In Section IV of the paper, remarks 4 iii)–v) as well as the proofs of Lemmas 4 and 5, to the extent that they depend on these remarks, are incorrect. While the remarks pertain to general functions in $\mathscr{L}_e^\infty$, the lemmas deal with more restrictive classes of functions. Hence, the statement of the lemmas themselves and consequently the proof of stability, are not affected.

Remark 4 iii) states that if two continuous functions $x(\cdot), y(\cdot)$ are in $\mathscr{L}_e^\infty$, then only one of the following conditions holds:

$$\sup_{\tau\leqslant t}|x(\tau)| \sim \sup_{\tau\leqslant t}|y(\tau)|,$$

$$\sup_{\tau\leqslant t}|x(\tau)| = o\left[\sup_{\tau\leqslant t}|y(\tau)|\right],$$

$$\sup_{\tau\leqslant t}|y(\tau)| = o\left[\sup_{\tau\leqslant t}|x(\tau)|\right].$$

This is not true in general, as shown by the following example.

Example: Let $\{t_i\}$, $\{t_i'\}$, and $\{T_i\}$ be three unbounded sequences in R^+ such that $t_i' < t_i < t_{i-1}'$, $T_0 = T_1 = 0$, $T_{2i+1} = (t_{i+1}' - t_i) + T_{2i-1}$, and $T_{2i} = (t_i - t_i') + T_{2i-2}$, where $i \in \{1,2,\cdots\}$. Let $x(\cdot), y(\cdot)$ be two functions defined as:

$$x(t) = t + (T_{2i-2} - t_i'), \quad y(t) = T_{2i-1} \qquad t_i' \leqslant t < t_i$$

$$x(t) = T_{2i}, \quad y(t) = t + (T_{2i-1} - t_i) \qquad t_i \leqslant t < t_{i-1}'.$$

Choosing the sequences $\{t_i\}$ and $\{t_i'\}$ in such a manner that $\lim_{i\to\infty}(T_{i+1}/T_i) = \infty$, we see that $x(\cdot)$ and $y(\cdot)$ do not satisfy 4 iii). Similar examples can also be given to invalidate remarks 4 iv) and v). In the proofs of Lemmas 4 and 5, Remark 4 is used to prove that a function $|f(t)| = O[\sup_{\tau\leqslant t}|g(\tau)|]$ by demonstrating that $|g(t)| \neq O[\sup_{\tau\leqslant t}|f(\tau)|]$. In what follows we present alternate proofs of Lemmas 4 and 5.

Lemma 4: Let $x(\cdot)$ and $y(\cdot)$ be the input and output, respectively, of a system with a rational transfer function whose zeros are in the open left half plane. If $x(\cdot)$ is such that i) $x(\cdot) \in \mathscr{L}_e^\infty$ and ii) $|\dot{x}(t)| = O[\sup_{\tau\leqslant t}|x(\tau)|]$, then $|x(t)| = O[\sup_{\tau\leqslant t}|y(\tau)|]$.

Proof: With $H_1(s)$, $H_2(s)$ and x_1 as defined in [1]

$$\sup_{\tau\leqslant t}|x_1(\tau)| = O\left[\sup_{\tau\leqslant t}|y(\tau)|\right].$$

If $h(t)$ is the impulse response of $H_2(s)$, then for a $\Delta > 0$ and any $t_i \in R^+$

$$x_1(t_i + \Delta) = c_0 x_1(t_i) + \int_{t_i}^{t_i+\Delta} h(t_i + \Delta - \tau)x(\tau)\,d\tau \tag{1}$$

where $c_0 x_1(t_i)$ is due to initial conditions. Let $|x(t)| = \sup_{\tau\leqslant t}|x(\tau)|$ for all $t \in [t_i, t_i + \Delta]$. Then

$$|x_1(t_i + \Delta)| \geqslant |\{c_2|x(t_i)| - |c_0 x_1(t_i)|\}| \tag{2}$$

where $c_2 > 0$ is as defined in the paper[1]. Further, by assumption ii) a constant $0 < c_3 < 1$ exists such that $|x(t_i)| \geqslant c_3|x(t_i + \Delta)|$. Hence, in (2), if $|x_1(t_i)| < M|x(t_i)|$, then $|x(t_i + \Delta)| \leqslant (1/(M+\delta))|x_1(t_i+\Delta)|$ where $M = (c_2c_3 - \delta)/(|c_0|c_3 + 1) > 0$ and $\delta > 0$. Since t_i is arbitrary, this condition holds for every t for which $|x(t)| = \sup_{\tau\leqslant t}|x(\tau)|$. Further, since $\sup_{\tau\leqslant t}|x_1(\tau)| \geqslant |x_1(t)|$ for all $t \in R^+$, it follows that

$$\sup_{\tau\leqslant t}|x(\tau)| = O\left[\sup_{\tau\leqslant t}|x_1(\tau)|\right] \quad \text{and hence} \quad O\left[\sup_{\tau\leqslant t}|y(\tau)|\right].$$

Corollary: Lemma 4 is valid when condition ii) is replaced by the condition $|\dot{x}(t)| = O[\sup_{\tau\leqslant t}\|z(\tau)\|]$ where $z^T(t) = [x(t), x_1(t)]$. The same arguments carry over since $|x_1(t_i)| < M|x(t_i)|$ implies $\|z(t_i)\| < \sqrt{1+M^2}\,|x(t_i)|$, and hence

$$|\dot{x}(t_i)| \leqslant M_1 \sup_{\tau\leqslant t_i}\|z(\tau)\| \leqslant M_1\sqrt{1+M^2}\sup_{\tau\leqslant t_i}|x(\tau)|.$$

Lemma 5: In the adaptive control loop $u(t)$, $v_i^{(1)}(t), v_i^{(2)}(t) = O[\sup_{\tau\leqslant t}|y_p(\tau)|]$ $i = 1,2,\cdots,n-1$.

Proof: Since $\dot{v}^{(1)}(t) = O[\sup_{\tau\leqslant t}\|\omega(\tau)\|]$ and $y_p(t) = O[\sup_{\tau\leqslant t}\|\omega(\tau)\|]$ we have $\dot{v}^{(1)}(t) = O[\sup_{\tau\leqslant t}\|v(\tau)\|]$ where $v^T(t) = [v^{(1)^T}(t), v^{(2)^T}(t)]$.

Since the plant transfer function $W_p(s)$ has zeros in the open left half plane and $W_p(s)Iv^{(1)}(t) = v^{(2)}(t)$, it follows from the above corollary that $\|v^{(1)}(t)\| = O[\sup_{\tau\leqslant t}\|v^{(2)}(\tau)\|] = O[\sup_{\tau\leqslant t}|y_p(\tau)|]$. Since $u(t) = \theta^T(t)\omega(t) + r(t)$ where $\|\theta(t)\|$ and $r(t)$ are uniformly bounded, $|u(t)| = O[\sup_{\tau\leqslant t}\|\omega(\tau)\|] = O[\sup_{\tau\leqslant t}|y_p(\tau)|]$.

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The Decentralized Quickest Detection Problem

D. TENEKETZIS AND P. VARAIYA

Abstract—**Two detectors making independent observations must decide when a Markov chain jumps from state 0 to state 1. The decisions are coupled through a common cost function. It is shown that the optimal decision is characterized by thresholds as in the decoupled case. However, the thresholds are time-varying and their computation requires the solution of two coupled sets of dynamic programming equations. A comparison to the decoupled case shows the structure of the coupling.**

I. INTRODUCTION

Two detectors make independent observations of a Markov chain x_t which jumps from state 0 to state 1 at the random time ϑ. Based on its own observation detector i declares that the jump occurred at time τ_i, $i = 1,2$. The problem is to find stopping times τ_i that minimize the expected cost $EJ(\tau_1, \tau_2, \vartheta)$.

If the cost is separable, $J(\tau_1, \tau_2, \vartheta) = J_1(\tau_1, \vartheta) + J_2(\tau_2, \vartheta)$, then the two decisions are decoupled. In this case, for certain costs J_i, the optimal decision τ_i^* is to stop when the "false alarm" probability drops below a *threshold* p_i^*, i.e.,

$$\tau_i^* = \min\{t | P(\vartheta > t | Y_t^i) \leqslant p_i^*\}, \tag{1.1}$$

where Y_t^i is the information available to i at time t. This threshold property holds for the cost function $J_i(\tau, \vartheta) = 1(\tau < \vartheta) + c(\tau - \vartheta)1(\tau \geqslant \vartheta)$. See [1], [2]. Here $1(A)$ is the indicator of A.

In this paper the cost function considered is

$$J(\tau_1, \tau_2, \vartheta) = 1(\tau_1 < \vartheta)1(\tau_2 < \vartheta) + c_1(\tau_1 - \vartheta)1(\tau_1 \geqslant \vartheta) + c_2(\tau_2 - \vartheta)1(\tau_2 \geqslant \vartheta). \tag{1.2}$$

This cost puts a constant penalty for false alarms and, for each detector, a penalty proportional to the delay in detecting the jump. Since this cost is not decoupled there is an interaction between the optimal decisions. The

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interaction is simple since there is no communication between the detectors. The same problem is considered in [3]. The proofs given here are simpler and the results are more general.

Theorem 3.1 states that the optimal stopping time τ_i^* is still characterized by a threshold $p_i^*(t)$, which however varies with time [unlike (1.1)]:

$$\tau_i^* = \min\{t|P(\vartheta > t|Y_t^i) \leq p_i^*(t)\}.$$

The computation of these thresholds requires a simultaneous solution of a pair of coupled dynamic programming equations.

If instead of (1.2) one considers the separable cost

$$J(\tau_1,\tau_2,\vartheta) = 1(\tau_1<\vartheta)+1(\tau_2<\vartheta)+c_1(\tau_1-\vartheta)1(\tau_1\geq\vartheta) + c_2(\tau_2-\vartheta)1(\tau_2\geq\vartheta), \quad (1.3)$$

then the optimal decisions τ_i^d are decoupled and characterized by constant thresholds p_i^d, say. Theorem 4.1 states that $p_i^d \leq p_i^*(t)$ for all t so that $\tau_i^d \geq \tau_i^*$. Theorem 4.2 states that under some additional conditions $p_i^*(t) \to p_i^d$ as $t\to\infty$.

The remainder of the paper is organized as follows. The formal model is presented in Section II. Section III is devoted to a proof of the threshold property, and Section IV developes some properties of the thresholds.

II. The Model

Let $\{x_t, t=1,2,\cdots\}$ be a Markov chain with values in $\{0,1\}$ and known transition probabilities

$$P\{x_{t+1}=1|x_t=0\}=q, \quad P\{x_{t+1}=1|x_t=1\}=1, \quad P\{x_1=0\}=p_0.$$

Thus, the chain makes a single jump to 1 at the random time $\vartheta := \min_t\{x_t = 1\}$.

Let $\{w_t^i\}$, $i=1,2$, be mutually independent iid sequences, also independent of $\{x_t\}$. Detector i's observation at t is $y_t^i = f^i(x_t, w_t^i)$. Let $Y_t^i := \sigma(y_s^i, s\leq t)$, and let τ_i denote Y_t^i stopping times. The problem is

$$\min_{\tau_1}\min_{\tau_2} E\left\{1(x_{\tau_1}=0)1(x_{\tau_2}=0)+c_1\sum_{t=1}^{\tau_1-1}1(x_t=1)+c_2\sum_{t=1}^{\tau_2-1}1(x_t=1)\right\}. \quad (2.1)$$

Fix τ_2, possibly at the optimum. Then detector 1's problem is

$$\min_{\tau_1} E\left\{\gamma(\tau_2-\vartheta)1(x_{\tau_1}=0)+c_1\sum_{t=1}^{\tau_1-1}1(x_t=1)\right\}, \quad (2.2)$$

where

$$\gamma(\tau_2-\vartheta) := 1(x_{\tau_2}=0) = 1(\tau_2<\vartheta). \quad (2.3)$$

Since attention is henceforth focused on problem (2.2), the index 1 is dropped. Thus the aim is to find a Y_t stopping time τ to minimize $EJ(\tau)$ where

$$EJ(\tau) := E\left\{\gamma(\tau_2-\vartheta)1(x_\tau=0)+c\sum_{t=1}^{\tau-1}1(x_t=1)\right\}. \quad (2.4)$$

Until Section IV the special form (2.3) of γ is not required. It is assumed only that $c>0$ and that γ is nonnegative and bounded.

III. Dynamic Programming

To express (2.4) in a more convenient form introduce the statistic

$$p_t := P\{x_t=0|Y_t\} = P\{\vartheta>t|Y_t\}, \quad (3.1)$$

and observe that

$$P\{x_{t+s}=0|Y_t\} = p_t(1-q)^s, \qquad s\geq 0. \quad (3.2)$$

Then the second term on the right in (2.4) equals $Ec\sum_{t=1}^{\tau-1}(1-p_t)$. The first term equals

$$E[1(x_\tau=0)\gamma(\tau_2-\vartheta)] = E[1(\tau<\vartheta)E\{\gamma(\tau_2-\vartheta)|Y_\tau V\sigma(\vartheta)\}] = E[1(\tau<\vartheta)g(\vartheta)]$$

where $g(\vartheta) := E\{\gamma(\tau_2-\vartheta)|\vartheta\}$. The last equality above follows from the fact that $P\{B|Y_\infty^1 V\sigma(\vartheta)\} = P\{B|\sigma(\vartheta)\}$ for $B\in Y_\infty^2$. Finally,

$$\begin{aligned}E[1(\tau<\vartheta)g(\vartheta)] &= E\left[\sum_{i=1}^{\infty} g(\tau+i)E\{1(\vartheta=\tau+i)|Y_\tau\}\right]\\ &= E\sum_{j=1}^{\infty} g(\tau+j)p_\tau(1-q)^{j-1}q, \qquad \text{using (3.2)}\\ &=: EG_\tau p_\tau, \qquad \text{where } G_t := \sum_{i=1}^{\infty}E\{\gamma(\tau_2-\vartheta)|\vartheta=t+i\}(1-q)^{i-1}q.\end{aligned} \quad (3.3)$$

Thus, the problem is to find a Y_t stopping time τ to minimize

$$EJ(\tau) = E\left\{c\sum_{t=1}^{\tau-1}(1-p_t)+G_\tau p_\tau\right\}.$$

Let $P_i(y)$ be the probability density of y_t conditioned on $x_t = i$, and

$$q(y|p) := p(1-q)P_0(y)+[pq+(1-p)]P_1(y),$$
$$\Phi(p,y) := p(1-q)P_0(y)/q(y|p).$$

A familiar argument using Bayes' rule gives the "updating" formulas

$$p(y_{t+1}|Y_t) = q(y_{t+1}|p_t), \qquad p_{t+1} = \Phi(p_t, y_{t+1}). \quad (3.4)$$

From (3.4) one can conclude using dynamic programming arguments that the optimal stopping time is based only on the "sufficient" statistic p_t. Furthermore, if G_t is constant then known results imply that the optimal rule is characterized by a threshold [1], [2]. For the time varying case it is necessary to study the value function. To do this it is convenient to first consider a finite horizon N and then let $N\to\infty$.

Finite Horizon

Fix $N<\infty$ and consider the problem

$$\min_{1\leq\tau\leq N} EJ(\tau). \quad (3.5)$$

Define the operator Ψ which transforms any function $W(p)$, $p\in[0,1]$, into

$$[\Psi W](p) := \int W(\Phi(p,y))q(y|p)\,dy, \quad (3.6)$$

and define the functions W_t by

$$W_N(p) = G_N p, \quad (3.7)$$
$$W_t(p) = \min\{G_t p, c(1-p)+[\Psi W_{t+1}](p)\}, t<N. \quad (3.8)$$

A dynamic programming argument then shows that W_t is the value function, i.e.,

$$W_t(p) = \min_{t\leq\tau\leq N} E\left\{G_\tau 1(x_\tau=0)+c\sum_{s=t}^{\tau-1}1(x_s=1)|p_t=p\right\}. \quad (3.9)$$

This indeed implies that the optimal decision at t need only be based on p_t. In fact it is optimal to stop at t if and only if

$$G_t p_t \leq c(1-p_t)+[\Psi W_{t+1}](p_t).$$

The threshold property is based on the following fact.

Lemma 3.1: $W_t(p)$ is a concave nonnegative function of $p, t = 1, \cdots, N$.

Proof: By (3.7) the assertion is true for $t = N$. Suppose W_{t+1} is concave. Then there is a collection of affine functions $\alpha_i p + \beta_i$, $i \in I$, such that,

$$W_{t+1}(p) = \inf_i \{\alpha_i p + \beta_i\}.$$

With this representation, and (3.6),

$$[\Psi W_{t+1}](p) = \int \inf_i \{\alpha_i \Phi(p, y_{t+1}) + \beta_i\} q(y_{t+1}|p)\, dy_{t+1}$$
$$= \int \inf_i \left\{\alpha_i \frac{p(1-q)P_0(y_{t+1})}{q(y_{t+1}|Y_t)} + \beta_i\right\} q(y_{t+1}|p)\, dy_{t+1}$$
$$= \int \inf_i \{\alpha_i p(1-q)P_0(y_{t+1}) + \beta_i q(y_{t+1}|p)\}\, dy_{t+1}.$$

Hence, $[\Psi W_{t+1}]$ is concave since the term within $\{\cdots\}$ is affine in p. From (3.8) it follows that W_t is concave. Since the G_t are nonnegative, so are the W_t. □

Since $c > 0$, $G_t p < c(1-p) + [\Psi W_{t+1}](p)$ at $p = 0$. This inequality and Lemma 3.1 imply the threshold property.

Infinite Horizon

To minimize (2.4) take $N \to \infty$ in (3.5). So let W_t^N $1 \leqslant t \leqslant N$, denote the value functions defined by (3.7), (3.8). Since the set of Y_t-stopping times τ, with $\tau \leqslant N$ a.s., increases with N it follows that $W_t^{N+1}(p) \leqslant W_t^N(p)$, and so the following limit is defined:

$$W_t(p) := \lim_{N \to \infty} W_t^N(p) = \inf_N W_t^N(p). \tag{3.10}$$

Theorem 3.1: The value functions $W_t(p)$ for detector 1 satisfy

$$W_t(p) = \min\{G_t p, c(1-p) + [\Psi W_{t+1}](p)\}, \qquad t = 1, 2, \cdots. \tag{3.11}$$

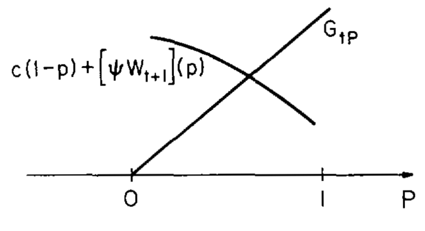
$W_t(p)$ is nonnegative and concave. Define $p^*(t)$ by

$$G_t p \leqslant c(1-p) + [\Psi W_{t+1}](p) \qquad \text{iff } p \leqslant p^*(t). \tag{3.12}$$

Then the optimal stopping time for detector 1 is

$$\tau^* = \min\{t | p_t \leqslant p^*(t)\}. \tag{3.13}$$

Proof: One gets (3.11) from (3.8); and concavity and nonnegativity follow from Lemma 3.1. Since W_t is concave (3.12) defines $p^*(t)$; see the figure below. From (3.11), the optimal decision at t is to stop iff $W_t(p_t) = G_t p_t$ which gives the rule (3.13). ■



It is interesting to note the uniqueness of the solution to (3.11).

Theorem 3.2: The value functions $\{W_t(p)\}$ give the unique solution to

$$V_t(p) = \min\{G_t p, c(1-p) + [\Psi V_{t+1}](p)\}, \qquad t = 1, 2, \cdots. \tag{3.14}$$

Proof: To show $V_t(p) \leqslant W_t(p)$, consider $W_t^N, t \leqslant N$. Then from (3.14)

$$W_N^N(p) = G_N p \geqslant V_N(p).$$

Suppose $W_{t+1}^N \geqslant V_{t+1}$. Then

$$W_t^N(p) = \min\{G_t p, c(1-p) + [\Psi W_{t+1}^N](p)\}$$
$$\geqslant \min\{G_t p, c(1-p) + [\Psi V_{t+1}(p)]\}$$
$$= V_t(p),$$

where the inequality follows from the fact that $f \geqslant g$ implies $\Psi f \geqslant \Psi g$. Letting $N \to \infty$ proves the inequality. To show $V_t(p) \geqslant W_t(p)$ fix t, p, and consider the problem starting at time t with $p_t = p$. Define the stopping time $\tau \geqslant t$ by

$$\tau = \min\{s \geqslant t | G_s p_s \leqslant c(1-p_s) + [\Psi V_{s+1}](p_s)\}.$$

Then

$$V_t(p_t) = c(1-p_t) + E\{V_{t+1}(p_{t-1}) | Y_t\},$$
$$\cdots$$
$$V_{\tau-1}(p_{\tau-1}) = c(1-p_{\tau-1}) + E\{V_\tau(p_\tau) | Y_{\tau-1}\},$$
$$V_\tau(p_\tau) = G_\tau p_\tau.$$

Adding and taking expectations conditioned on Y_t gives

$$V_t(p_t) = E\left\{\sum_{s=t}^{\tau-1} c1(x_s = 1) + G_\tau 1(x_\tau = 0) | Y_t\right\} \geqslant W_t(p_t)$$

since W_t is the minimum cost. ■

Remark: Thus, the solution to (2.1) is obtained by solving a pair of coupled equations similar to (3.14). The coupling arises from the fact that for detector i the penalty for false alarm at time t is G_t^i which depends on τ_j^* [see (3.3)]. Second, it should be clear that a pair of stopping times that satisfy these coupled equations only guarantee person-by-person optimality. Third, to prove the existence of a person-by-person optimum one may argue as follows. Obtain sequences $\{G_t^i(n), W_t^i(n)\}$ for detector i, $n = 1, 2, \cdots$ such that $\{G_t^1(n), W_t^1(n)\}$ are optimal for $\{G_t^2(n), W_t^2(n)\}$ and $\{G_t^2(n+1), W_t^2(n+1)\}$ are optimal for $\{G_t^1(n), W_t^1(n)\}$. The $G_t^i(n)$ are uniformly bounded, and the $W_t^i(n)$ are uniformly bounded and equicontinuous in p. There will be a subsequence along which these functions converge. It is not difficult to see that these limit functions define a person-by-person optimal pair.

IV. Properties of the Threshold

Assume henceforth that γ is given by (2.3),

$$\gamma(\tau_2 - \vartheta) = 1(\tau_2 < \vartheta). \tag{4.1}$$

Lemma 4.1: $E\{1(\tau_2 < \vartheta) | \vartheta = t\}$ is nondecreasing in t.

Proof:

$$E\{1(\tau_2 < \vartheta) | \vartheta = t\} = E\{1(\tau_2 \leqslant t-1) | \vartheta = t\}$$
$$= E\{1(\tau_2 \leqslant t-1) | x_s = 0, s \leqslant t-1, x_t = 1\}.$$

Now $\{\tau_2 \leqslant t-1\} \in Y_{t-1}^2$ and the fields Y_{t-1}, $\sigma(x_s, s \geqslant t)$ are independent given $\sigma(x_s, s \leqslant t-1)$. Hence

$$E\{1(\tau_2 \leqslant t-1) | x_s = 0, s \leqslant t-1, x_t = 1\}$$
$$= E\{1(\tau_2 \leqslant t-1) | x_s = 0, s \leqslant t, x_{t+1} = 1\}$$
$$= E\{1(\tau_2 \leqslant t-1) | \vartheta = t+1\}$$
$$\leqslant E\{1(\tau_2 \leqslant t) | \vartheta = t+1\}. \quad ■$$

Corollary 4.1:

$$G_t \leqslant G_{t+1} \leqslant 1. \tag{4.2}$$

Proof: From (3.2) and (4.1)

$$G_t = \sum_{l=1}^{\infty} g(t+1)(1-q)^{l-1} q$$
$$= \sum_{l=1}^{\infty} E\{1(\tau_2 < \vartheta) | \vartheta = t+1\}(1-q)^{l-1} q \leqslant G_{t+1},$$

by Lemma 4.1. Finally, since $\gamma \leqslant 1$, $G_t \leqslant 1$, by (3.3). ∎

Corollary 4.2:

$$W_t \leqslant W_{t+1}. \tag{4.3}$$

Proof: From (3.9)

$$W_t^N(p) = \min_{1 \leqslant \tau \leqslant N-t+1} E\left\{ G_{\tau+t-1} 1(x_\tau = 0) + c \sum_{s=1}^{\tau-1} 1(x_\tau = 0) | p_1 = p \right\}.$$

It follows from (4.2) that $W_t^N(p) \leqslant W_{t+1}^{N+1}(p)$. Now let $N \to \infty$. ∎

Comparison to Decoupled Case

With the cost given by (1.3) the two decisions are decoupled and detector 1 seeks to

$$\min_{1 \leqslant \tau < \infty} E\left\{ 1(x_\tau = 0) + c \sum_{t=1}^{\tau-1} 1(x_t = 1) \right\}$$

which is a special case of (2.4) with $\gamma = 1$. For this case it is known that the dynamic programming equation has a stationary solution [1], [2]. Therefore, from Theorem 3.1 the optimal value function W^d, threshold p^d, and stopping time τ^d are given by

$$W^d(p) = \min\{p, c(1-p) + [\Psi W^d](p)\}$$

$$p \leqslant c(1-p) + [\Psi W^d](p) \qquad \text{iff } p \leqslant p^d$$

$$\tau^d = \min\{t | p_t \leqslant p^d\}.$$

Theorem 4.1: $p^*(t) \geqslant p^d$ for all t. Hence, $\tau^* \leqslant \tau^d$ with probability 1.

Proof: From (3.11) and (3.12), $p > p^*(t)$ iff

$$p > G_t^{-1} W_t(p) = G_t^{-1} \min_{\tau \geqslant t} E\left\{ G_\tau 1(x_\tau = 0) + c \sum_{s=t}^{\tau-1} 1(x_s = 1) | p_t = p \right\}$$

$$\geqslant \min_{\tau \geqslant t} E\left\{ 1(x_\tau = 0) + c \sum 1(x_s = 1) | p_t = p \right\} \qquad \text{using (4.2)}$$

$$= W^d(p),$$

and so $p > p^d$. ∎

Thus, the alarm is declared later if the decisions are decoupled. This is intuitive since the false alarm penalty is larger, although the argument is more subtle since it depends on (4.2).

Theorem 4.2: $p^*(t) \to p^d$ if and only if $Pr\{\tau_2 < \vartheta | \vartheta = t\} \to 1$ as $t \to \infty$.

Proof: Clearly $p^*(t) \to p^d$ if and only if $G_t \to 1$ as $t \to \infty$, and then the result follows from (3.4) and Lemma 4.1 ∎

The condition of Theorem 4.2 will hold if detector 2's observations are poor. In the extreme case, if detector 2 makes no observations at all, then $\tau_2 = T_2$ will be a fixed stopping time and so $Pr\{\tau_2 < \vartheta | \vartheta = t\} = Pr\{\tau_2 < t\} = 1$ for $t > T_2$. In the other extreme, if $y_t^2 = x_t$, so that detector 2 has perfect observations, then clearly $\tau_2^* = \vartheta$, $Pr\{\tau_2 < \vartheta | \vartheta = t\} = 0$, and so $p^*(t)$ will be bounded away from p^d.

V. Conclusions

In the situation considered here there is no communication between the two detectors. Thus, this is a team problem with static information structure. Even in this simple case, the coupling induced by the cost structure causes considerable complexity in the optimal stopping rules. However, the fact that the false alarm penalty in the coupled case is smaller than in the decoupled case permits a comparison between the two sets of stopping rules and may suggest some simple suboptimal rules for the coupled problem.

A more interesting problem than the one considered here would be to allow communication between the two detectors. Specifically, suppose that whenever a detector decides to declare the alarm, this decision is conveyed to the other one. The information structure is no longer static since the information available to 1 at time t is now described by the σ-field generated by Y_t^1 and the sets $\{\tau_2 < s, s \leqslant t\}$, which depends on the decision of detector 2. This problem is more complex to analyze. Indeed it is no longer clear that τ_i^* is based only on p_t^i.

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Linear-Quadratic Reversed Stackelberg Differential Games with Incentives

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Abstract—**In this technical note the linear-quadratic Stackelberg differential game with reversed information structure is considered. The leader is confined to stroboscopic (or snap-decision) strategies and necessary and sufficient conditions are then given for the leader to be able to impose, with the help of side payments, the (optimal) team solution.**

I. Introduction

In this technical note we discuss deterministic, two-player, linear-quadratic, dynamic continuous-time differential games with a fixed horizon, and, from the information structure point of view, we are in the realm of closed-loop Stackelberg games with reversed information structure. In addition, we also incorporate into the model a side-payment transfer (viz. an incentive).

Loosely speaking, two agents, the leader and the follower, control a linear dynamic system and are interested in optimizing (viz. in minimizing) their respective quadratic loss functionals. The follower employs a feedback strategy, whereas the leader's strategy is a mapping from the space of follower decisions into the decision space of the leader. In addition, with a view to inducing desirable results, the leader announces his strategy (in conjunction with the formula for the side-payment transfer from the follower to the leader) in advance; whereas the follower moves first and informs the leader on his instantaneous decision. A similar formulation was adopted in [1] in a discrete-time setting. Our formulation for the continuous-time problem has recourse to a strategy concept (for the leader) introduced in [2], viz. the concept of a "stroboscopic" strategy.

Thus, in Section II a concise formulation for the game is presented and the minimum-energy problem is discussed. Necessary and sufficient con-

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