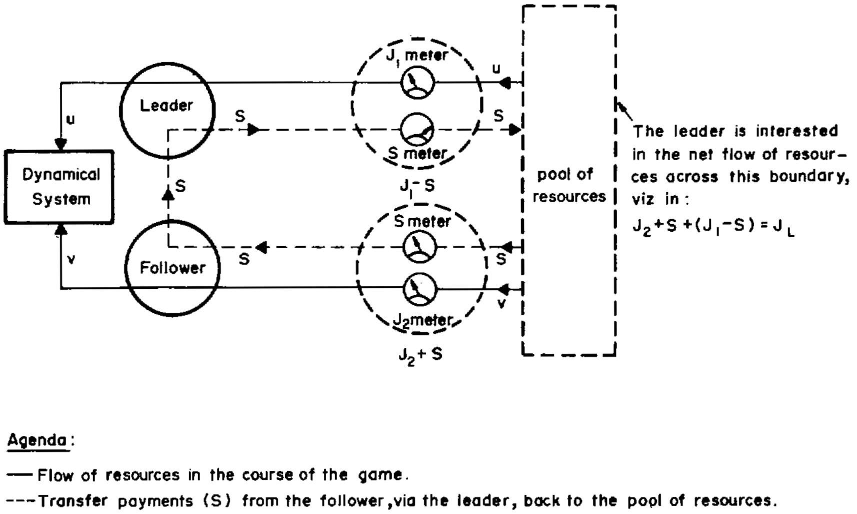

Fig. 1. The leader–follower system.

## IV. CONCLUDING REMARKS

In this technical note we have discussed fixed horizon, continuous-time, linear-quadratic, closed-loop "reversed" Stackelberg games with side payments. Having confined our attention to the class of leader strategies (9) which are "stroboscopic" (or "snap-decision" strategies), we were able then to derive necessary and sufficient conditions for leader enforceability of the desirable team solution (by declaring in advance his strategy and the side-payment formula). These conditions, in turn, are related to the (necessary and sufficient) conditions for max–min controllability in linear differential games as expounded in [2], where strategies of the form (9) are employed by the pursuer; or in [3, Theorem 4.3], where the evader's admissible strategies include linear feedback control laws.

We finally remark that the situation is different in the discrete-time case; while our analysis [with stroboscopic strategies of the form (9)] can be easily extended to include the discrete-time case, the discrete-time analog of stroboscopic-(state)feedback strategies then warrants special attention and is currently (see e.g., [1] and also [7] and [8]) under investigation.

## ACKNOWLEDGMENT

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# On the Interactions of Incentive and Information Structures

Y. C. HO AND D. TENEKETZIS

*Abstract*—**In this note we illustrate by way of a simple example the interactions and intricacies between the information and incentive structures of a problem.**

## I. INTRODUCTION

We have argued previously [1]–[3] on the importance of information structure in multiperson optimization problems. More recently, incentive problems have gained attention as a particularly interesting class of multiperson optimization problems with nonnested information structure [4]–[10], [12]–[14]. Briefly, "incentive" is concerned with the ability of one decision maker, the leader, in influencing the decision of another, the follower, by appropriate modification of the latter's payoff. There are several somewhat different versions of the incentive problem. The principal-agent problem has its main difficulty in the fact that the leader or principal cannot observe directly and exactly the decisions of the follower or agent [12, p. 12 and footnote 7]. On the other hand, the incentive-compatibility problem is concerned mainly with the fact that the follower has private information not known to the leader; hence, the leader cannot ascertain independently whether or not the decision of the follower which he observes is in fact the correct one [5]. A third version of the incentive problem gives the leader the ability to modify the information structure rather than the payoff structure of the follower [14]. It is through the manipulation of the information structure (e.g., letting you read only my propaganda) that the leader hopes to induce the appropriate decision from the follower. The purpose of this paper is primarily concerned with the second version of the problem, but with the added feature of partial information feedback and multistage considerations.

By assuming that decision makers are optimizers of their own payoff

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functions, we translate conditions for the existence of solutions to the incentive problem via a pair of coupled optimization problems to conditions on the incentive and information structure of the problem. In other words, we hope to gain insight to the questions: "Who must know what?" and "Who must be able to do what to whom?" (in order for the incentives to be effective). This note represents a first step in this direction to illustrate the interactions and the importance of incentive and information structure. We accomplish this by way of a simple example where our intuition helps to confirm our analysis.

## II. PROBLEM STATEMENT

There are two forces, the red (attacking) and the blue (defending), who are engaged in a battle. The red force may decide to launch a strong or a diversionary attack. This is modeled as a binary random variable and represented as the state of nature $x = 1$ or 0. The field commander of the blue force (the follower) sensing $x$ will have to decide to report the state to the central headquarter (the leader). This decision is denoted by $v = 1$ or 0. Upon receiving the report $v$ from the follower, the leader must decide to send reinforcement to the field commander or not ($u = 1$ or 0). The desired decision of the leader is to send reinforcement only when the field commander is experiencing a strong attack. This can be represented by a payoff matrix as in Fig. 1.

On the other hand, the payoff of the follower is different from that of the leader in that there is the temptation to misrepresent the state of nature; extra reinforcement is always welcomed. We represent this again by the payoff matrix (Fig. 2) where the constant $a$ is introduced for the purpose of parametric study later.

The leader must decide to send or not send reinforcement based only on information he possesses which always includes the follower's report (decision $v$), but may or may not include the follower's information (the state $x$). The leader may sometimes learn about the value of $x$ "after the fact"; it is too late to help the decision to send or not send reinforcement, but not too late for postmortem analysis. In order to induce "truthful reporting," the leader may be allowed to issue additional reward (or punishment), denoted as $p$, based on the information at his disposal. Thus, the incentive problem can be stated as the determination of the incentive strategies ($u$ and $p$ as functions of the leader's information) so that the leader achieves his desired objective, namely, send help only when needed. We are interested in the nature of the solution of this incentive problem as the information available to the leader varies. For added insight, we shall also consider repeating this game over two stages (attacks) so that history of the leader's information (i.e., the follower's decisions history) can be used to implement incentives.[1]

In the multistage case, we shall use subscripts $i = 1, 2$ to differentiate the variables at different stages.

## III. A TALE OF 11 INFORMATION STRUCTURES

Given the above setup, we shall discuss a series of cases of the incentive problem, each with different information available to the leader. These cases will illustrate the various insights and make concrete our intuitions about the incentive problem.

In all cases below, the information of the follower $z_F$ is always the state of nature $x$. If there are two stages, then the follower's information $z_{F1}$ and $z_{F2}$ are $x_1$ and $x_2$, respectively. The states are always independent from stage to stage.

*Case 1:* Single-stage problem with $z_L = v$ and $p = 0$.

This is the baseline case used to establish the simple fact that without means to independently verify the truth of the follower's report and to reward the follower separately using $p$, it is impossible to induce truth.

*Case 2:* Same as Case 1, but $p$ is free.

This is the opposite of Case 1, showing that with $p$ it is possible to induce truth without being able to verify what truth is.

*Case 3:* Two-stage problem with $z_{L1} = v_1$, and $z_{L2} = v_1, v_2$, $p_1 = 0$, and $p_2$ free.

This case illustrates the advantages of memory.

*Case 4:* Same as Case 3, except $z_{L2} = v_1, v_2$, and $x_1$.

$$L_L \qquad \begin{array}{c|c||c|c} & & \multicolumn{2}{c}{u} \\ & & 0 & 1 \\ \hline\hline x & 0 & 1 & 0 \\ \cline{2-4} & 1 & 0 & 1 \end{array}$$

Fig. 1.

$$L_F \qquad \begin{array}{c|c||c|c} & & \multicolumn{2}{c}{u} \\ & & 0 & 1 \\ \hline\hline x & 0 & a & 0.5 \\ \cline{2-4} & 1 & 0 & 1 \end{array} \qquad a < 0.5$$

Fig. 2.

Here the leader at stage 2 can verify the truth of the follower's report at stage 1. This "information feedback" permits additional flexibility in the design of incentives (cf. remarks in Section II).

*Case 5:* Same as Case 4, but with $p_i = 0$, $i = 1, 2$.

Here again we illustrate the impossibility of asking the decisions $u_i$ to do double duty, i.e., inducing the truth (influencing the follower's payoff) and optimizing the leader's payoff.

*Case 6:* Same as Case 5, but we are only interested in inducing truth at the first stage.

We show that decisions at later stages $u_2$ can be used in place of $p$ to influence the follower's decision at an earlier stage as long we do not have to worry about the impact of $u_2$ on the leader's payoff of the second stage.

*Case 7:* Same as Case 6, except $z_{L2} = v_1, v_2$, but not $x_1$.

Again, the importance of information feedback.

*Case 8:* Same as Case 7, but with two followers $A$ and $B$, correspondingly $x_{1A}$ and $x_{1B}$; $z_{L1} = v_{1A}, v_{1B}$; $z_{L2} = v_{1A}, v_{2A}, v_{1B}, v_{2B}$.

This case illustrates the possibility of playing off one follower against the other when their information ($x_A$ and $x_B$) are correlated.

*Case 9:* Same as Case 2, but with uncertain values in the payoff matrices.

*Cases 10 and 11:* Same as Case 2, but with two followers and correlated information.

Cases 9–11 illustrate how the range of workable incentives is effected by uncertainties and partial information.

## IV. ANALYSIS OF THE CASES[2]

*Case 1:* Since the leader's information is the decision of the follower, the leader has only four possible strategies: always send reinforcement, always not send, send whenever asked, and send only when not asked. It is clear that there is no loss of generality to assume that $u = v$ is an optimal strategy for the leader. Consequently, the decision problem of the follower is represented by the matrix in Fig. 3.

It is clear the strategy to report $x = 1$, i.e., $v = 1$, is a dominant choice and truth inducing is not possible. Note also that there is no need to consider a two-stage problem since there can be no coupling between the two stages.

*Case 2:* Once again we assume that $u = v$. However, with $p$ free, the payoff matrix of the follower becomes as shown Fig. 4.

The incentive problem is to choose $p_d$ and $p_s$ such that the diagonal terms of the above matrix dominate the off-diagonal terms. This is clearly possible, e.g., $p_s = 0$ and any $a < p_d < 1$ will do. This case points out the importance of a separate reward (punishment) decision variable in incentive problems.

*Case 3:* With $u_i = v_i$ and $p_1 = 0$, the two-stage payoff matrix for the follower is as shown Fig. 5.

To induce truth, we need to choose $p^1 - p^4$ such that once again the diagonal terms dominate. One feasible choice is $p^4 = a$, $p^3 = 0.55 + a = p^2$, $p^1 = 1 + a$. This example shows that memory helps to the extent that we only need to have one reward/punishment variable for both stages.

*Case 4:* With the added information $x_1$ at stage two in addition to $v_1$

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[1] Recall the case of "the boy who cried wolf."

[2] Please refer to Section III for a statement and comment on the cases.

| | | v=u | |
|---|---|---|---|
| | | 0 | 1 |
| x | 0 | $a$ | 0.5 |
| | 1 | 0 | 1 |

Fig. 3.

| | | v=u | |
|---|---|---|---|
| | | 0 | 1 |
| x | 0 | $a+P_d$ | $0.5+P_s$ |
| | 1 | $P_d$ | $1+P_s$ |

$P_d \triangleq P_2(v=0)$

$P_s \triangleq P_2(v=1)$

Fig. 4.

| | | $v_1 v_2 = u_1 u_2$ | | | |
|---|---|---|---|---|---|
| | | 00 | 01 | 10 | 11 |
| $x_1 x_2$ | 00 | $2a+p^1$ | $0.5+a+p^2$ | $0.5+a+p^3$ | $1+p^4$ |
| | 01 | $a+p^1$ | $1+a+p^2$ | $0.5+p^3$ | $1.5+p^4$ |
| | 10 | $a+p^1$ | $0.5+p^2$ | $1+a+p^3$ | $1.5+p^4$ |
| | 11 | $p^1$ | $1+p^2$ | $1+p^3$ | $2+p^4$ |

$p^1 \triangleq p_2(v_1=0,\ v_2=0)$

$p^2 \triangleq p_2(v_1=0,\ v_2=1)$

$p^3 \triangleq p_2(v_1=1,\ v_2=0)$

$p^4 \triangleq p_2(v_1=1,\ v_2=1)$

Fig. 5.

| | | $v_1 v_2 = u_1 u_2$ | | | |
|---|---|---|---|---|---|
| | | 00 | 01 | 10 | 11 |
| $x_1 x_2$ | 00 | $2a+p^1$ | $0.5+a+p^2$ | $0.5+a+p^3$ | $1+p^4$ |
| | 01 | $a+p^1$ | $1+a+p^2$ | $0.5+p^3$ | $1.5+p^4$ |
| | 10 | $a+p^5$ | $0.5+p^6$ | $1+a+p^7$ | $1.5+p^8$ |
| | 11 | $p^5$ | $1+p^6$ | $1+p^7$ | $2+p^8$ |

$p_1^1 \triangleq p_2(v_1=0, v_2=0,\ x_1=0)$

$p^2 \triangleq p_2(v_1=0, v_2=0,\ x_1=0)$

$\vdots$

$p^8 \triangleq p_2(v_1=1,\ v_2=1,\ x_1=1)$

Fig. 6.

| | | $(v_1=u_1,\ v_2)$ | |
|---|---|---|---|
| | | 00 | 01 |
| $(x_1 x_2)$ | 00 | $a+0.5\gamma_2(0,v_2,x_1=v_1)$ $+a(1-\gamma_2(0,v_2,x_1=v_1))$ | $a+0.5\gamma_2(0,v_2,x_1=v_1)$ $+a(1-\gamma_2(0,v_2,x_1=v_1))$ |
| | 01 | $a+\gamma_2(0,v_2,x_1=v_1)$ | $a+\gamma_2(0,v_2,x_1=v_1)$ |

a

(Note: The convention is that $\gamma_2=1$ or 0 depending upon the value of $v_2$ in these figures.)

| | | $(v_1=u_1,\ v_2)$ | |
|---|---|---|---|
| | | 10 | 11 |
| $(x_1 x_2)$ | 00 | $0.5+0.5\gamma_2(1,v_2,x_1\neq v_1)$ $+a(1-\gamma_2(1,v_2,x_1\neq v_1))$ | $0.5+0.5\gamma_2(1,v_2,x_1\neq v_1)$ $+a(1-\gamma_2(1,v_2\ x_1\neq v_1))$ |
| | 01 | $0.5+\gamma_2(1,v_2,x_1\neq v_1)$ | $0.5+\gamma_2(1,v_2,x_1\neq v_1)$ |

b

| | | $(v_1=u_1,\ v_2)$ | |
|---|---|---|---|
| | | 00 | 01 |
| $(x_1 x_2)$ | 10 | $0.5\gamma_2(0,v_2,x_1\neq v_1)$ $+a(1-\gamma_2(0,v_2,x_1\neq v_1))$ | $0.5\gamma_2(0,v_2,x_1\neq v_1)$ $+a(1-\gamma_2(0,v_2,x_1\neq v_1))$ |
| | 11 | $\gamma_2(0,v_2,x_1\neq v_1)$ | $\gamma_2(0,v_2,x_1\neq v_1)$ |

c

| | | $(v_1=u_1,v_2)$ | |
|---|---|---|---|
| | | 10 | 11 |
| $(x_1 x_2)$ | 10 | $1+0.5\gamma_2(1,v_2,x_1=v_1)$ $+a(1-\gamma_2(1,v_2,x_1=v_1))$ | $1+0.5\gamma_2(1,v_2,x_1=v_1)$ $+a(1-\gamma_2(1,v_2,x_1=v_1))$ |
| | 11 | $1+\gamma_2(1,v_2,x_1=v_1)$ | $1+\gamma_2(1,v_2,x_1=v_1)$ |

d

Fig. 7.

and $v_2$, we now can choose eight (instead of the four as in Case 3) parameters to induce truth in both stages. This is because $p_2$ can now be functions of three binary variables. The pertinent payoff matrix is now as in Fig. 6.

To achieve $v_2 = x_2$, we have to solve four subproblems, each one of the type of Case 2 as indicated in Fig. 6. To achieve $v_1 = x_1$, we want in addition, (1,1)-element > (1,3)-element, (2,2)-element > (2,4)-element, (3,3)-element > (3,1)-element, and (4,4)-element > (4,2)-element for the above matrix. A feasible choice is $p^1 = 2.25+a$, $p^2 = 1.75+a$, $p^3 = 1.25+a$, $p^4 = 0.75+a$, $p^5 = 2+a$, $p^6 = 1.5+a$, $p^7 = 1.25+a$, and $p^8 = 0.75+a$. This example illustrates the importance of information feedback, i.e., in this case, of being able to verify the truth of the follower's report. We submit that a large class of human behavior is governed by the consideration that "I will have to do business with you in the future." The implication here is that the appropriateness of my action ($v$) can be verified "after the fact" and used to reward or punish in the future. With information feedback, it is less important for the leader to know the follower's payoff function precisely since we no longer need to rely on it exclusively to induce the desired behavior.

*Case 5:* We attempt to push further here the power of information feedback by requiring $p_2 = 0$. In other words, can $u_2$ be used to induce truth as well as to optimize the leader's payoff? Letting $u_1 = v_1$, we can write the payoff matrix of the follower as in Fig. 7. To achieve $v_2 = x_2$, we see that we have to solve four separate problems, each one of the type of Case 1, which is known to be impossible.

*Case 6:* Despite the negative result of Case 5, it is nevertheless interesting to consider if it is possible to induce $v_1 = x_1$ using $u_2$ in place of $p_2$. In other words, we shall use the second-stage payoff as determined by the incentive strategy $u_2 = (v_1, v_2, x_1)$ to replace the $p^1 - p^4$ in Case 4. To achieve this, we require (cf. the matrix in Case 5)

$$\begin{aligned}\gamma_2(0, v_2, x_1 = v_1) &\geqslant 1 + \gamma_2(1, v_2, x_1 \neq u_1)\\ \gamma_2(0, v_2, x_1 = v_1) &> \gamma_2(1, v_2, x_1 \neq v_1) + (0.5 - a)\\ 1 + \gamma_2(1, v_2, x_1 = v_1) &> \gamma_2(0, v_2, x_1 \neq v_1)\\ 1 + (0.5 - a)\gamma_2(1, v_2, x_1 = v_1) &> (0.5 - a)\gamma_2(0, v_2, x_1 = v_1)\end{aligned}$$

where the inequalities are derived by requiring that the (1,1) element of Fig. 7(a) be greater than the (1,1) element of Fig. 7(b),$\cdots$, etc.

We can satisfy all requirements by the following strategy:[3]

$$u_2 = \begin{cases} 0 & \text{if } v_1 \neq x_1\\ 1 & \text{if } v_1 = x_1 = 0\\ v_2 & \text{if } v_1 = x_1 = 1\end{cases}$$

which can be interpreted as the embodiment of the "boy who cried wolf fable" incentive.

*Case 7:* In this case, we attempt to extend this idea of Case 6 further by eliminating information feedback form the first stage. The following self-explanatory analysis shows that it is no longer possible to use $u_2$ to induce truthful reporting at the first stage (Fig. 8).

To achieve $v_1 = x_1$, we require (cf. Case 6)

$$\begin{aligned}\gamma_2(0, v_2) &> 1 + \gamma_2(1, v_2)\\ \gamma_2(0, v_2) &> \gamma_2(1, v_2) + (0.5 + a)\\ 0.5 + (0.5 - a)\gamma_2(1, v_2) &> (0.5 - a)\gamma_2(0, v_2)\\ 1 + \gamma_2(1, v_2) &> \gamma_2(0, v_2).\end{aligned}$$

But the first and the fourth inequalities are not compatible. Thus, without information feedback, it is not possible even to induce $u_1 = x_1$ at the first stage using $u_2$.

---

[3] We assume that everything else being equal, people will choose to tell the truth

| | | $v_1=u_1$, $v_2$ | | | |
|---|---|---|---|---|---|
| | | 00 | 01 | 10 | 00 |
| $(x_1\, x_2)$ | 00 | $a+0.5\gamma_2(0,v_2)$ <br> $+a(1-\gamma_2(0,v_2))$ | $a+0.5\gamma_2(0,v_2)$ <br> $+a(1-\gamma_2(0,v_2))$ | $0.5+0.5\gamma_2(1,v_2)$ <br> $+a(1-\gamma_2(1,v_2))$ | $0.5+0.5\gamma_2(1,v_2)$ <br> $+a(1-\gamma_2(1,v_2))$ |
| | 01 | $a+\gamma_2(0,v_2)$ | $a+\gamma_2(0,v_2)$ | $0.5+\gamma_2(1,v_2)$ | $0.5+\gamma_2(1,v_2)$ |
| | 10 | $0.5\gamma_2(0,v_2)$ <br> $+a(1-\gamma_2(0,v_2))$ | $0.5\gamma_2(0,v_2)$ <br> $+a(1-\gamma_2(0,v_2))$ | $1+0.5\gamma_2(1,v_2)$ <br> $a(1-\gamma_2(1,v_2))$ | $1+0.5\gamma_2(1,v_2)$ <br> $+a(1-\gamma_2(1,v_2))$ |
| | 11 | $\gamma_2(0,v_2)$ | $\gamma_2(0,v_2)$ | $1+\gamma_2(1,v_2)$ | $1+\gamma_2(1,v_2)$ |

Fig. 8.

*Case 8:*[4] Case 7, however, can be rescued if we introduce partial information feedback by the device of a second follower who has correlated information with the first. To introduce correlated information, let us assume that the state of nature $(x_A, x_B)$ can take on values (0,1) or (1,0) with equal probability, i.e., the red force can launch a strong attack against either $A$ or $B$, but not both. Thus, at the first stage, there are two possible states (0,1) or (1,0). For every one of the four possible response pairs by $A$ and $B$, i.e., $v_{1A}$ and $v_{1B}$, there are again two possible states for the second stage. It is clear that at the second stage, the equilibrium strategy for both followers is to report $v=1$ since nothing is lost by lying (there is no third stage to punish the followers). Consequently, the expected cost to the followers at the first stage averaged over the two possible states of nature at the second stage can be displayed as two bimatrix games in Fig. 9. Assuming that everything being equal, one prefers to tell the truth, it can be easily seen from Fig. 9 that truth telling is an equilibrium strategy in both cases. Comparing to Case 7, we see the possibility of playing off one follower against another. (See [10] for a more general situation.)

*Case 9:* This case is the same as Case 2, except that we modify the follower's payoff matrix to that of Fig. 10.

Exactly similar analysis shows that the range of feasible incentives has now been reduced to

$$p_s = 0, \qquad p_d = b_{\min} - a_{\max}.$$

*Cases 10 and 11:* In these two cases, we have two followers $A$ and $B$. The states of nature are correlated according to: a) Case 10: three states (0,0, (0,1), and (1,0) with equal probability; and b) Case 11: two states (0,0) and (1,1) with equal probability. We wish to show that under a), the range of feasible incentives is the same as that of Case 2, while under b), the range is considerably increased. The intuitive difference between a) and b) is that if one of the followers lies, the leader can ascertain which one lied under b), but not under a). We leave it as an exercise for the reader to complete the analysis.

## V. Conclusion

The subject of incentive control theory (i.e., control theory where the control is exerted indirectly) is still in its infancy. In contradistinction to most of the economic incentive literature,[5] the problem is dynamic or multistage (see [3, sec. II]). Similarly to the decentralized stochastic optimal control, it has an intimate connection to the information structure of the problem. Incredible complexity exists even in the simplest problems. However, unlike the case of decentralized control where fundamental computational difficulties exist [11], incentive control offers the possibility of solving genuine problems with a nonnested information structure. New conceptual, theoretical, and computational issues are awaiting further development.

| | | $v_{1B}$ | |
|---|---|---|---|
| | | 0 | 1 |
| $v_{1A}$ | 0 | $\frac{3a}{2}$ <br> $\frac{a}{2}$ | $\frac{3a}{2}$ <br> $1+\frac{a}{2}$ |
| | 1 | $0.5+\frac{a}{2}$ <br> $\frac{a}{2}$ | $\frac{3a}{2}$ <br> $\frac{a}{2}$ |

State (0,1)

| | | $v_{1B}$ | |
|---|---|---|---|
| | | 0 | 1 |
| $v_{1A}$ | 0 | $\frac{a}{2}$ <br> $\frac{3a}{2}$ | $\frac{a}{2}$ <br> $0.5+\frac{a}{2}$ |
| | 1 | $1+\frac{a}{2}$ <br> $\frac{3a}{2}$ | $\frac{a}{2}$ <br> $\frac{3a}{2}$ |

State (1,0)

Fig. 9.

| | | $v=u$ | |
|---|---|---|---|
| | | 0 | 1 |
| $x$ | 0 | $a+p_d$ | $b+p_s$ |
| | 1 | $p_d$ | $1+p_s$ |

$a \in [a_{\min}, a_{\max}]$

$b \in [b_{\min}\ b_{\max}]$

$a_{\max} < b_{\min}$

Fig. 10.

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# A Recursive Algorithm for Computing the Partial Fraction Expansion of Rational Functions Having Multiple Poles

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***Abstract*—A recursive algorithm is derived which easily permits the determination of the terms in a partial fraction expansion associated with multiple poles. The algorithm is readily programmed on a digital computer.**

## I. Introduction

This note is concerned with the partial fraction expansion of a rational function possessing multiple poles. Specifically, a recursive algorithm is derived which permits one to easily determine the constants $c_{m-\ell}$ in the expansion

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[4] We thank P. Luh for pointing out that an even better strategy which can induce truthful reporting at both stages for this case is to send reinforcement as requested, except when $A$ and $B$ obviously lied by reporting strong attacks simultaneously.

[5] One exception is [13] where one aspect of a simple dynamic problem is analyzed.

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