breadth-first traversal visits the tree level by level

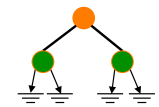
- how would you implement breadth-first traversal?

Tree Sizes

A tree may be empty

External node: an empty node with no children

Internal node: a node with children



Leaf node: an internal node whose children are all external nodes

[often, outside this course, internal node simply means non-leaf node]

In general, how many external nodes does an N-ary tree with n internal nodes have?

N-ary tree: a tree with degree N
(each node can have a maximum of N children)

Tree Sizes

How many external nodes does an N-ary tree with n internal nodes have?

n	binary	tertiary	4-ary
0	1	1	1
1	2	3	4
2	3	5	7

Every new internal node replenishes one external node and brings with it $N–1$ new external nodes

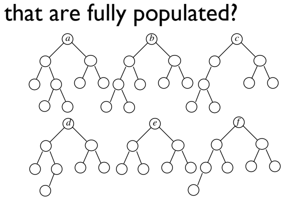
For n internal nodes, we have $1+n(N–1)$ external nodes

For binary tree, n internal nodes means $n+1$ external nodes ⇒ maximum $\text{ceil}(n/2)$ leaf nodes

How many internal nodes does an N-ary tree with m external nodes have?

Study Questions

1. How many links are there in an N-ary tree with n internal nodes?
2. What is the maximum height of a binary tree of n internal nodes?
3. How many internal nodes does it take to fully populate level l of a binary tree?
4. What do you call a tree of l levels that are fully populated?
5. Identify any proper, perfect, and complete binary tree in the figure:

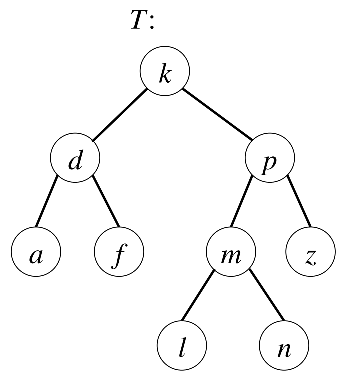


6. How many internal nodes are there in a perfect binary tree of height h ($h+1$ levels)?
7. How many levels of a binary tree are needed to hold n internal nodes?
8. What is the minimum height of a binary tree of n internal nodes?
9. Is the height of the root node of a subtree the same as the depth of the subtree?

Binary Search Trees (BST)

A BST is a binary tree that

- has a key associated with each of its internal node, and that
- the key in any node is $>$ the keys in all nodes in its left subtree and
- $<$ the keys in all nodes in its right subtree,
- where '$<$' and '$>$' can be user defined

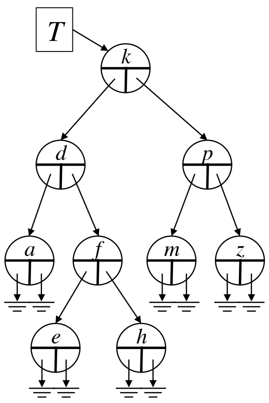


Implements sorted dictionary with $O(\log N)$ complexity for both insert and search

Binary Search Trees Representation

A binary search tree can be represented as a linked structure:

```
struct Node {
   Item item;
   Node *left, *right;
};
typedef Node *Link;
```



Efficient for moving down a tree to search for an item

How to remove a node from, and add a node to, a binary search tree?

BST Search: Recursive

```
Item BST::
rsearch(Node *root, Key &searchkey)
{





}
```

`BST::rsearch()` called with pointer to root and key

BST Search: Iterative

```
Item BST::
isearch(Node *root, Key &searchkey)
{






}
```

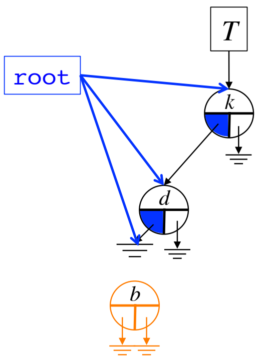
`BST::isearch()` with minimal change to `BST::rsearch()`

We will refer to both as `BST::search()`

BST Insert 1st (Bad) Attempt

- If new item has a key smaller than root's, recursive call on left subtree
- Else recursive call on right subtree
- Insert new item as leaf node

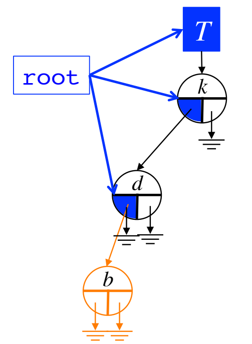
Example: `insert(root, Item('b'));`




```
void BST::
insert(Node *root, Item newitem)
{
  if (root == NULL) {
   new Node(newitem);
   // how to update parent
   // to point to this new child?
   return;
  }
  if (newitem.key < root->item.key)
   insert(root->left, newitem);
  else if (newitem.key > root->item.key)
   insert(root->right, newitem);
}
```

BST Insert

`typedef Node *Link;`



What to do with duplicates?

```
void BST::
insert(Link &root, Item newitem)
{
  if (root == NULL) {
   root = new Node(newitem);
   return;
  }
  if (newitem.key < root->item.key)
   insert(root->left, newitem);
  else if (newitem.key > root->item.key)
   insert(root->right, newitem);
}
```

`BST::insert()` called with double pointer to root and item to be inserted

if at leaf, and only at leaf, insert
(note the nifty use of reference args!)

if new item has a key smaller than that of root's, recursive call on left subtree

else recursive call on right subtree

BST Removal

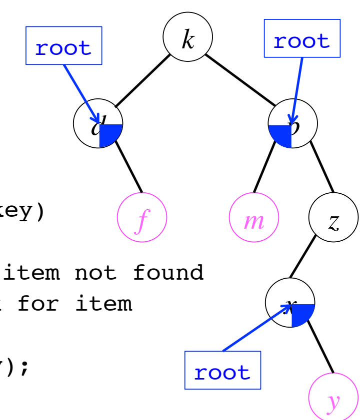
After the removal of a node, the tree must remain a BST

1. Find the node to be removed
2. If node is a leaf node, remove, done
3. If node has a single child, replace node to be removed with child, done
4. If node has 2 children, find the smallest element in right child, called the in-order successor (`find_ios()`)
5. Swap with in-order successor, repeat Steps 2 and 3

(Instead of in-order successor, Steps 4 and 5 can also use in-order predecessor, the largest element in the left child)

BST Removal

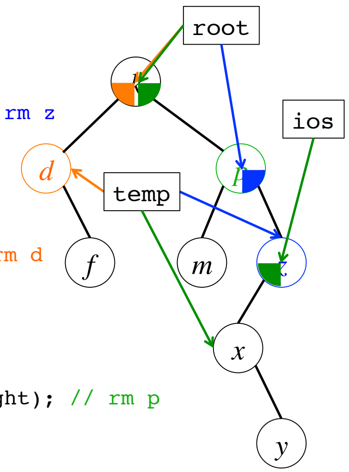
After the removal of a node, the tree must remain a BST




```
void BST::
remove(Link &root, Key &searchkey)
{
  if (root == NULL) return; // item not found
  key = root->item.key; // look for item
  if (searchkey < key)
   remove(root->left, searchkey);
  else if (searchkey > key)
   remove(root->right, searchkey);
  else if (searchkey == key)
   if (isleaf(root)) { // e.g., rm f, m, or y
    delete root; root = NULL;
   } else { // what to do?  see next page
}
```

BST Removal

```
  else {
    if (root->right == NULL) { // rm z
      Node *temp = root;
      root = root->left;
      delete temp; return;
    }
    if (root->left == NULL) { // rm d
      Node *temp = root;
      root = root->right;
      delete temp; return;
    }
    Link *ios = find_ios(root->right); // rm p
    Node *temp = *ios;
    root->item = temp->item;
    *ios = temp->right; // null ok
    delete temp;
    // or swap root and *ios instead of copying item
  }
 }
}
```



BST Search Times

Average case search times:

	successful	unsuccessful
linked lists	$n/2$	n
hashing	$1+L/2$	L
BST	$\log n$	$\log n$

expressed in terms of depth:
successful search on BST takes O(depth of found node)
unsuccessful search on BST takes O(depth of tree)

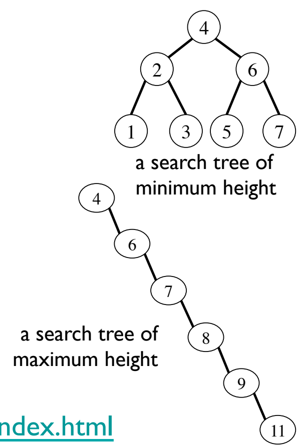
Worst-case successful search time on BST: $O(n)$
Worst-case unsuccessful search time on BST: $O(n)$

Worst-Case BST Performance

Exercise:

- insert 4, 2, 6, 3, 7, 1, 5
- remove 2, insert 8, remove 5, insert 9, remove 1, insert 11, remove 3

Moral: even a balanced tree can become unbalanced after a number of insertions and removals



Demo: http://people.ksp.sk/~kuko/bak/index.html

Sorted Dictionary

What kind of operations can we not do with an unsorted dictionary?

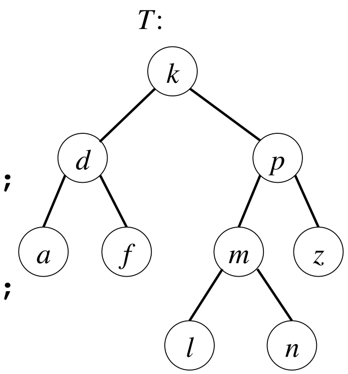
Sort: return the values in order
- example: return search results by item's popularity

Rank search: return the k-th largest item
- example: return the next building to be completed in a strategy game

Range search: return values between h and k
- example: return all the restaurants within 100 m of user

BST Sort

```
void BST::
printsorted(Link root)
{
  if (root == NULL) return;
  printsorted(root->left);
  print(root);
  printsorted(root->right);
}
```



What kind of tree traversal does `BST::printsorted()` perform?

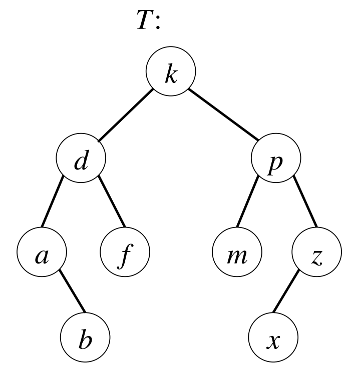
BST Rank Search

Given a BST, where is the smallest item?

Where is the largest item?

Would the node containing the largest/smallest item always be a leaf node?

How would you find the k-th largest item? E.g., find 2^{nd}, 3^{rd}, and 6^{th} largest items

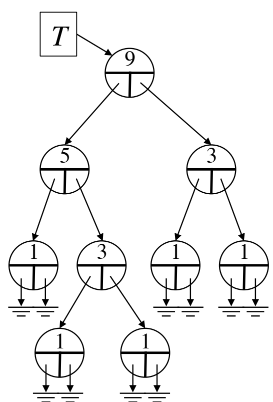


BST Add Count

A BST node with count:

```
struct Node {
 Item item;
 int count;
 Node *left, *right;
};
typedef Node *Link;
```

Let count counts the number of a node's descendants (nodes at and below the current node in tree)



BST Insert with Count

How would you modify `BST::insert()` to keep track of the count?

```
void BST::
insert(Link &root, Item newitem)
{
  if (root == NULL) {
   root = new Node(newitem);
   return;
  }
  if (newitem.key < root->item.key)
   insert(root->left, newitem);
  else insert(root->right, newitem);
}
```

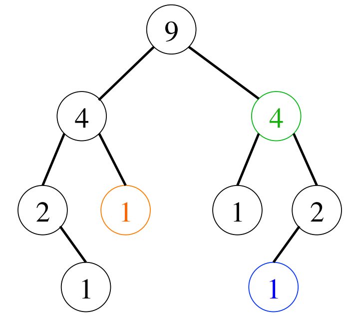
BST Rank Search: Idea

A: If there are more than k items in the right subtree, the k-th largest item must be in the right subtree

B: Else, there are m ($<\ k$) items in the right subtree, if the k-th item is not in the root node, find the (k–m–1)-th largest item in the left subtree

BST Rank Search

T:




```
Node.rightcount() returns
right->count if right is
non-null, else returns 0

Link BST::
findkth(Link root, int rank)
{
  if (root == NULL) return root;
```

A:
```
  if (root->rightcount() - rank >= 0)
   return findkth(root->right, rank);
  } else {
```
B:
```
   rank -= (root->rightcount() + 1);
   if (rank == 0) return root;
   else return findkth(root->left, rank);
  }
}
```

Find 2^{nd}, 3^{rd}, and 6^{th} largest items

Find 2^{nd} smallest item?