The decentralized disruption problem with linear penalties on false alarms

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Two detectors take independent noisy observations of a two-state ($\{0, 1\}$) Markov chain and have to decide when the Markov chain jumps from state 0 to state 1. The detectors do not communicate. Their decisions are coupled through a cost function where delays in the detection of the jump as well as false alarms are linearly penalized. It is shown that the optimal decision rules of the detectors are characterized by thresholds. These thresholds are time varying and their computation requires the solution of two interdependent dynamic programming equations. A numerical solution of these equations is very difficult and has not yet been achieved; thus, the paper provides a qualitative characterization of the optimal decision rules of the detectors. However, a comparison with the thresholds of a class of single-detector finite horizon disruption problems, where delays in the detection of the jump as well as false alarms are linearly penalized, is possible. Such a comparison provides lower bounds on the member by member optimal thresholds of the decentralized problem.

1. Introduction

Two detectors make independent observations of a Markov chain x_t which jumps from state 0 to state 1 at some random time θ. Each detector has to detect the time of the jump based on its own noisy measurements; let τ_i be the time detector i declares that the jump has occurred. The problem is to find stopping times τ_i which minimize the expected cost $EJ(\tau_1, \tau_2, \theta)$.

Such a problem arises in surveillance. Consider, for example, two radars attempting to detect quickly and accurately the appearance of an airplane in a surveillance area; assume that for reasons such as compartmentalization the radars are not allowed to communicate between each other. The appearance of the airplane in the surveillance area can be modelled as a jump from state 0 (no airplane present) to state 1 (airplane present) of a Markov chain. The quick and accurate detection of the appearance of the plane is the common objective of the radars; such an objective can be described by $EJ(\tau_1, \tau_2, \theta)$.

If the cost is separable, i.e. $J(\tau_1, \tau_2, \theta) = J(\tau_1, \theta) + J(\tau_2, \theta)$, then the decisions of the two detectors are decoupled. In such a case, for certain costs $J(\tau_i, \theta)$ (see [1], [2], [9]–[12]) the optimal decision τ_i^* is to stop and declare the jump the first time instant the probability of false alarm drops below a time-invariant threshold l_i, i.e.

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$$\tau_i^* = \min \{t \mid \text{Prob}(\theta > t \mid Y^{it}) \leq l_i\} , \tag{1.1}$$

where Y^{it} is the information available to detector i at time t. The threshold property holds for the cost functions

$$J(\tau, \theta) = c(\tau - \theta)1(\tau \geq \theta) + 1(\tau < \theta) \tag{1.2}$$

and

$$J(\tau, \theta) = c(\tau - \theta)1(\tau \geq \theta) + k(\theta - \tau)1(\tau < \theta) , \tag{1.3}$$

where c and k are constants.

If the cost $J(\tau_1, \tau_2, \theta)$ is not separable, then there is an interaction between the optimal decisions. Detection problems with nonseparable costs have been previously investigated in refs. [3]–[8]. The problem investigated in this paper has a nonseparable cost but is essentially different from those of refs. [5]–[8] which are not sequential; it is also different from the problem of [4] where the Markov chain is frozen in one of two states; it is similar to the problem considered in [3], but it has a cost function different from that of [3]. The cost function considered in [3] was

$$J(\tau_1, \tau_2, \theta) = c(\tau_1 - \theta)1(\tau_1 \geq \theta) + c(\tau_2 - \theta)1(\tau_2 \geq \theta) + 1(\tau_1 < \theta)1(\tau_2 < \theta) , \tag{1.4}$$

where c is a constant. This cost puts a constant penalty for false alarms and, for each detector, a penalty proportional to the delay in detecting the jump. The cost function considered in this paper is

$$J(\tau_1, \tau_2, \theta) = k(\theta - \tau_1 + \theta - \tau_2)1(\tau_1 < \theta)1(\tau_2 < \theta) + c(\tau_1 - \theta)1(\tau_1 \geq \theta) + c(\tau_2 - \theta)1(\tau_2 \geq \theta) , \tag{1.5}$$

where k and c are constants. The motivation for studying the problem with cost (1.5) is primarily theoretical. Even for the centralized disruption problem the threshold property of the optimal solution is sensitive with respect to the cost $J(\tau, \theta)$. The fact that in [3] the threshold property was proved for the cost (1.4) does not automatically imply that this property holds for *any arbitrary cost* $J(\tau_1, \tau_2, \theta)$. Thus, it is interesting, from a theoretical viewpoint, to determine the form of cost functions $J(\tau_1, \tau_2, \theta)$ for which either the threshold property holds or, more generally, it is possible to obtain a characterization of the qualitative properties of the member by member optimal solutions. The results of this paper contribute a positive result in that direction.

The cost of (1.5) puts, for each detector, a penalty proportional to the delay in detecting the jump and a penalty proportional to how early false alarms occurred. Thus, the coupling of the detectors (described by the term $k(\theta - \tau_1 + \theta - \tau_2)1(\tau_1 < \theta)1(\tau_2 < \theta)$) is different from that of [3]. Because of the coupling through the cost function, there is an interaction between the optimal decisions of the detectors. The interaction is simple since there is no communication between the detectors.

In this paper it is shown that the member by member optimal (*m.b.m.o.*) decisions of the detectors are described by thresholds as in [3]. These thresholds are time-varying, as in [3], and their computation requires the solution of two interdependent dynamic programming equations. Numerical solution of these equations has not yet been achieved. Thus, the paper provides only a qualitative characterization of the optimal decision rules of the detectors. However, it is possible to compare the *m.b.m.o.* thresholds with the thresholds of a class of single-detector finite horizon disruption problems. We prove that for each instant of time t the *m.b.m.o.* thresholds lie above the thresholds at t of a class of finite horizon $N(N > t)$ single-detector disruption problems where delays in detecting the jump as well as false alarms are linearly penalized. The thresholds of these single-detector problems are time varying. A similar comparison was achieved in [3]. However, whereas in [3] the *m.b.m.o.* thresholds were compared to a stationary threshold, such a comparison is impossible for the present problem.

The remainder of the paper is organized as follows. The formal model is presented in Section 2. The characterization of the *m.b.m.o.* solutions of the decentralized disruption problem is presented in Section 3. In Section 4 the comparison of the *m.b.m.o.* thresholds with the thresholds of a class of finite horizon single-detector disruption problems is made. Concluding remarks appear in Section 5.

2. The model

Consider a Markov chain $\{x_t, t = 1, 2, \dots\}$ with values in $\{0, 1\}$, known transition probabilities

$$\text{Prob}\,(x_{t+1} = 1 \mid x_t = 0) = p\,, \tag{2.1}$$

$$\text{Prob}\,(x_{t+1} = 1 \mid x_t = 1) = 1\,, \tag{2.2}$$

and

$$\text{Prob}\,(x_1 = 0) = r\,. \tag{2.3}$$

Thus, the chain makes a jump to state 1 at the random time $\theta = \min\{t{:}x_t = 1\}$ and remains at that state afterwards. Detector i's observation at time t is

$$y_t^i = g^i(x_t, w_t^i), \quad i = 1, 2\,, \tag{2.4}$$

where it is assumed that $\{w_t^i\}$, $i = 1, 2$, are mutually independent i.i.d. sequences which are also independent of $\{x_t\}$. The detectors do not communicate.

The problem the detectors are faced with is the following:

$$\left.\begin{aligned} &\min_{\tau_1, \tau_2} E\left\{k(\theta - \tau_1 + \theta - \tau_2)1(x_{\tau_1} = 0)1(x_{\tau_2} = 0) + c\sum_{t=1}^{\tau_1 - 1} 1(x_t = 1) + c\sum_{t=1}^{\tau_2 - 1} 1(x_t = 1)\right\} \quad (2.5), \\ &\text{s.t. } (2.1)\text{–}(2.4)\,, \end{aligned}\right\} \tag{P}$$

where τ_i is a Y^{it} stopping time and

$$Y^{it} \triangleq \sigma(y_s^i, s \leq t)\,. \tag{2.6}$$

Problem (P) is a team problem. In this paper we derive certain qualitative properties of the solution of problem (P). We prove that the *m.b.m.o.* stopping rules of the detectors are characterized by time-varying thresholds whose computation requires the solution of two interdependent dynamic programming equations.

3. Characterization of the optimal solution

In this section the qualitative properties of the *m.b.m.o.* solutions of problem (P) are derived. The main result of the section and the paper can be summarized by the following theorem:

Theorem 3.1. *The m.b.m.o. decision rules of the detectors are characterized by thresholds as in the case of a single detector. However, the thresholds are time varying and their computation requires the solution of two interdependent dynamic programming equations.*

The proof of this theorem proceeds in various steps. Fix $\tau_2 \triangleq \tau_2^*$ (possibly at the optimum). Then, the problem faced by detector 1 is

$$\min_{\tau_1} EJ(\tau_1) = \min_{\tau_1} E\left\{k(\theta - \tau_1 + \theta - \tau_2^*)1(x_{\tau_2^*} = 0)1(x_{\tau_1} = 0) + c\sum_{t=1}^{\tau_1 - 1} 1(x_t = 1)\right\}. \tag{3.1}$$

Define

$$\pi_t^1 = \text{Prob}\,(x_t = 0 \mid Y^{1t}) = \text{Prob}\,(\theta > t \mid Y^{1t}). \tag{3.2}$$

Then, we can alternatively write the cost in (3.1) as follows:

$$EJ(\tau_1) = E\left\{c\sum_{t=1}^{\tau_1 - 1} (1 - \pi_t^1) + H_{\tau_1}^1 \pi_{\tau_1}^1\right\}, \tag{3.3}$$

where

$$\begin{aligned} H_t^1 &= k\sum_{l=1}^{\infty} l(1-p)^{l-1} pE\{1(\tau_2^* < \theta) \mid \theta = t + l\} + k\sum_{s=0}^{t} \text{Prob}\,(\tau_2^* \le s \mid \theta > s) \\ &\quad + k\sum_{l=1}^{\infty} \text{Prob}\,(\tau_2^* \le t + l \mid \theta > t + l)(1-p)^l \\ &= k\sum_{l=1}^{\infty} (1-p)^{l-1}\, \text{Prob}\,(\theta > \tau_2^* \mid \theta > t + l - 1) \\ &\quad + k\left[\frac{1}{p} + E\{[t - \tau_2^*]^+ \mid \theta > \tau_2^* \vee t\}\right] \text{Prob}\,(\theta > \tau_2^* \mid \theta > t) \end{aligned} \tag{3.4}$$

and

$$[z]^+ = \max\,(0, z), \qquad \tau_2^* \vee t = \max\,(t, \tau_2^*). \tag{3.5}$$

The equality of the cost functions in (3.1) and (3.3) is shown in Appendix A. Thus, detector 1 has to determine a Y^{1t}-stopping time to minimize (3.3). Since the detectors do not communicate and τ_2 is fixed, a Y^{1t}-stopping time for detector 1 can be determined by dynamic programming.

3.1. *Dynamic programming: Finite horizon*

Fix the final time $T (T < \infty)$ and consider the problem

$$\min_{1 \le \tau_1 \le T} EJ(\tau_1). \tag{3.6}$$

The dynamic programming argument for this problem shows that the value function $V_t^{1T}(\pi)$, i.e.

$$V_t^{1T}(\pi) = \min_{t \le \tau_1 \le T} E\left\{H_{\tau_1}^{1T} 1(x_{\tau_1} = 0) + c\sum_{m=t}^{\tau_1 - 1} 1(x_m = 1) \middle| \pi_t^1 = \pi\right\}, \tag{3.7}$$

is obtained by

$$V_T^{1T}(\pi) = H_T^{1T}\pi \tag{3.8}$$

$$V_t^{1T}(\pi) = \min\{H_t^{1T}\pi, (LV_{t+1}^{1T})(\pi) + c(1-\pi)\}, \quad t < T, \tag{3.9}$$

where

$$(LV)(\pi) = \int V(A(\pi, y))q(y \mid \pi)\mathrm{d}y, \tag{3.10}$$

$$q(y \mid \pi) = \pi(1-p)p_0^1(y) + \pi p p_1^1(y) + (1-\pi)p_1^1(y), \tag{3.11}$$

$$A(\pi, y) = \pi(1-p)p_0^1(y)/q(y \mid \pi), \tag{3.12}$$

$$H_t^{1T} = \frac{k}{B}\sum_{l=1}^{T-t}(1-p)^{l-1}\operatorname{Prob}(\theta > \tau_2^* \mid \theta > t+l) + \frac{k}{B}\sum_{l=0}^{t}\operatorname{Prob}(\tau_2^* \leq l \mid \theta > l)$$
$$+ \frac{k}{B}\sum_{l=1}^{T-t}\operatorname{Prob}(\tau_2^* \leq t+l \mid \theta > t+l)(1-p)^l, \tag{3.13}$$

$$B = \sum_{l=1}^{T}(1-p)^{l-1}p, \tag{3.14}$$

and $p_i^1(y_t^1)$ is the probability density of the measurement y_t^1 under the assumption that $x_t = i$. The term $H_t^{1T}\pi$ in (3.9) describes the cost incurred by the decision to stop at time t and the term $c(1-\pi) + (LV_{t+1}^{1T})(\pi)$ describes the cost incurred by the decision at t to continue taking measurements. Thus, it is optimal for detector 1 to stop if and only if

$$H_t^{1T}\pi \leq c(1-\pi) + (LV_{t+1}^{1T})(\pi). \tag{3.15}$$

The value function $V_t^{1T}(\pi)$ has the following important property:

Lemma 3.1. *$V_t^{1T}(\pi)$ is a non-negative concave function of $\pi (t = 1, 2, \ldots, T)$. $(LV_{t+1}^{1T})(\pi)$ is also a non-negative concave function of π $(t = 1, 2, \ldots, T)$.*

Proof. The proof is the same as that of Lemma 3.1 in [3]. □

Lemma 3.2. *At $\pi = 0$:*

$$H_t^{1T}\pi < c(1-\pi) + (LV_{t+1}^{1T})(\pi). \tag{3.16}$$

At $\pi = 1$:

$$H_t^{1T}\pi > c(1-\pi) + (LV_{t+1}^{1T})(\pi). \tag{3.17}$$

Proof. See Appendix B. □

Lemmas 3.1 and 3.2 imply the threshold property of agent 1's optimal policy for fixed (arbitrary) τ_2 and for the finite horizon problem.

3.2. *Dynamic programming: Infinite horizon*

To minimize (3.3) let $T \to \infty$. Since the set of stopping times τ_1 of detector 1 increases as T increases, it follows that

$$V_t^{1(T+1)}(\pi) \leq V_t^{1T}(\pi), \tag{3.18}$$

and since the value functions $V_t^{1T}(\pi)$ are non-negative, the following limit is well defined:

$$V_t^1(\pi) \triangleq \lim_{T\to\infty} V_t^{1T}(\pi) = \inf_T V_t^{1T}(\pi). \tag{3.19}$$

We can show that the value function $V_t^1(\pi)$ has the following properties.

Lemma 3.3. *The value function $V_t^1(\pi)$ satisfies the equation*

$$V_t^1(\pi) = \min\{H_t^1\pi, c(1-\pi) + (LV_{t+1}^1)(\pi)\}, \quad t = 1, 2, \ldots, \tag{3.20}$$

where H_t^1 is given by (3.4). $V_t^1(\pi)$ as well as $(LV_{t+1}^1)(\pi)$ are non-negative and concave functions of π. Moreover, at $\pi = 0$ and $\pi = 1$, the following inequalities hold.

At $\pi = 0$:

$$H_t^1\pi < c(1-\pi) + (LV_{t+1}^1)(\pi). \tag{3.21}$$

At $\pi = 1$:

$$H_t^1\pi > c(1-\pi) + (LV_{t+1}^1)(\pi). \tag{3.22}$$

Proof. Equation (3.20) is obtained by (3.9). The non-negativity and concavity of $V_t^1(\pi)$ and $(LV_{t+1}^1)(\pi)$ follow from Lemma 3.1. The inequalities (3.21) and (3.22) follow from Lemma 3.2. □

Furthermore, it is possible to show that the value functions have the following additional property.

Lemma 3.4. *The value functions $\{V_t^1(\pi)\}$ are the unique solution of*

$$Z_t(\pi) = \min\{H_t^1\pi, c(1-\pi) + (LZ_{t+1})(\pi)\}, \quad t = 1, 2, \ldots. \tag{3.23}$$

Proof. See [3, Theorem 3.2]. □

The properties of the value function $V_t^1(\pi)$ imply the following characterization of agent 1's optimal policy for fixed τ_2.

Lemma 3.5. *For fixed τ_2 the optimal stopping time for detector 1 is*

$$\tau_1^* = \min\{t: \pi_t^1 \leq l_t^1\}. \tag{3.24}$$

The threshold l_t^1 is defined by the solution of:

$$H_t^1\pi = c(1-\pi) + (LV_{t+1}^1)(\pi). \tag{3.25}$$

Proof. The concavity of $(LV_{t+1}^1)(\pi)$ and the inequalities (3.21) and (3.22) imply that the functions $H_t^1\pi$ and $c(1-\pi) + (LV_{t+1}^1)(\pi)$ intersect at one point. The solution of (3.25) defines this point (see Fig. 1). From (3.20) the optimal decision rule of agent 1 at time t is to stop if and only if

$$V_t^1(\pi) = H_t^1\pi. \tag{3.26}$$

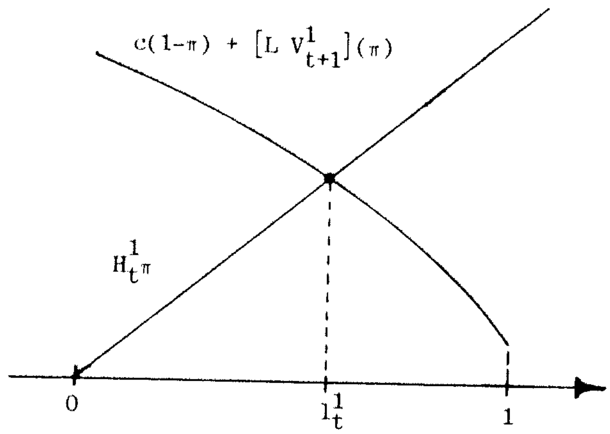

Fig. 1.

Equation (3.26) gives the rule (3.24). Q.E.D. □

We are now in a position to prove the main result of this paper which was summarized by Theorem 3.1.

Proof of Theorem 3.1. Since the optimum policy of agent 1 is characterized by thresholds for any arbitrary fixed policy of agent 2, it will also be characterized by thresholds when the policy of agent 2 is the optimum. By symmetry, the optimal policy of agent 2 is characterized by thresholds. Hence, the *m.b.m.o.* policies of the agents are characterised by thresholds $\{l_t^{1*}, l_t^{2*}\}$. These thresholds are determined by the solution of a set of interdependent dynamic programming equations which are of the form:

$$H_t^1(\{l_t^{2*}\})l_t^{1*} = c(1-l_t^{1*}) + (LV_{t+1}^1)(l_t^{1*}), \quad t = 1, 2, \ldots, \tag{3.27}$$

$$H_t^2(\{l_t^{1*}\})l_t^{2*} = c(1-l_t^{2*}) + (LV_{t+1}^1)(l_t^{2*}), \quad t = 1, 2, \ldots, \tag{3.28}$$

The proof of Theorem 3.1 is now complete. □

Remark. The existence of *m.b.m.o.* solutions can be shown as follows: Obtain sequences $\{H_t^i(n), V_t^i(n)\}$ for detector i, $n = 1, 2, \ldots$ such that $\{H_t^1(n), V_t^1(n)\}$ are optimal for $\{H_t^2(n), V_t^2(n)\}$ and $\{H_t^2(n+1), V_t^2(n+1)\}$ are optimal for $\{H_t^1(n), V_t^1(n)\}$. For fixed t, the $H_t^i(n)$ are uniformly bounded and the $V_t^i(n)$ are uniformly bounded and equicontinuous in π^i. Then, for all t there will be a subsequence along which the functions $H_t^i(n)$ and $V_t^i(n)$ converge. The limit functions then define a *m.b.m.o.* pair.

The results presented in this section can be extended to the following situation. Consider $N (N > 2)$ detectors, each one taking its own observations and attempting to detect the time a two-state $\{0, 1\}$ Markov chain, described by (2.1)–(2.3), jumps from state 0 to state 1. The detectors do not communicate and their decisions are coupled through the cost function

$$J(\tau_1, \tau_2, \ldots, \tau_N, \theta) = c\sum_{i=1}^{N} \sum_{l=1}^{\tau_1 - 1} 1(x_l = 1)$$
$$+ k(\theta - \tau_1 + \theta - \tau_2 + \cdots + \theta - \tau_N)1(\tau_1 < \theta)1(\tau_2 < \theta)\cdots 1(\tau_N < \theta). \tag{3.29}$$

The detectors have to determine the stopping rules that minimize the cost (3.29). It can be shown, by arguments similar to those presented in this section, that the *m.b.m.o.* decision rules are described by thresholds which can be determined by the solution of N interdependent dynamic programming equations.

Numerical solution of the interdependent dynamic programming equations (3.27) and (3.28) has not yet been achieved; thus, the results of this paper provide only a qualitative characterization of the optimal decision rules of the detectors. It is possible, however, to compare the thresholds $\{l_t^{1*}, l_t^{2*}\}$ with the thresholds of single-detector finite horizon disruption problems, as shown in the next section.

4. Comparison with a class of single agent problems

In this section it is shown that for each time t it is possible to compare the *m.b.m.o.* thresholds of problem (P) with the thresholds of finite horizon single-detector disruption problems. The thresholds of these single-detector problems are time varying. Thus, the results obtained in this section are different from those of [3] where the comparison of the *m.b.m.o.* thresholds with a time-invariant threshold was possible.

Let (l_s^{1*}, l_s^{2*}) be two m.b.m.o. thresholds for problem (P) at time s. Consider the following finite horizon disruption problem. A single detector takes noisy observations of the two state $\{0, 1\}$ Markov chain $\{x_t, t = 1, 2, \ldots, N\}$ described by (2.1)–(2.3) and attempts to detect the time of the jump from state 0 to state 1. The observations of the detector are the same as the observations for the first detector in the team problem, that is,

$$y_t = g^1(x_t, w_t^1), \quad t = 1, 2, \ldots, N, \tag{4.1}$$

where g and $\{w_t^1\}$ are the same as in (2.4) for $i = 1$ and $\{w_t^1\}, t = 1, 2, \ldots, N$, is a sequence of i.i.d. random variables which are independent of x_t. Assume that the final time $N > s$. Consider a cost where both delays in the detection of the jump and false alarms are linearly penalized, namely

$$J(\tau, \theta) = E\left\{G_\tau^N 1(\tau < \theta) + c\sum_{t=1}^{\tau-1} 1(x_t = 1)\right\}, \tag{4.2}$$

where

$$G_t^N = k\left\{\frac{1}{b}\sum_{l=1}^{N-t} l(1-p)^{l-1}p + N + 2\frac{1}{p}\right\}, \tag{4.3}$$

$$b = \sum_{l=1}^{N} l(1-p)^{l-1}p, \tag{4.4}$$

and the constants c and k are the same as in (2.5). The detector has to choose the stopping rule which minimizes the cost (4.2).

It is well known [1, 2] that the optimal stopping rule for the above problem is described by thresholds $\{\lambda_t^N\}_{t=1}^N$ which are determined by the solution of a set of equations of the form:

$$G_t^N \lambda_t^N = c(1 - \lambda_t^N) + [LM_{t+1}^N](\lambda_t^N), \tag{4.5}$$

where $M_t^N(\pi)$ is defined by

$$M_N^N(\pi) = k\left(N + 2\frac{1}{p}\right)\pi, \tag{4.6}$$

$$M_t^N(\pi) = \min\{G_t^N\pi, c(1-\pi) + [LM_{t+1}^N](\pi)\}, \quad t = 1, 2, \ldots, N-1, \tag{4.7}$$

$$\pi_t = \text{Prob}\,(\theta > t \mid Y^{1t})\,, \tag{4.8}$$

$$Y^{1t} \triangleq \sigma(y^1_\sigma, \sigma \leqslant t)\,, \tag{4.9}$$

and $[LM^N_{t+1}](\pi)$ is defined in a way analogous to (3.10)–(3.12).

The main result of this section is summarized by the following theorem:

Theorem 4.1. *Let (l^{1*}_s, l^{2*}_s) be two m.b.m.o. thresholds for problem (P) at time s, and let λ^N_s be the optimal threshold at time s for the single detector disruption problem with final time $N > s$, observations described by* (4.1), *and cost given by* (4.2). *Then for all $s < N$*

$$\lambda^N_s < l^{1*}_s\,. \tag{4.10}$$

The proof of the theorem proceeds in various steps. First, we prove the following lemmas:

Lemma 4.1. *For any t, s, $(s, t < N)$, $t > s$,*

$$G^N_t < G^N_s\,.$$

Proof. Follows directly from the definition of G^N_t. □

Lemma 4.2. *For any t, s, $t > s$,*

$$H^1_t > H^1_s\,, \tag{4.11}$$

where H^1_t is defined by (3.4).

Proof. See Appendix C.

Lemma 4.3. *For all $t < N$,*

$$H^1_t < G^N_t\,. \tag{4.12}$$

Proof. Follows directly from the definitions of H^1_t and G^N_t. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. It suffices to prove that for any fixed stopping rule τ^f_2 of the second detector the corresponding threshold l^1_s of the first detector satisfies

$$\lambda^N_s < l^1_s, \quad s = 1, 2, \ldots, N-1\,. \tag{4.13}$$

From Lemma 3.5, $\pi > l^1_s$ if and only if

$$\pi > (H^1_t)^{-1} \min_{\tau \geqslant s} E\left\{H^1_\tau 1(x_\tau = 0) + c \sum_{m=s}^{\tau-1} 1(x_m = 0) \mid \pi_s = \pi\right\}. \tag{4.14}$$

The minimum in the right-hand side of (4.14) is achieved for some finite τ, $\tau = \tau(\pi)$ [1]. Let $N \geqslant \tau$ Then, because of Lemmas 4.1–4.3 and the selection of N, (4.14) gives

$$\pi > \min_{s \le \tau \le N} E\left\{ \frac{G_\tau^N}{G_s^N} 1(x_\tau = 0) + (G_s^N)^{-1} c \sum_{m=s}^{\tau-1} 1(x_m = 1) \mid \pi_s = \pi \right\}. \tag{4.15}$$

Equation (4.15) implies that

$$G_s^N \pi > M_s^N(\pi). \tag{4.16}$$

Consequently, (4.14)–(4.16) show that

$$l_s^1 > \lambda_s^N, \quad s = 1, 2, \dots, N-1, \; Q.E.D.. \tag{4.17}$$

Consider next a single-detector finite horizon $N(N > s)$, disruption problem with the cost (2.2)–(2.4) and observations

$$y_t = g^2(x_2, w_t^2), \tag{4.18}$$

where g^2 and $\{w_t^2\}$ are the same as in (2.4) for $i = 2$ and $\{w_t^2\}$, $t = 1, 2, \dots, N$, is a sequence of i.i.d. random variables which are independent of x_t. If μ_s^N is the optimal threshold at time s for that single-detector disruption problem, it can be shown, by the same arguments as above, that

$$\mu_s^N < l_s^2. \tag{4.19}$$

The result of Theorem 4.1 is similar to that of [3]. However, whereas in [3] comparison with a stationary threshold of a single-detector problem is possible, such a comparison is impossible for problem (P). Intuitively, this can be explained as follows. For the problem considered in [3] at each instant of time each detector has to account for an average cost due to a false alarm of the other detector. Such a cost varies with time, but always remains less than one. This feature makes possible the comparison of the decentralized *m.b.m.o.* thresholds with a stationary threshold resulting from a single-detector disruption problem. For the problem considered in this paper the average cost due to false alarms of each detector's team-mate is not necessarily bounded. This feature of the cost makes the comparison of thresholds a more subtle problem. Comparison with the threshold of a disruption problem with a single detector is only possible if that threshold corresponds to a cost which has similar features as the average cost due to a false alarm in problem (P). This can be accomplished if one restricts attention to finite horizon single-detector disruption problems which result in time-varying thresholds.

5. Concluding remarks

The problem considered in this paper is similar to those of [3]–[8] in that there is no communication between the two detectors. Even though the problem treated here is one of the simplest decentralized sequential detection problems, the coupling induced by the cost structure causes (as in [3] and [4]), considerable complexity in the optimal stopping rules. Computation of the optimal thresholds is very difficult since it requires the solution of two interdependent dynamic programming equations. It is possible, however, to lowerbound the optimal decentralized thresholds by the thresholds of well-known finite horizon single-detector disruption problems. The thresholds of the single-detector disruption problems are time varying. A similar comparison was achieved in [3]. However, whereas in [3] the optimal decentralized thresholds are compared to a stationary threshold, such a comparison is impossible for the present problem.

The results of this paper provide only a qualitative characterization of the optimal decision rules of the detectors. The qualitative properties of the optimal decision rules as well as the comparison between the optimal decentralized thresholds and the thresholds of a class of single-detector disruption problems may provide the basis (as in the case of the Decentralized Wald problem [4]) for the development of practical algorithms for the problem studied in this paper.

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Appendix A: Proof of the equality of the cost functions (3.1) and (3.3)

Consider first the term,

$$E\left\{c\sum_{l=1}^{\tau_1-1} 1(x_l=1)\right\}, \tag{A.1}$$

in (3.1).

Using

$$E(f(x)) = E\{E(f(x)\mid Y)\}, \tag{A.2}$$

we can write

$$E\left\{c\sum_{l=1}^{\tau_1-1} 1(x_l=1)\right\} = E\left\{c\sum_{l=1}^{\tau_1-1}(1-\pi_l^1)\right\}, \tag{A.3}$$

Next, consider the term

$$E\{k(\theta-\tau_1+\theta-\tau_2^*)1(x_{\tau_2^*}=0)1(x_{\tau_1}=0)\}. \tag{A.4}$$

Since henceforth attention is focused on (A.4) with fixed τ_2^*, the index 1 is dropped. Thus, the aim is to show that

$$E\{k(\theta-\tau+\theta-\tau_2^*)1(x_{\tau_2^*}=0)1(x_\tau=0)\} = E\{H_\tau^1\pi_\tau^1\}, \tag{A.5}$$

where H_τ^1 is given by (3.4)–(3.5). Consider first the term $E\{k(\theta-\tau)1(x_{\tau_2^*}=0)1(x_\tau=0)\}$:

$$\begin{aligned} E\{k(\theta-\tau)1(x_{\tau_2^*}=0)1(x_\tau=0)\mid Y^{1\tau}\} &= E\{k(\theta-\tau)1(\tau<\theta)1(\tau_2^*<\theta)\mid Y^{1\tau}\} \\ &= E\{k(\theta-\tau)1(\tau<\theta)h_1(\theta)\mid Y^{1\tau}\}, \end{aligned} \tag{A.6}$$

where

$$h_1(\theta) = E\{1(\tau_2^*<\theta)\mid\theta\}. \tag{A.7}$$

The last equality in (A.6) holds because the observations of the detectors are independent conditioned on the time of the jump.

We can write

$$E\{k(\theta-\tau)1(\tau<\theta)h_1(\theta)\mid Y^{1\tau}\}=k\sum_{l=\tau+1}^{\infty}h_1(l)(l-\tau)\operatorname{Prob}(\theta=l\mid Y^{1\tau})$$
$$=k\sum_{l=1}^{\infty}h_1(\tau+l)l\operatorname{Prob}(\theta=\tau+l\mid Y^{1\tau})\,. \tag{A.8}$$

But

$$\operatorname{Prob}(\theta=\tau+l\mid Y^{1\tau})=\operatorname{Prob}(\theta>\tau\mid Y^{1\tau})\operatorname{Prob}(\theta=\tau+l\mid x_\tau=0)=\pi_\tau^1(1-p)^{l-1}p\,. \tag{A.9}$$

Consequently, (A.6) and (A.8) give:

$$kE\{(\theta-\tau)1(x_{\tau_2^*}=0)1(x_\tau=0)\mid Y^{1\tau}\}=k\sum_{l=1}^{\infty}l(1-p)^{l-1}pE\{1(\tau_2^*<\theta)\mid\theta=\tau+l\}\pi_\tau^1\,. \tag{A.10}$$

Using the equality

$$\operatorname{Prob}(\tau_2^*<\theta\mid\theta>\tau)=\sum_{l=1}^{\infty}\operatorname{Prob}(\tau_2^*<\theta\mid\theta=\tau+l)(1-p)^{l-1}p\,, \tag{A.11}$$

we can also write (A.10) as

$$kE\{(\theta-\tau)1(x_{\tau_2^*}=0)1(x_\tau=0)\mid Y^{1\tau}\}=k\sum_{l=1}^{\infty}(1-p)^{l-1}\operatorname{Prob}(\theta>\tau_2^*\mid\theta>\tau+l-1)\pi_\tau^1\,. \tag{A.12}$$

Finally, consider the term $E\{k(\theta-\tau_2^*)1(x_{\tau_2^*}=0)1(x_\tau=0)\}$:

$$E\{k(\theta-\tau_2^*)1(x_{\tau_2^*}=0)1(x_\tau=0)\mid Y^{1\tau}\}=kE\{(\theta-\tau_2^*)1(\tau_2^*<\theta)1(\tau<\theta)\mid Y^{1\tau}\}$$
$$=kE\{1(\tau<\theta)h_2(\theta)\mid Y^{1\tau}\}\,, \tag{A.13}$$

where

$$h_2(\theta)=E\{(\theta-\tau_2^*)1(\tau_2^*<\theta)\mid\theta\}\,. \tag{A.14}$$

The last equality in (A.13) holds because the observations of the detectors are independent conditioned on the time of jump.

Using (A.13) and (A.9) we can write

$$E\{(\theta-\tau_2^*)1(\tau_2^*<\theta)1(\tau<\theta)\mid Y^{1\tau}\}=\sum_{l=1}^{\infty}E\{(\theta-\tau_2^*)1(\tau_2^*<\theta)\mid\theta=\tau+l\}(1-p)^{l-1}p\pi_\tau^1$$
$$=\sum_{l=1}^{\infty}\left\{\sum_{m=1}^{\tau+l-1}(\tau+l-m)\operatorname{Prob}(\tau_2^*=m\mid\theta=\tau+l)\right\}(1-p)^{l-1}p\pi_\tau^1$$
$$=\sum_{l=1}^{\infty}\left\{\sum_{m=1}^{\tau+l-1}\operatorname{Prob}(\tau_2^*\le m\mid\theta=\tau+l)\right\}(1-p)^{l-1}p\pi_\tau^1$$
$$=\left[\sum_{m=1}^{\tau}\operatorname{Prob}(\tau_2^*\le m\mid\theta>\tau)+\sum_{l=1}^{\infty}\operatorname{Prob}(\tau_2^*\le\tau+l\mid\theta>\tau+l)(1-p)^{l}\right]\pi_\tau^1\,. \tag{A.15}$$

Note that when $m \leq \tau$,

$$\operatorname{Prob}(\tau_2^* \leq m \mid \theta > \tau) = \operatorname{Prob}(\tau_2^* \leq m \mid \theta > m) . \tag{A.16}$$

Consequently, (A.13), (A.15) and (A.16) give

$$\begin{aligned} kE\{(\theta - \tau_2^*)1(x_{\tau_2^*} = 0)1(x_\tau = 0) \mid Y^{1\tau}\} = \Big[&\sum_{m=0}^{\tau} \operatorname{Prob}(\tau_2^* \leq m \mid \theta > m) \\ &+ \sum_{l=1}^{\infty} \operatorname{Prob}(\tau_2^* \leq \tau + l \mid \theta > \tau + l)(1-p)^l \Big] \pi_\tau^1 . \end{aligned} \tag{A.17}$$

An alternative expression for $kE\{(\theta - \tau_2^*)1(x_{\tau_2^*} = 0)1(x_\tau = 0) \mid Y^{1\tau}\}$ can be derived as follows. Using (A.9) and (A.13) we can write

$$\begin{aligned} &kE\{(\theta - \tau_2^*)1(\tau_2^* < \theta)1(\tau < \theta) \mid Y^{1\tau}\} \\ &= k\sum_{l=1}^{\infty} E\{(\theta - \tau_2^*)1(\tau_2^* < \theta) \mid \theta = \tau + l\} \operatorname{Prob}(\theta = \tau + l \mid \theta > \tau)\pi_\tau^1 \\ &= kE\{(\theta - \tau_2^*)1(\tau_2^* < \theta) \mid \theta > \tau)\}\pi_\tau^1 \\ &= kE\{(\theta - \tau_2^*) \mid \theta > \tau_2^* \vee \tau\} \operatorname{Prob}(\theta > \tau_2^* \mid \theta > \tau)\pi_\tau^1 , \end{aligned} \tag{A.18}$$

where

$$\tau_2^* \vee \tau = \max(\tau_2^*, \tau) . \tag{A.19}$$

But

$$\begin{aligned} E\{\theta \mid \theta > \tau_2^* \vee \tau\} &= E\{\theta \mid \theta > \tau, \tau \geq \tau_2^*\} \operatorname{Prob}(\tau \geq \tau_2^* \mid \theta > \tau_2^* \vee \tau) \\ &\quad + \sum_{l=1}^{\infty} E\{\theta \mid \theta > \tau_2^*, \tau_2^* = \tau + l\} \operatorname{Prob}(\tau_2^* = \tau + l \mid \theta > \tau_2^* \vee \tau) \\ &= \left(\frac{1}{p} + \tau\right) \operatorname{Prob}(\tau \geq \tau_2^* \mid \theta > \tau_2^* \vee \tau) \\ &\quad + \sum_{l=1}^{\infty} \left[\frac{1}{p} + (\tau + l)\right] \operatorname{Prob}(\tau_2^* = \tau + l \mid \theta > \tau_2^* \vee \tau) \\ &= \frac{1}{p}\left[\operatorname{Prob}(\tau_2^* \leq \tau \mid \theta > \tau_2^* \vee \tau) + \operatorname{Prob}(\tau_2^* > \tau \mid \theta > \tau_2^* \vee \tau)\right] \\ &\quad + \tau \operatorname{Prob}(\tau \geq \tau_2^* \mid \theta > \tau_2^* \vee \tau) + \sum_{l=1}^{\infty} (\tau + l) \operatorname{Prob}(\tau_2^* = \tau + l \mid \theta > \tau_2^* \vee \tau) \\ &= \frac{1}{p} + E\{\tau \vee \tau_2^* \mid \theta > \tau_2^* \vee \tau\} . \end{aligned} \tag{A.20}$$

Because of (A.20) and the first equality in (A.13), (A.18) gives

$$kE\{(\theta - \tau_2^*)1(x_{\tau_2^*} = 0)1(x_\tau = 0) \mid Y^{1\tau}\}$$

$$= k\left\{\frac{1}{p} + E[\tau \vee \tau_2^* - \tau_2^* \mid \theta > \tau_2^* \vee \tau]\right\} \operatorname{Prob}(\theta > \tau_2^* \mid \theta > \tau)\pi_\tau^1$$

$$= k\left\{\frac{1}{p} + E[(\tau - \tau_2^*)^+ \mid \theta > \tau_2^* \vee \tau]\right\} \operatorname{Prob}(\theta > \tau_2^* \mid \theta > \tau)\pi_\tau^1 , \tag{A.21}$$

where

$$[x]^+ = \max(x, 0) . \tag{A.22}$$

Because of (A.2), (A.10), (A.12), (A.17) and (A.21) we get

$$E\{k(\theta - \tau + \theta - \tau_2^*)1(x_{\tau_2^*} = 0)1(x_\tau = 0)\} = E\{H_\tau^1 \pi_\tau^1\} , \tag{A.23}$$

where H_τ^1 is defined by (3.4)–(3.5). Combining (A.3) and (A.23) we obtain the equality of the cost functions (3.1) and (3.3).

Appendix B: Proof of Lemma 3.2

The inequality (3.14) at $\pi = 0$ is true because $c > 0$ and $(LV_{t+1}^{1T})(\pi)$ is non-negative.
To prove (3.15) note that because of (3.9)–(3.12),

$$(LV_{t+1}^{1T})(\pi) \leq (1-p)H_{t+1}^{1T}\pi . \tag{B.1}$$

Thus, to complete the proof of the lemma, it is enough to show that

$$(1-p)H_{t+1}^{1T} \leq H_t^{1T} . \tag{B.2}$$

We can easily show, following the arguments in Appendix A, that

$$H_t^{1T} = \frac{k}{B}\sum_{l=1}^{T-t}(1-p)^{l-1}\operatorname{Prob}(\theta > \tau_2^* \mid \theta > t+l) + \frac{k}{B}\sum_{l=0}^{t}\operatorname{Prob}(\tau_2^* \leq l \mid \theta > l)$$
$$+ \frac{k}{B}\sum_{l=1}^{T-t}\operatorname{Prob}(\tau_2^* \leq t+l \mid \theta > t+l)(1-p)^l , \tag{B.3}$$

where

$$B = \sum_{l=1}^{T}(1-p)^{l-1}p . \tag{B.4}$$

The inequality (B.2) then follows directly from (B.3) and (B.4). Q.E.D.

Appendix C: Proof of Lemma 4.2

By definition (equation (3.4))

$$H_t^1 = k \sum_{l=1}^{\infty} l(1-p)^{l-1} p E\{1(\tau_2^* < \theta) \mid \theta = t + l\}$$
$$+ k \sum_{r=0}^{t} \text{Prob}(\tau_2^* \leq r \mid \theta > r) + k \sum_{l=1}^{\infty} \text{Prob}(\tau_2^* \leq t + l \mid \theta > t + l)(1-p)^l . \tag{C.1}$$

When $t > s$, then for any l,

$$E\{1(\tau_2^* < \theta) \mid \theta = t + l\} = \text{Prob}(\tau_2^* < \theta \mid \theta = t + l)$$
$$> \text{Prob}(\tau_2^* < \theta \mid \theta = s + l) = E\{1(\tau_2^* < \theta) \mid \theta = s + l\} . \tag{C.2}$$

Moreover, for $t > s$,

$$\sum_{r=0}^{t} \text{Prob}(\tau_2^* \leq r \mid \theta > r) + \sum_{l=0}^{\infty} \text{Prob}(\tau_2^* \leq t + l \mid \theta > t + l)(1-p)^l$$
$$> \sum_{r=0}^{s} \text{Prob}(\tau_2^* \leq r \mid \theta > r) + \sum_{l=1}^{\infty} \text{Prob}(\tau_2^* \leq t + l \mid \theta > t + l)(1-p)^l . \tag{C.3}$$

From (C.1)–(C.3) it is clear that $H_t^1 > H_s^1$ for $t > s$.

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