SIAM J. CONTROL AND OPTIMIZATION
Vol. 32, No. 6, pp. 1726–1751, November 1994

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INFORMATION STRUCTURES, CAUSALITY, AND NONSEQUENTIAL STOCHASTIC CONTROL II: DESIGN-DEPENDENT PROPERTIES*

MARK S. ANDERSLAND† AND DEMOSTHENIS TENEKETZIS‡

Abstract. In control theory, the usual notion of causality—that, at all times, a system's output (action) only depends on its past and present inputs (observations)—presupposes that all inputs and outputs can be ordered, a priori, in time. In reality, many distributed systems (those subject to deadlock, for instance), are not *sequential* in this sense.

In a previous paper (part I) [*SIAM J. Control Optim.*, 30 (1992), pp. 1447–1475], the relationship between a less restrictive notion of causality, *deadlock-freeness*, and the design-independent properties of a potentially *nonsequential* generic stochastic control problem formulated within the framework of Witsenhausen's intrinsic model was explored. In the present paper (part II) the properties of individual designs are examined. In particular, a property of a design's *information partition* that is necessary and sufficient to ensure its deadlock-freeness is identified and shown to be sufficient to ensure its possession of an expected reward. It is also shown, by example, that there exist nontrivial deadlock-free designs that cannot be associated with any deadlock-free information structure.

The first result provides an intuitive design-dependent characterization of the cause/effect notion of causality and suggests a framework for the optimization of constrained nonsequential stochastic control problems. The second implies that this characterization is finer than existing design-independent characterizations, including properties C (Witsenhausen) and CI (part I).

Key words. information structures, causality, deadlock-freeness, nonsequential stochastic control

AMS subject classifications. 93E03, 93E05, 90D15, 93A15, 93A14, 93E20

1. Introduction.

In control theory the usual notion of causality—that, at all times a system's output (action) only depends on its past and present inputs (observations)—presupposes that all inputs and outputs can be ordered a priori in time. In reality, many controlled systems—including distributed data [5], communication [6], manufacturing [3], and detection networks [2]—need not be *sequential* [10] in this sense.

Consider, for example, a simple detection network in which three decentralized detectors D^1, D^2, and D^3 (perhaps radars or inspectors) each make a noisy observation of the same uncertain event (plane or product). Suppose that each detector forms and transmits a one-bit hypothesis concerning the event (e.g., friend/foe or pass/fail) to a silent coordinator. Moreover, suppose that each detector may elect to monitor the others' transmissions before forming its hypothesis. Then, depending on the detectors' control laws (termed the *design*) and the particular event that occurs, 64 different dependencies are possible, 39 of which deadlock.[1] For instance, D^3 may wait for D^1, and depending on D^1's transmission, perhaps D^2, but D^3 and D^1 may not wait for each other because then neither can act.

* Received by the editors September 9, 1991; accepted for publication (in revised form) May 12, 1993. This research was supported in part by National Science Foundation grant ECS-8517708, Office of Naval Research grant N00014-87-K-0540, Defense Advanced Research Projects Agency grant N00174-91-C-0116, National Science Foundation grant NCR-9204419, and a Hewlett Packard Faculty Development Award.

† Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, Iowa 52242-1595.

‡ Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, Michigan 48109-2122.

[1] Each of the three detectors may wait for: none, one, the other, or both detectors; hence there are 4^3 possibilities. By case analysis, 39 of these deadlock.

This example illustrates two key differences between sequential and *nonsequential* systems, namely: i) that the order in which a nonsequential system's actions occur may explicitly depend on the system's uncontrolled inputs and the actions taken, and ii) that when two or more of a nonsequential system's actions are interdependent, no "causal" ordering of the actions is possible. Due to i), deadlock-free designs that exploit a system's nonsequentiality can outperform those that do not (see [2], Appendix A). This should not be surprising; unlike sequential systems, the dependencies among a nonsequential system's actions can change dynamically. Due to ii), the problem of identifying these "good" designs is difficult to formulate as a well-defined stochastic control problem. In particular, a design that deadlocks need not possess an expected reward,[2] and when it does, it may be mathematically optimal despite the fact that it is "not causal." This raises the question: Under what conditions is it possible to pose well-defined nonsequential stochastic control problems?

In a previous paper [2] (part I), we addressed this question by defining a nonsequential system to be "causal" when, independent of its design, it is deadlock-free. We then identified a property of a potentially nonsequential generic stochastic control problem's *information structure* (property CI) that is necessary and sufficient to ensure deadlock-freeness, and sufficient to ensure that *all* of the problem's designs possess expected rewards. This result subsumes Witsenhausen's design-independent causality condition (property C, in [9], [11]) and provides a framework for the recursive optimization of *unconstrained* nonsequential stochastic control problems [1].

In the present paper (part II) we explore the relationship between deadlock-freeness and the properties of individual designs. Our work is motivated by the fact that when the observations available to a nonsequential system's decision-making agents (e.g., the detectors) are specified independently, the resulting information structure need not be causal in the C or CI sense, although many admissible designs may be deadlock-free. This presents systems designers with a dilemma. If the existence of noncausal designs is ignored, formal optimization may not be possible. On the other hand, if the agents' information is constrained to ensure design-independent causality—by forcing sequentiality, for instance—the designer may limit the system's possible performance.

An obvious alternative to either "fix" is to identify necessary and sufficient conditions for individual designs to be causal. Once again, Witsenhausen's intrinsic model [9], [11] provides the framework for our work. Within this framework, we identify design-dependent analogues of the causality properties C and CI. Specifically, we introduce properties of a design's *information partition* (properties C* and CI*) that are necessary and sufficient to ensure that the design is deadlock-free, and sufficient to ensure that it possesses an expected reward. Moreover, we show by example that there exist deadlock-free designs that cannot be associated with any deadlock-free information structure.

The first result provides an intuitive, design-dependent characterization of the cause/effect notion of causality, and suggests a framework for the optimization of *constrained* nonsequential stochastic control problems. The second implies that for $N > 2$ agents, this characterization is finer than existing design-independent characterizations, including properties C and CI. Because our conditions are based on what a nonsequential system's decision-making agents may know as opposed to what they may do, they are substantially different than those derived using event sequence-based

[2] To compute the reward we must break the deadlock, but the reward may vary depending on how this is done (see [2, §2.3]).

representations such as finite-state automata [7], or Petri nets [8].

The remainder of the paper is organized as follows. In §2 we briefly review the structure of Witsenhausen's intrinsic model and our generic stochastic control problem. In §3 we introduce the design-dependent analogues of the deadlock-freeness, well-posedness, and causality properties in [2] and [9], [11], and relate a design's possession of these properties to its deadlock-freeness and possession of an expected reward. In §4 we examine the relationship between the design-independent and dependent properties, and establish, by example, that the design-dependent properties are finer. Section 5 contains our conclusions.

2. Problem formulation.

The generic stochastic control problem considered in this paper is identical to that in [2] (part I). As before, the problem is posed within the framework of Witsenhausen's intrinsic model [9], [11]. This model, which is interpreted in [2], has three components.

1. An *information structure* $\mathcal{I} := \{(\Omega,\mathcal{B}),(U^k,\mathcal{U}^k),\mathcal{J}^k : 1 \le k \le N\}$ specifies the system's allowable decisions and distinguishable events.

   (a) $N \in \mathbb{N}$ denotes the number of control actions to be taken.

   (b) $(\Omega,\mathcal{B})$ denotes the measurable space from which a random input ω is drawn.

   (c) $(U^k,\mathcal{U}^k)$ denotes the measurable space from which u^k, the kth control action, is selected. Card(U^k) is assumed to be greater than one, and $\mathcal{U}^k$ is assumed to contain the singletons of U^k. The measurable product space containing the N-tuple of control actions, $u := (u^1,u^2,\ldots,u^N)$, is denoted by $(U,\mathcal{U}) := (\prod_{i=1}^N U^i, \bigotimes_{i=1}^N \mathcal{U}^i)$.

   (d) $\mathcal{J}^k \subset \mathcal{B}\otimes\mathcal{U}$ characterizes the maximal information that can be used to select the kth control action.

2. A design constraint set Γ_C constrains N-tuples of control laws $\gamma := (\gamma^1,\gamma^2,\ldots,\gamma^N)$, $\gamma^k : (\Omega\times U,\mathcal{J}^k) \to (U^k,\mathcal{U}^k)$, $k = 1,2,\ldots,N$, called *designs*, to a nonempty subset of $\Gamma := \prod_{i=1}^N \Gamma^i$, where Γ^k, $k=1,2,\ldots,N$, denotes the set of all $\mathcal{J}^k/\mathcal{U}^k$-measurable functions.

3. A probability measure P on $(\Omega,\mathcal{B})$ determines the statistics of the random input.

When posed within this framework the generic problem takes the following form [2].

(P). Given an information structure $\mathcal{I}$, a design constraint set Γ_C, a probability measure P, and a bounded, nonnegative $\mathcal{B}\otimes\mathcal{U}$-measurable reward function V, identify a design γ in Γ_C that achieves $\sup_{\gamma\in\Gamma_C} E_\omega[V(w,u_\omega^\gamma)]$ exactly, or within $\epsilon > 0$.[3]

3. Design-dependent properties.

Problem (P) is well defined when it is: i) *causal*, i.e., every $\gamma \in \Gamma_C$ is deadlock-free; and ii) *well posed*, i.e., every $\gamma\in\Gamma_C$ possesses an expected reward. As in part I, our objective is to identify properties necessary and sufficient to ensure that (P) is causal and well-posed. Here, however, we permit the problem's design constraint set $\Gamma_C \subset \Gamma$ to be arbitrary, and focus on developing *design-dependent* properties (properties that may only hold for *specific* $\gamma \in \Gamma$), as opposed to *design-independent* properties (properties that hold for *all* $\gamma \in \Gamma$).

3.1. Deadlock-freeness, solvability, and solvability-measurability: properties DF*, S*, and SM*.

The identification of the design-dependent analogues of the deadlock-freeness property DF [2], and the well-posedness properties S (solvability [9]) and SM (solvability-measurability [9]), is straightforward. To ensure the

[3] The notation u_ω^γ indicates that u depends on ω through γ (see Definitions 2 and 3).

deadlock-freeness of the control problem, it is necessary and sufficient to require that each $\gamma \in \Gamma_C$ possess property DF* (cf. [2, Def. 1]).

DEFINITION 1. *A design γ possesses* property DF* (deadlock-freeness) *when for every $\omega \in \Omega$ there exists an ordering of γ's N control laws, say $\gamma^{s_1(\omega)}$, $\gamma^{s_2(\omega)}$, ..., $\gamma^{s_N(\omega)}$, such that no control action depends on itself or the control actions that follow; i.e., $u^{s_i(\omega)}$ does not depend on $u^{s_j(\omega)}$ for $j \geq i$.*

When a design γ possesses property DF*, it is deadlock-free in the sense that, given ω, $u^{s_1(\omega)}$ can be determined; given ω and $u^{s_1(\omega)}$, $u^{s_2(\omega)}$ can be determined; and so on.

To ensure well-posedness, it suffices to require that each $\gamma \in \Gamma_C$ possess properties S* and SM* (cf. [9, §4]).

DEFINITION 2. *A design γ possesses* property S* (solvability) *when for every $\omega \in \Omega$ there exists a unique $u := (u^1, u^2, \dots, u^N) \in U$ satisfying the system of equations $u^k = \gamma^k(\omega, u)$, $k = 1, 2, \dots, N$.*

DEFINITION 3. *A design γ possesses* property SM* (solvability-measurability) *when γ possesses property S*, and the solution map $\Sigma^\gamma : \Omega \to U$ induced by the system of equations $u = \gamma(\omega, u)$ (i.e., $\Sigma^\gamma(\omega) = u^\gamma_\omega$, where $u^\gamma_\omega = \gamma(\omega, u^\gamma_\omega)$) is $\mathcal{B}/\mathcal{U}$-measurable.*

Properties S* and SM* ensure that γ's reward $V(\cdot, \Sigma^\gamma(\cdot))$ is $\mathcal{B}$-measurable, and consequently, that $E_\omega[V(w, \Sigma^\gamma(w))]$ is well defined.

3.2. Design-dependent causality: property C*. When all designs $\gamma \in \Gamma_C$ possess property SM*, (P) is well posed. However, just as property SM need not imply property C ([9, Thm. 2]), a design's possession of property SM* need not ensure that it is deadlock-free.

Example 1.[4] Suppose, for instance, that

$$\gamma^1(\omega, u^1, u^2, u^3) = \begin{cases} 1 & \omega\bar{u}^2u^3 = 1, \\ 0 & \text{else}, \end{cases}$$

$$\gamma^2(\omega, u^1, u^2, u^3) = \begin{cases} 0 & \omega\bar{u}^3u^1 = 1, \\ 1 & \text{else}, \end{cases} \tag{3.1}$$

and

$$\gamma^3(\omega, u^1, u^2, u^3) = \begin{cases} 1 & \omega\bar{u}^1u^2 = 1, \\ 0 & \text{else},^5 \end{cases}$$

are the component control laws of an admissible design $\gamma := (\gamma^1, \gamma^2, \gamma^3)$ for a three-agent problem in which

$$\Omega = U^1 = U^2 = U^3 = \{0, 1\}, \tag{3.2}$$

and

$$\mathcal{B} = \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}. \tag{3.3}$$

Because

$$\begin{aligned} G^\gamma &:= \{(\omega, u) : \ \gamma(\omega, u) = u\} \\ &= \{(0, 0, 1, 0), (1, 0, 1, 1)\}, \end{aligned} \tag{3.4}$$

[4] This example is a variation of the example used to prove Theorem 2 of [9].

[5] $\overline{u}$ denotes the binary complement of $u \in \{0, 1\}$, i.e., $\overline{u} = 1 - u$.

γ possesses property SM*. Nonetheless, when $\omega = 1$, γ^1 depends on u^2 and u^3, γ^2 depends on u^3 and u^1, and γ^3 depends on u^1 and u^2. Accordingly, no agent can act without precognition.

Clearly, Witsenhausen's design-independent causality property C [9], [11] provides a condition sufficient to ensure that individual designs $\gamma \in \Gamma$ do not experience such deadlocks. This condition is not necessary, however, because it imposes constraints on all events that the agents can distinguish (i.e., the sets in the information fields $\mathcal{J}^k$, $k = 1, 2, \ldots, N$), not just those distinguishable given a particular design γ (i.e., those in the restriction of the *information partitions* $\mathcal{J}^{\gamma^k} := \{[\gamma^k]^{-1}(u^k) : u^k \in U^k\}$, $k = 1, 2, \ldots, N$, to the *graph* $\mathrm{G}^\gamma := \{(\omega, u) : \gamma(\omega, u) = u\}$ of γ).

These observations suggest that for fixed $\gamma \in \Gamma$, a design-dependent analogue to property C might be constructed by substituting $\mathcal{J}^{\gamma^k}$ for $\mathcal{J}^k$ and G^γ for $\Omega \times U$ in C (cf. [9, §5] or [11, §2]).

DEFINITION 4. *A design $\gamma \in \Gamma$ possesses* property c* *when $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$ and there exists at least one map $\psi : \mathrm{G}^\gamma \to S_N$ such that for all $s := (s_1, s_2, \ldots, s_k) \in S_k$ and $k = 1, 2, \ldots, N$,*

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [T_k^N \circ \psi]^{-1}(s) \subset \mathcal{F}(T_{k-1}^k(s)) \bigcap \mathrm{G}^\gamma. \tag{3.5}$$

Here, as in [2], S_k, $k = 1, 2, \ldots, N$, denotes the set of all k-action orderings (i.e., all injections of $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, N\}$); $T_j^k : S_k \to S_j$, $j = 0, 1, \ldots, N$, $k = j, j+1, \ldots, N$, denotes a truncation map that returns the ordering of the first j agents of a k-action ordering (i.e., T_j^k restricts $s \in S_k$ to the domain $\{1, 2, \ldots, j\}$ or to $\emptyset$ when $j = 0$); $\mathcal{P}_s$, $s := (s_1, s_2, \ldots, s_k) \in S_k$, $k = 1, 2, \ldots, N$, denotes the projection of $\Omega \times (\prod_{i=1}^N U^i)$ onto $\Omega \times (\prod_{i=1}^k U^{s_i})$ (i.e.,

$$\mathcal{P}_s(\omega, u) := (\omega, u^{s_1}, u^{s_2}, \ldots, u^{s_k}), \tag{3.6}$$

when $s \neq \emptyset$ and (ω) when $s = \emptyset$); and

$$\mathcal{F}(s) := [\mathcal{P}_s]^{-1} \left(\mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_i}) \right), \tag{3.7}$$

$s := (s_1, s_2, \ldots, s_k) \in S_k$, $k = 1, 2, \ldots, N$, denotes the cylindrical extension of $\mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_i})$ to $\Omega \times U$.

To interpret (3.5) note that

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [T_k^N \circ \psi]^{-1}(s) := \{A \bigcap [T_k^N \circ \psi]^{-1}(s) : A \in \mathcal{J}^{\gamma^{s_k}}\} \tag{3.8}$$

is the restriction of the set of events distinguishable by agent s_k under γ to the subset of outcomes $(\omega, u) \in \mathrm{G}^\gamma$ that are mapped by ψ into action orders in which the order of the first k agents is $s \in S_k$. Similarly, $\mathcal{F}(T_{k-1}^k(s)) \bigcap \mathrm{G}^\gamma$ is the restriction, to G^γ, of the set of events that can be induced by ω and the actions of the first $k-1$ agents in s. Accordingly, (3.5) asserts that the set of events that agent s_k can distinguish under γ, for *known* G^γ, given that the ordering of the first k agents as determined by ψ is s, must be a subset of the events that can be induced on G^γ by ω and the actions of the first $k-1$ agents in s.

Consider, for instance, the design γ in Example 1. Because for all $k = 1, 2, 3$, and $s \in S_k$, $\mathcal{F}(T_{k-1}^k(s)) \bigcap \mathrm{G}^\gamma$ is the power set of G^γ, all events that can be distinguished by s_k under any $\psi : \mathrm{G}^\gamma \to S_3$ can be induced by $\omega, \ldots, u^{s_{k-1}}$. Hence γ satisfies property c*.

Although a design's possession of property c* implies that it possesses an expected reward (just as $\mathcal{I}$'s possession of property C implies that all designs $\gamma \in \Gamma$ possess expected rewards [9]), property c* does not imply deadlock-freeness.

LEMMA 1. *For fixed $\gamma \in \Gamma$, property c* implies property* SM*, *although property c* need not imply property* DF*.

Proof. See Appendix A. □

The proof that c* implies SM* parallels the proof that C implies SM in [9, Thm. 1]. Property c*'s failure to ensure deadlock-freeness can be explained as follows. Property C is too restrictive to characterize the deadlock-freeness of individual designs because it requires that there exist a causal ordering for all outcomes in $\Omega \times U$, not just those that can occur (i.e., the outcomes in G^γ). Property c* is not restrictive enough because, for fixed $s \in S_k$, it implicitly permits the s_kth agent to possess information about its own action and the actions of its successors in s—i.e., because the domain of ψ is G^γ, $\mathcal{J}^{\gamma^{s_k}} \bigcap [T_k^N \circ \psi]^{-1}(s)$ unavoidably constrains $\mathcal{J}^{\gamma^{s_k}}$ along axes corresponding to agents that are not among the first $k-1$ agents in s. For instance, as previously noted, the design in Example 1 trivially satisfies property c* although it is not deadlock-free.

One compromise between these extremes is to continue to restrict the domain of ψ to G^γ. However, another is to only require, for all $s \in S_k$ and $k = 1, 2, \ldots, N$, that the inclusion in (3.5) hold when $\mathcal{J}^{\gamma^{s_k}}$ and $\mathcal{F}(T_{k-1}^k(s))$ are restricted to, respectively,

$$[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))) \tag{3.9}$$

and

$$[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\mathrm{G}^\gamma)), \tag{3.10}$$

the smallest subsets of $\Omega \times U$ containing $[T_k^N \circ \psi]^{-1}(s)$ and G^γ that can be constructed without knowledge of the decisions of agents that are not among the first $k-1$ agents in s.

DEFINITION 5. *A design $\gamma \in \Gamma$ possesses* property C* (causality) *when $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$, and there exists at least one map $\psi : \mathrm{G}^\gamma \to S_N$ such that for all $s := (s_1, s_2, \ldots, s_k) \in S_k$ and $k = 1, 2, \ldots, N$,*

$$\begin{aligned} &\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))) \\ &\quad \subset \mathcal{F}(T_{k-1}^k(s)) \bigcap [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\mathrm{G}^\gamma)). \end{aligned} \tag{3.11}$$

Because the restriction of $\mathcal{J}^{\gamma^{s_k}}$ to $[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s)))$ in (3.11) does not provide information to agent s_k concerning its action or the actions of its successors, in addition to ensuring that a design possesses an expected reward, property C* also implies deadlock-freeness.

THEOREM 1. *If a design $\gamma \in \Gamma$ possesses property* C*, *then*

(i) *γ possesses property* SM*, *and*

(ii) *γ possesses property* DF*.

Proof. See Appendix B. □

The proof of (i) follows from Lemma 1 and the fact that property C* implies property c*. Part (ii) is an immediate consequence of C*'s definition.

3.3. Design-dependent causality: property CI*. By Theorem 1, when all $\gamma \in \Gamma_C$ possess property C*, problem (P) is causal and well posed. It is not clear, however, that the converse implication holds. In particular, it would seem that the measurability constraints that property C* imposes on ψ are unnecessary to ensure deadlock-freeness. Regardless of ψ's measurability, γ should be deadlock-free if ψ orders the agents, for all outcomes $(\omega, u) \in \mathrm{G}^\gamma$, such that at (ω, u), each agent's action only depends on ω and its predecessors' actions. This suggests the following design-dependent analogue of property CI.

DEFINITION 6. *A design* $\gamma \in \Gamma$ *possesses* property CI* (causal implementability) *when* $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$ *and there exists at least one map* $\psi : \mathrm{G}^\gamma \to S_N$ *such that for all* $k = 1, 2, \ldots, N$, *and* $(\omega, u) \in \mathrm{G}^\gamma$,

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \subset \{\emptyset, [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u))\} \tag{3.12}$$

when $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$.

As in property C*, for fixed $\gamma \in \Gamma$, the ψ in property CI* is a function that maps every outcome in G^γ into an N-agent ordering. Unlike property C*, however, this ψ is not constrained to be measurable in any sense. Instead, for all outcomes $(\omega, u) \in \mathrm{G}^\gamma$, the cylinder set

$$[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) = [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) \tag{3.13}$$

induced on $\Omega \times U$ by ω and the actions of the first $k-1$ agents in $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$ is constrained to be a subset of all events containing (ω, u) in the information partition $\mathcal{J}^{\gamma^{s_k}}$ induced by the s_kth agent's control law γ^{s_k}—i.e., no event in $\mathcal{J}^{\gamma^{s_k}}$ containing (ω, u), may depend on u^{s_k}, $u^{s_{k+1}}$, $\ldots$, or u^{s_N} (cf. [2, Def. 2]). Accordingly, property CI* ensures that for all outcomes $(\omega, u) \in \mathrm{G}^\gamma$, there exists an action order $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$ such that for all $k = 1, 2, \ldots, N$, the s_kth agent's action at the point (ω, u) does not depend on itself or the actions of its successors in s.

Clearly, the design in Example 1 does not satisfy this condition—when $\omega = 1$, all three agents' actions are interdependent. Such is not the case in the following three-agent example.

Example 2. Suppose that

$$\Omega = U^1 = U^2 = U^3 = [0, 1], \tag{3.14}$$

$$\mathcal{B} = \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \mathrm{Borel}[0, 1], \tag{3.15}$$

and

$$\begin{aligned}
\gamma^1(\omega, u^1, u^2, u^3) &= \begin{cases} 0 & \text{when } \omega \in [0, \frac{1}{2}), \\ \frac{1}{2} & \text{when } (\omega, u^2) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1], \\ 1 & \text{else} \end{cases} \\
\gamma^2(\omega, u^1, u^2, u^3) &= \begin{cases} 0 & \text{when } \omega \in [\frac{1}{2}, 1], \\ \frac{1}{2} & \text{when } (\omega, u^1) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ 1 & \text{else,} \end{cases} \\
\gamma^3(\omega, u^1, u^2, u^3) &= \begin{cases} 0 & \text{when } \omega \in [0, \frac{1}{2}), \\ 1 & \text{else} \end{cases}
\end{aligned} \tag{3.16}$$

are the component policies of an admissible design $\gamma = (\gamma^1, \gamma^2, \gamma^3)$. It is straightforward to verify that

$$\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \mathcal{P}_\emptyset\left([0, \tfrac{1}{2}) \times \{(0, 1, 0)\} \bigcup [\tfrac{1}{2}, 1] \times \{(1, 0, 1)\}\right) = \Omega, \tag{3.17}$$

and that (3.12) is satisfied for all $k = 1, 2, 3$ and $(\omega, u) \in \mathrm{G}^\gamma$ when

$$\bar{\psi}(\omega, u^1, u^2, u^3) = \begin{cases} (1,2,3) & \text{when } \omega \in [0, \frac{1}{2}), \\ (2,1,3) & \text{else.} \end{cases} \tag{3.18}$$

Hence γ possesses property CI*.

Property CI* is of interest because it implies property SM* and provides a complete characterization of γ's deadlock-freeness.

THEOREM 2. *Let γ be an arbitrary design in Γ. Then*

(i) *γ possesses property* SM* *if γ possesses property* CI*, *and*

(ii) *γ possesses property* DF* *if and only if γ possesses property* CI*.

Proof. See Appendix C. □

Theorem 2 ensures that (P) is causal and well posed if and only if all designs $\gamma \in \Gamma_C$ possess property CI*. Its proof, like that of property CI [2], hinges on the following observation. When ψ is an order function such that γ possesses property CI*, for arbitrary but fixed $(\omega, u) \in \Omega \times U$, and $k = 1, 2, \ldots, N$, (3.12) and the fact that $\mathcal{U}^k$ contains the singletons of U^k imply that, at the point (ω, u), γ^{s_k}, $s = \psi(\omega, u)$, does not depend on the s_kth, s_{k+1}th, or s_Nth components of u. This suggests that for fixed $\gamma \in \Gamma$, a unique $\mathcal{B}$-measurable solution $\Sigma^\gamma : \Omega \to U$ to the closed-loop equation $u = \gamma(\omega, u)$ can be obtained by the following recursion.

Fix $\omega \in \mathcal{P}_\emptyset(\mathrm{G}^\gamma)$ and $u^\gamma_\omega \in \mathrm{G}^\gamma|_\omega$. Let $r \in U$ be an arbitrary reference element, let π_U and π_Ω denote the canonical projections of $\Omega \times U$ onto, respectively, U and Ω, let $L^\gamma : \Omega \times U \to \Omega \times U$ be defined as

$$L^\gamma(\omega, r) := (\omega, \gamma(\omega, r)), \tag{3.19}$$

and let $L^\gamma_k : \Omega \times U \to \Omega \times U$ be a k-fold composition of L^γ—i.e.,

$$L^\gamma_k(\omega, r) := (\underbrace{L^\gamma \circ \cdots \circ L^\gamma}_{k \text{ times}})(\omega, r). \tag{3.20}$$

Although (3.19) and (3.20) are nearly identical to (3.6) and (3.7) of [2], because the domain of ψ is G^γ (as opposed to $\Omega \times U$), the arguments following (3.7) in [2] no longer suffice to ensure $\pi_U \circ L^\gamma_N$ is the closed-loop solution map Σ^γ induced by γ. In particular, because $L^\gamma_k(\omega, r)$ need not belong to G^γ for all $r \in U$ and $k = 1, 2, \ldots, N$, a somewhat different argument is required to show that at least one agent's decision becomes invariant after every iteration. Formally, we have the following.

1. After one iteration, the components of $L^\gamma_1(\omega, r)$ corresponding to agents whose actions at the point (ω, r) do not depend on r become invariant to subsequent iterations. By property CI*, the set $\mathcal{A}_1(\omega) \subset \{1, 2, \ldots, N\}$ indexing (by agent) these components is nonempty since, at the point $(\omega, u^\gamma_\omega)$, at least agent $(\psi(\omega, u^\gamma_\omega))_1$'s action does not depend on r. Moreover, since r is arbitrary,

$$\mathcal{P}_i(L^\gamma_1(\omega, r)) = \mathcal{P}_i(L^\gamma_1(\omega, u^\gamma_\omega)) = \mathcal{P}_i(\omega, u^\gamma_\omega), \tag{3.21}$$

for all $i \in \mathcal{A}_1(\omega)$.

2. After two iterations, the components of $L^\gamma_2(\omega, r)$ corresponding to agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$ whose actions at the point $L^\gamma_1(\omega, r)$ do not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$ become invariant to subsequent iterations.[6]

[6] For sets $A, B \subset X$, $A \setminus B := \{x \in A : x \notin B\}$, the complement of B relative to A.

By property CI*, the set $\mathcal{A}_2(\omega)$ indexing (by agent) these components is nonempty when $\mathrm{card}(\mathcal{A}_1(\omega)) < N$ since, at the point $(\omega, u^\gamma_\omega)$, at least agent $(\psi(\omega, u^\gamma_\omega))_j$'s action,

$$j = \min \{m \in \{1, 2, \ldots, N\} : \ (\psi(\omega, u^\gamma_\omega))_m \notin \mathcal{A}_1(\omega)\}, \tag{3.22}$$

does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$, and by (3.21), the remaining components of $(\omega, u^\gamma_\omega)$ are identical to those of $L^\gamma_1(\omega, r)$. As before, since r is arbitrary,

$$\mathcal{P}_i(L^\gamma_2(\omega, r)) = \mathcal{P}_i(L^\gamma_2(\omega, u^\gamma_\omega)) = \mathcal{P}_i(\omega, u^\gamma_\omega) \tag{3.23}$$

for all $i \in \mathcal{A}_1(\omega) \cup \mathcal{A}_2(\omega)$.

$\vdots$

k. After k iterations, the components of $L^\gamma_k(\omega, r)$ corresponding to agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$ whose decisions at the point $L^\gamma_{k-1}(\omega, r)$ do not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$ become invariant to subsequent iterations. By property CI*, the set $\mathcal{A}_k(\omega)$ indexing (by agent) these components is nonempty when $\mathrm{card}(\bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)) < N$ since, at the point $(\omega, u^\gamma_\omega)$, at least agent $(\psi(\omega, u^\gamma_\omega))_j$'s action,

$$j = \min \left\{ m \in \{1, 2, \ldots, N\} : \ (\psi(\omega, u^\gamma_\omega))_m \notin \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega) \right\}, \tag{3.24}$$

does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$, and by the preceding iterations (e.g., (3.23)), the remaining components of $(\omega, u^\gamma_\omega)$ are identical to those of $L^\gamma_{k-1}(\omega, r)$. Once again, since r is arbitrary,

$$\mathcal{P}_i(L^\gamma_k(\omega, r)) = \mathcal{P}_i(L^\gamma_k(\omega, u^\gamma_\omega)) = \mathcal{P}_i(\omega, u^\gamma_\omega) \tag{3.25}$$

for all $i \in \bigcup_{i=1}^{k} \mathcal{A}_i(\omega)$.

$\vdots$

And so on....

Because property CI* ensures that, until all agents' components are invariant, at least one new component becomes invariant after every iteration, the recursive procedure must converge in, at most, N iterations—i.e., the unique solution to the closed-loop equation $u = \gamma(\omega, u)$ is $\pi_U(L^\gamma_N(\omega, r))$, where $r \in U$ is an arbitrary "seed" that starts the recursive solution process. Because π_Ω, π_U, and γ are, respectively, $\mathcal{B}\otimes\mathcal{U}/\mathcal{B}$-, $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$- and $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable, L^γ, and by composition, L^γ_k and $\pi_U \circ L^\gamma_N$, are, respectively, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, and $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable. It follows, because all u-sections of $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable functions are $\mathcal{B}/\mathcal{U}$-measurable, that the induced solution map $\Sigma^\gamma = \pi_U \circ L^\gamma_N|_r$ is necessarily $\mathcal{B}/\mathcal{U}$-measurable.

The preceding recursion has the same physical interpretation as the recursion in [2]. If for all k we ignore all components of $\pi_U(L^\gamma_k(\omega, r))$ except those corresponding to the agents indexed in $\mathcal{A}_k(\omega)$, the preceding recursion outlines the partial ordering of agent actions that a passive observer would record, given ω, if the design γ were implemented in a "maximally" concurrent fashion. Although the recursion implicitly demonstrates that property CI* implies property DF*, it is far easier to establish sufficiency by a direct appeal to property CI*. For all $(\omega, u) \in \mathrm{G}^\gamma$ and $k = 1, 2, \ldots, N$, property CI* implies that at the point (ω, u), agent s_k's action does not depend on the s_kth, s_{k+1}th, ..., and s_Nth components of u. Consequently, no agent's action depends on its own action or the actions of its successors—i.e., γ must be deadlock-free.

The fact that γ must deadlock when property CI* fails to hold is also a direct consequence of property CI*'s definition. When $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) \neq \Omega$, for some $\omega \in \Omega$, the closed-loop equation has no solution; consequently, for that ω, γ has no implementation (let alone a deadlock-free implementation). Alternatively, suppose that there exists at least one outcome $(\omega, u) \in \mathrm{G}^\gamma$ such that for all N-agent orderings $s := (s_1, s_2, \ldots, s_N) \in S_N$, (3.12) fails for at least one $k \in \{1, 2, \ldots, N\}$, say k_s. Then, for all orderings $s \in S_N$, the s_{k_s}th agent's action, at the point (ω, u), always depends on itself and or the actions of its successors in s, and once again, γ is not deadlock-free.

3.4. Are properties C* and CI* equivalent? By Theorems 1(ii) and 2(ii), property C* implies DF*, which in turn implies property CI*. Consequently, we have the following.

COROLLARY 1. *Property* C* *implies property* CI*.

Proof. See Appendix D for a direct proof. □

Are properties C* and CI* equivalent? When $N = 1$, the answer is yes (this follows from Definition 7 and Theorem 3). When $N > 1$, it is not known (in general) whether property CI* implies property C*. In particular, attempts to establish a design-dependent analogue of Corollary 2 in [2] (i.e., that CI* implies C* when $N = 2$) are complicated by the fact that S* need not imply CI* (or C*) under any circumstances (cf. [9, Thm. 2]). Consider, for instance, the following one-agent example.

Example 3. Suppose that $\Omega = \{0, 1\}$ and $U = \{0, 1, 2\}$, and let

$$\gamma(\omega, u) = \begin{cases} 2 & \text{if } (\omega, u) \in \{(1,1), (1,2)\}, \\ 1 & \text{if } (\omega, u) = (1, 0), \\ 0 & \text{else.} \end{cases} \tag{3.26}$$

Because

$$\begin{aligned} \mathrm{G}^\gamma &:= \{(\omega, u) : \ \gamma(\omega, u) = u\} \\ &= \{(0,0), (1,2)\}, \end{aligned} \tag{3.27}$$

γ possesses property S*. But $[\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset(1,2)) = \{(1,0), (1,1), (1,2)\}$ and $[\gamma]^{-1}(1) = \{(1,0)\}$; consequently,

$$[\gamma]^{-1}(1) \bigcap [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset(1,2)) = \{(1,0)\} \notin \{\emptyset, \{(1,0), (1,1), (1,2)\}\}. \tag{3.28}$$

Hence γ does not process property CI*.

Properties CI* and C* are equivalent in at least two important cases: when γ is *sequential* (Theorem 3), and when the measurable structure underlying (P) is *discrete*, i.e., when $\mathcal{B} \otimes \mathcal{U}$ contains the singletons of $\Omega \times U$ and $\Omega \times U$ is a countable set (Theorem 4).

DEFINITION 7. *A design $\gamma \in \Gamma$ is said to be* sequential *when property* CI* *holds for some constant order function* ψ.

THEOREM 3. *All constant order functions ψ such that a design $\gamma \in \Gamma$ possesses property* CI* *are order functions such that γ possesses property* C*.

Proof. See Appendix E. □

THEOREM 4. *When Ω and U^k, $k = 1, 2, \ldots, N$, are countable sets, and $\mathcal{B}$ contains the singletons of Ω, all order functions ψ such that a design $\gamma \in \Gamma$ possesses property* CI* *are order functions such that γ possesses property* C*.

Proof. See Appendix F. □

When $\gamma \in \Gamma$ is nonsequential and (P)'s measurable structure is not discrete, it is far more difficult to prove that property CI* implies property C* because, even if γ possesses property C*, order functions for which γ possesses property CI* need not be order functions for which γ possesses property C*.

Example 4. Consider again the three-agent design of Example 2. Although the design γ defined in (3.16) possesses properties CI* and C*, when A is any nonmeasurable subset of $[0, \frac{1}{2})$ (such a set always exists [4]),

$$\psi(\omega, u^1, u^2, u^3) = \begin{cases} (1,2,3) & \text{when } \omega \in [0, \frac{1}{2})/A, \\ (3,1,2) & \text{when } \omega \in A, \\ (2,1,3) & \text{else} \end{cases} \tag{3.29}$$

is an order function such that γ possesses property CI*, but not property C*. To see this, note that (3.12) holds for all $k = 1, 2, 3$, and $s \in S_k$, whereas (3.11) fails, for instance, when $k = 1$ and $s = 3 \in S_1$, since

$$\begin{aligned} [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset([T_1^3 \circ \psi]^{-1}(3))) &= A \times U \\ &\notin \mathcal{F}(\emptyset) \bigcap [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset(\mathrm{G}^\gamma)) \\ &= \mathcal{B} \otimes \{\emptyset, U\}. \end{aligned} \tag{3.30}$$

The fact that there exist nonsequential designs $\gamma \in \Gamma$ and order functions ψ such that γ possesses property CI*, but not property C*, implies that general proofs that property CI* implies property C* (if such exist) must be constructive—i.e., to prove that property CI* implies property C*, given a ψ such that γ possesses property CI*, but not property C*, we must be able to construct a new order function $\widehat{\psi}$ (obviously distinct from ψ), such that γ possesses property C*. To date, no such constructions are known.

4. Design-independence vs. design-dependence. In this section we briefly examine the relationships between the design-independent properties introduced in [2] (part I) and the design-dependent properties introduced here (part II).

4.1. Basic relationships.

THEOREM 5. *Let $\mathcal{I}$ be an arbitrary information structure. Then*

(i) *all $\gamma \in \Gamma$ possess property* S* *if and only if $\mathcal{I}$ possesses property* S,
(ii) *all $\gamma \in \Gamma$ possess property* SM* *if and only if $\mathcal{I}$ possesses property* SM,
(iii) *all $\gamma \in \Gamma$ possess property* CI* *if and only if $\mathcal{I}$ possesses property* CI, *and*
(iv) *all $\gamma \in \Gamma$ possess property* C* *if $\mathcal{I}$ possesses property* C.

Proof. See Appendix G. □

Parts (i) and (ii) are immediate consequences of the definitions of properties S, SM, S*, and SM*. Part (iii) follows from the fact that properties CI and CI* are necessary and sufficient conditions for, respectively, all designs $\gamma \in \Gamma$ and particular designs $\gamma \in \Gamma$, to be deadlock-free. If properties C and C* were known to provide necessary and sufficient conditions for, respectively, all designs and particular designs to be deadlock-free (as is the case, for instance, when Ω and U are countable sets and $\mathcal{B}$ contains the singletons of Ω), the proof of part (iv), with the *if* replaced by *if and only if,* would also be immediate. In the absence of such knowledge it is necessary to prove (iv)—and if possible, the converse of (iv)—constructively. The forward construction is straightforward. Given a ψ such that $\mathcal{I}$ possesses property C, simply let $\psi^\gamma = \psi|_{G^\gamma}$ (the restriction of ψ to G^γ) for each $\gamma \in \Gamma$. The reverse construction (if such exists) is not obvious since there does not seem to be any way of relating the set of order functions

$$\bigcup_{\gamma \in \Gamma} \{\psi^\gamma : \ \gamma \text{ possesses property C* given } \psi^\gamma\} \tag{4.1}$$

to an order function ψ such that $\mathcal{I}$ possesses property C.

4.2. Design-dependent characterizations are finer. By Theorem 5, an information structure $\mathcal{I}$ cannot possess the design-independent property CI (respectively, C, SM, or S) if any one of its designs $\gamma \in \Gamma$ fails to possess the design-dependent property CI* (respectively, C*, SM*, or S*). This suggests that the design-dependent properties provide a finer characterization of a design's closed-loop solvability and deadlock-freeness, than the design-independent properties.

THEOREM 6. *For $N > 2$, there exist designs possessing property* C* (*and consequently, properties* CI*, SM*, *and* S*) *that cannot be associated with any deadlock-free information structure possessing property* S, *let alone properties* SM, CI, *or* C.

Proof. Since C* $\Rightarrow$ CI* $\Rightarrow$ SM* $\Rightarrow$ S* (by Corollary 1, Theorem 2, and Definition 3), and since C $\Rightarrow$ CI $\Rightarrow$ SM $\Rightarrow$ S (by [2, Cor. 1 and Thm. 2] and [9, §4]), it suffices to construct a design possessing property C* that cannot be associated with any information structure possessing property S.

Example 5. Consider a nonsequential $\mathcal{I}$ of the following form:

$$\begin{aligned} &N = 3,\\ &\Omega = U^1 = U^2 = U^3 = \{0,1\},\\ &\mathcal{B} = \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \Big\{\emptyset, \{0\}, \{1\}, \{0,1\}\Big\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathcal{J}^1 &= \Big\{\emptyset, \{(\omega,u) : \ \omega = 0\}, \{(\omega,u) : \ \omega = 1\}, \Omega \times U\Big\},\\ \mathcal{J}^2 &= \Big\{\emptyset, \{(\omega,u) : \ \max(\bar\omega \bar u^1 \bar u^3, u^1 u^3) = 0\},\\ &\qquad \{(\omega,u) : \ \max(\bar\omega \bar u^1 \bar u^3, u^1 u^3) = 1\}, \Omega \times U\Big\}, \end{aligned} \tag{4.3}$$

and

$$\mathcal{J}^3 = \Big\{\emptyset, \{(\omega,u) : \ \omega u^2 = 0\}, \{(\omega,u) : \ \omega u^2 = 1\}, \Omega \times U\Big\}.$$

Since the closed-loop equations for the design $\widehat\gamma = (\widehat\gamma^1, \widehat\gamma^2, \widehat\gamma^3)$,

$$\begin{aligned} \widehat\gamma^1(\omega, u^1, u^2, u^3) &= \begin{cases} 1 & \omega = 1,\\ 0 & \text{else}, \end{cases}\\ \widehat\gamma^2(\omega, u^1, u^2, u^3) &= \begin{cases} 1 & \max(\bar\omega \bar u^1 \bar u^3, u^1 u^3) = 1,\\ 0 & \text{else}, \end{cases} \end{aligned} \tag{4.4}$$

$$\widehat{\gamma}^3(\omega, u^1, u^2, u^3) = \begin{cases} 1 & \omega u^2 = 1, \\ 0 & \text{else}, \end{cases}$$

exhibit two distinct outcomes when $\omega = 1$—i.e.,

$$\begin{aligned} \mathrm{G}^{\widehat{\gamma}} &:= \{(\omega, u) : \widehat{\gamma}(\omega, u) = u\} \\ &= \{(0,0,1,0), (1,1,0,0), (1,1,1,1)\} \end{aligned} \tag{4.5}$$

—$\widehat{\gamma}$ does not possess property S*. Consequently, no information structure (including $\mathcal{I}$) that can be associated with $\widehat{\gamma}$ can possess property S (Theorem 5)—i.e., no information structure

$$\widehat{\mathcal{I}} := \{(\Omega, \mathcal{B}), (U^k, \mathcal{U}^k), \widehat{\mathcal{J}}^k : \ 1 \le k \le 3\} \tag{4.6}$$

such that

$$[\widehat{\gamma}^k]^{-1}(\mathcal{U}^k) = \mathcal{J}^k \subset \widehat{\mathcal{J}}^k, \qquad 1 \le k \le 3 \tag{4.7}$$

can possess property S.

Consider, however, the design $\gamma = (\gamma^1, \gamma^2, \gamma^3)$,

$$\begin{aligned} \gamma^1(\omega, u^1, u^2, u^3) &= \begin{cases} 0 & \omega = 1, \\ 1 & \text{else}, \end{cases} \\ \gamma^2(\omega, u^1, u^2, u^3) &= \widehat{\gamma}^2(\omega, u^1, u^2, u^3), \\ \gamma^3(\omega, u^1, u^2, u^3) &= \widehat{\gamma}^3(\omega, u^1, u^2, u^3). \end{aligned} \tag{4.8}$$

This design possesses property S*—i.e.,

$$\begin{aligned} \mathrm{G}^{\gamma} &:= \{(\omega, u) : \ \gamma(\omega, u) = u\} \\ &= \{(0,1,0,0), (1,0,0,0)\}. \end{aligned} \tag{4.9}$$

Moreover, when

$$\psi(\omega, u^1, u^2, u^3) = \begin{cases} (1,3,2) & (\omega, u^1, u^2, u^3) = (0,1,0,0), \\ (1,2,3) & (\omega, u^1, u^2, u^3) = (1,0,0,0), \end{cases} \tag{4.10}$$

it can also be shown to possess property C*.[7] But for all $k = 1,2,3$, γ^k and $\widehat{\gamma}^k$ both induce the same information subfield $\mathcal{J}^k$ (i.e., $\mathcal{J}^k = [\gamma^k]^{-1}(\mathcal{U}^k) = [\widehat{\gamma}^k]^{-1}(\mathcal{U}^k)$, for all $k = 1,2,3$). Accordingly, even though γ possesses property C*, it cannot be associated with any information structure possessing property S. This proves the theorem. □

Heuristically, the three-agent information structure that appears in the preceding example can be viewed as a synthesis, parameterized by agent 1's $\mathcal{F}(\emptyset)$-measurable decision[8] of two different two-agent information structures for agents 2 and 3. The first of these structures, $\mathcal{I}^C$, corresponds to the restriction of agent 2 and agent 3's information subfields to the u^1-sections of $\mathcal{J}^2$ and $\mathcal{J}^3$ induced when $u^1 = 0$—i.e.,

$$\mathcal{I}^C := \Big\{(\Omega, \mathcal{B}), \ (U^i, \mathcal{U}^i), \mathcal{J}^i|_{u^1=0} : \ 2 \le i \le 3\Big\}, \tag{4.11}$$

[7] In this case it is somewhat easier to check property CI* and then apply Theorem 4.

[8] By (4.2), $\mathcal{J}^1 \subset \mathcal{F}(\emptyset) := \{\emptyset, \{0\} \times U, \{1\} \times U, \Omega \times U\}$.

where

$$\mathcal{J}^2|_{u^1=0} = \Big\{\emptyset, \{(\omega, u^2, u^3) : \bar{\omega}\bar{u}^3 = 0\}, \{(\omega, u^2, u^3) : \bar{\omega}\bar{u}^3 = 1\}, \Omega \times U^2 \times U^3\Big\} \tag{4.12}$$

and

$$\mathcal{J}^3|_{u^1=0} = \Big\{\emptyset, \{(\omega, u^2, u^3) : \omega u^2 = 0\}, \{(\omega, u^2, u^3) : \omega u^2 = 1\}, \Omega \times U^2 \times U^3\Big\}. \tag{4.13}$$

The second of these structures, $\mathcal{I}^{NS}$, corresponds to the restriction of agent 2 and agent 3's information subfields to the u^1-sections of $\mathcal{J}^2$ and $\mathcal{J}^3$ induced when $u^1 = 1$—i.e.,

$$\mathcal{I}^{NS} := \Big\{(\Omega, \mathcal{B}), (U^i, \mathcal{U}^i), \mathcal{J}^i|_{u^1=1} : 2 \leq i \leq 3\Big\}, \tag{4.14}$$

where

$$\mathcal{J}^2|_{u^1=1} = \Big\{\emptyset, \{(\omega, u^2, u^3) : u^3 = 0\}, \{(\omega, u^2, u^3) : u^3 = 1\}, \Omega \times U^2 \times U^3\Big\} \tag{4.15}$$

and

$$\mathcal{J}^3|_{u^1=1} = \Big\{\emptyset, \{(\omega, u^2, u^3) : \omega u^2 = 0\}, \{(\omega, u^2, u^3) : \omega u^2 = 1\}, \Omega \times U^2 \times U^3\Big\}. \tag{4.16}$$

It is not difficult to verify that $\mathcal{I}^C$ possesses property C when

$$\psi(\omega, u^2, u^3) = \begin{cases} (3,2) & \omega = 0, \\ (2,3) & \text{else.} \end{cases} \tag{4.17}$$

To see this note that

$$\mathcal{J}^2\,|_{u^1=0} \bigcap [T_1^2 \circ \psi]^{-1}(2) = \Big\{\emptyset, \{0\} \times U^2 \times U^3\Big\} \subset \mathcal{F}(\emptyset)\,|_{u^1=0} \tag{4.18}$$

and

$$\mathcal{J}^3\big|_{u^1=0} \bigcap [T_1^2 \circ \psi]^{-1}(3) = \Big\{\emptyset, \{1\} \times U^2 \times U^3\Big\} \subset \mathcal{F}(\emptyset)\,|_{u^1=0}\,. \tag{4.19}$$

$\mathcal{I}^{NS}$, however, does not even possess property S. For instance,

$$\begin{aligned} \mathrm{G}^\gamma &:= \{(\omega, u^2, u^3) : \gamma(\omega, u^2, u^3) = (u^2, u^3)\} \\ &= \{(0,0,0), (1,0,0), (1,1,1)\} \end{aligned} \tag{4.20}$$

when

$$\begin{aligned} \gamma^2(\omega, u^2, u^3) &= \begin{cases} 1 & u^3 = 1, \\ 0 & \text{else,} \end{cases} \\ \gamma^3(\omega, u^2, u^3) &= \begin{cases} 1 & \omega u^2 = 1, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.21}$$

It follows, because agent 1's decision determines whether agent 2 and agent 3's interdependence is characterized by $\mathcal{I}^C$ ($u^1 = 0$) or $\mathcal{I}^{NS}(u^1 = 1)$, that agent 1's control law determines whether nontrivial designs for the synthesized system (4.4) possess property C* or do not possess property S*. Specifically, all designs such that

$$\gamma^1(\omega, u^1, u^2, u^3) = 1 \tag{4.22}$$

or

$$\gamma^1(\omega, u^1, u^2, u^3) = \begin{cases} 1 & \omega = 1, \\ 0 & \text{else,} \end{cases} \tag{4.23}$$

and neither γ^2 nor γ^3 is a constant policy (there are 8 such designs since card($\mathcal{J}^k$) > 2 and card(U^k) = 2 for $k = 2, 3$), do not possess property S*. All remaining designs (there are 56) possess property C*.

Clearly, the preceding heuristic can be used to synthesize far more complicated information structures that fail to possess property S, but nonetheless admit nontrivial designs possessing property C*. For instance, noncausal and causal 2-agent information structures can be combined, when parameterized by two additional agents' decisions, to form a 4-agent information structure that fails to possess property S but admits nontrivial designs possessing property C*; similarly, this 4-agent information structure and a second 4-agent information structure can be combined, when parameterized by three additional agents' decisions, to form a 7-agent information structure that fails to possess property S but admits nontrivial designs possessing property C*; and so on. It follows that there exist a myriad of designs whose deadlock-freeness and closed-loop solvability can not be characterized using any design-independent property.

5. Conclusions. In this paper we have introduced conditions (properties C* and CI*) necessary and sufficient to ensure the deadlock-freeness (property DF*) and measurable closed-loop solvability (property SM*) of a nonsequential design γ represented within the framework of Witsenhausen's intrinsic model. We have also shown, by example, that there exist nontrivial, deadlock-free designs that cannot be associated with any deadlock-free information structure.

Our conditions, which are the design-dependent analogues of conditions in [2] and [9] (properties CI and C), provide an intuitive characterization of the cause/effect notion of causality in terms of the events that a system's decision-making agents can distinguish, and suggest a framework for the optimization of constrained nonsequential stochastic control problems.

The existence of deadlock-free designs that cannot be associated with any deadlock-free information structure is not surprising. Many network routing, flow, and concurrency control systems are seen to be deadlock-free under some designs and deadlock-prone under others. In fact, unless the specification of a nonsequential system's agents' information subfields is coordinated (in practice physical constraints, complexity and/or cost may preclude such coordination) it is unlikely that the system's information structure will possess any design-independent property. Moreover, as illustrated by Example 5, the deadlock-freeness and closed-loop solvability of the admissible designs for such systems may hinge on the control laws of a small fraction of the agents. The only difference between the designs $\widehat{\gamma}$ and γ of Example 5, for instance, is that $\widehat{\gamma}^1$'s decision is the binary complement of γ^1's decision. Nonetheless, although $\widehat{\gamma}$ does not possess any design-dependent property, γ possesses all of the known design-dependent properties. Simply put, the inappropriate use of information by a single agent can give rise to deadlocks.

One final note. In [9, p. 159] it is remarked that the "physical interpretation" of information structures possessing property SM, but not property C, "appears difficult" (the difficulty being the host of paradoxes that arise when effects precede their causes). In light of Example 5, it would seem, rather, that it is the physical interpretation of *designs* possessing property SM* but not property CI* that may be difficult.

Appendix A.

Proof of Lemma 1. Fix $\gamma \in \Gamma$ and suppose that $\psi : \mathrm{G}^\gamma \to S_N$ is an order function such that γ possesses property c*. Except for the restriction of ψ's domain to G^γ, the proof that c* implies SM* parallels the proof that C implies SM in [9, Thm. 1]. Note, however, that unlike Witsenhausen's kth umpire update map [9, §7], the analogous update map, $M_k^\gamma : \mathrm{G}^\gamma \to \mathrm{G}^\gamma$ with

$$\text{(A.1)}\ \mathcal{P}_\alpha(M_k^\gamma(\omega,u)) := \begin{cases} (\omega, \gamma^\alpha(\omega,u)) & \text{when } \alpha = (\psi(\omega,u))_k, \\ \mathcal{P}_\alpha(\omega, u_\omega^\gamma) & \text{when } \alpha = (\psi(\omega,u))_j, \quad j = k+1,\dots,N, \\ \mathcal{P}_\alpha(\omega,u) & \text{otherwise} \end{cases}$$

for all $\alpha \in \{\emptyset, 1, \dots, N\}$, cannot be used to establish γ's deadlock-freeness because the restriction of M_k^γ to G^γ permits the umpire to know the actions of agents before they have acted.

To see that c* need not imply property DF*, note that although the design in Example 1 is not deadlock-free, for all $\psi : \mathrm{G}^\gamma \to S_3$, it trivially satisfies property c* because, as pointed out in §3.2, for all $k = 1,2,3$, and $s \in S_k$, $\mathcal{F}(T_{k-1}^k(s)) \bigcap \mathrm{G}^\gamma$ is the power set of G^γ (see (3.4)). □

Appendix B. Proof of Theorem 1. To prove Theorem 1 we need the following facts.

FACT 1. *For all $s \in S_k$, $k = 1,2,\dots,N$, if $\mathcal{P}_s(\omega,u) = \mathcal{P}_s(\bar\omega,\bar u)$ for some (ω,u) and $(\bar\omega,\bar u) \in \mathrm{G}^\gamma$, then no set in $\mathcal{F}(s)\bigcap \mathrm{G}^\gamma$ contains (ω,u) but not $(\bar\omega,\bar u)$.*

Proof of Fact 1. Suppose that the fact fails for some $s \in S_k$, $(\omega,u) \in \mathrm{G}^\gamma$, and $(\bar\omega,\bar u) \in \mathrm{G}^\gamma$. Then because

$$\text{(B.1)} \qquad \mathcal{F}(s) := [\mathcal{P}_s]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^k \mathcal{U}^{s_i}\right)\right),$$

there exists a set $A \in \mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_i})$ such that

$$\text{(B.2)} \qquad (\omega,u) \in [\mathcal{P}_s]^{-1}(A) \bigcap \mathrm{G}^\gamma$$

and

$$\text{(B.3)} \qquad (\bar\omega,\bar u) \notin [\mathcal{P}_s]^{-1}(A) \bigcap \mathrm{G}^\gamma.$$

It follows that ω and $\bar\omega$, or least one of the s_1th through s_kth components of u and $\bar u$, must differ. But $\mathcal{P}_s(\omega,u) = \mathcal{P}_s(\bar\omega,\bar u)$, a contradiction. Accordingly, the fact holds. □

FACT 2. *Property* C* *implies property* c*.

Proof of Fact 2. Fix $\gamma \in \Gamma$ and suppose that ψ is an order function such that γ possesses property C*. Because property C* ensures that $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$, it suffices to show that ψ is also an order function such that γ possesses property c*.

The restriction of (3.11) to G^γ yields the desired result—(3.5) of property c*—if, for all $k = 1, 2, \ldots, N$, and $s \in S_k$,

$$[\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) \bigcap G^\gamma = [T^N_k \circ \psi]^{-1}(s). \tag{B.4}$$

By construction, the right side of (B.4) is a subset of the left. Suppose that the converse inclusion fails for some $k \in \{1, 2, \ldots, N\}$ and $s \in S_k$. Then there exists an outcome

$$(\bar{\omega}, \bar{u}) \in [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) \bigcap G^\gamma \tag{B.5}$$

that is not in $[T^N_k \circ \psi]^{-1}(s)$. Moreover,

$$\mathcal{P}_{T^k_{k-1}(s)}(\bar{\omega}, \bar{u}) \in \mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)). \tag{B.6}$$

It follows from (B.6) that there exists an outcome $(\omega, u) \in [T^N_k \circ \psi]^{-1}(s)$ such that

$$\mathcal{P}_{T^k_{k-1}(s)}(\omega, u) = \mathcal{P}_{T^k_{k-1}(s)}(\bar{\omega}, \bar{u}), \tag{B.7}$$

and (ω, u) and $(\bar{\omega}, \bar{u})$ differ in one or more of the components of u not indexed in $T^k_{k-1}(s)$.

But this is impossible. By property C*, the sets

$$[\gamma^{s_k}]^{-1}(\bar{u}^{s_k}) \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) \bigcap G^\gamma \tag{B.8}$$

and

$$[\gamma^{s_k}]^{-1}(u^{s_k}) \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) \bigcap G^\gamma \tag{B.9}$$

are elements of $\mathcal{F}(T^k_{k-1}(s)) \bigcap G^\gamma$. If $\bar{u}^{s_k} \neq u^{s_k}$, $[\gamma^{s_k}]^{-1}(\bar{u}^{s_k})$ and $[\gamma^{s_k}]^{-1}(u^{s_k})$, and consequently the sets in (B.8) and (B.9), are disjoint. Since $(\bar{\omega}, \bar{u})$ and (ω, u) satisfy (B.7), this contradicts Fact 1.

Similarly, if, for $\bar{s} = T^N_j(\psi(\omega, u))$, $j > k$,

$$\mathcal{P}_{T^j_{j-1}(\bar{s})}(\omega, u) = \mathcal{P}_{T^j_{j-1}(\bar{s})}(\bar{\omega}, \bar{u}), \tag{B.10}$$

and $\bar{u}^{\bar{s}_j} \neq u^{\bar{s}_j}$, then by property C*,

$$[\gamma^{\bar{s}_j}]^{-1}(\bar{u}^{\bar{s}_j}) \bigcap [\mathcal{P}_{T^j_{j-1}(\bar{s})}]^{-1}(\mathcal{P}_{T^j_{j-1}(\bar{s})}([T^N_j \circ \psi]^{-1}(\bar{s}))) \bigcap G^\gamma \tag{B.11}$$

and

$$[\gamma^{\bar{s}_j}]^{-1}(u^{\bar{s}_j}) \bigcap [\mathcal{P}_{T^j_{j-1}(\bar{s})}]^{-1}(\mathcal{P}_{T^j_{j-1}(\bar{s})}([T^N_j \circ \psi]^{-1}(\bar{s}))) \bigcap G^\gamma \tag{B.12}$$

are disjoint sets in $\mathcal{F}(T^j_{j-1}(\bar{s})) \bigcap G^\gamma$. Since $(\bar{\omega}, \bar{u})$ and (ω, u) satisfy (B.10), Fact 1 is once again contradicted. It follows, by induction, that $(\omega, u) = (\bar{\omega}, \bar{u})$. Hence, Fact 2 is proved. □

Proof of Theorem 1. Fix $\gamma \in \Gamma$ and suppose that $\psi : G^\gamma \to S_N$ is an order function such that γ possesses property C*. The proof of (i) follows from Lemma 1 and Fact 2.

To prove (ii) it suffices to show that all agents can act without precognition for all outcomes in G^γ. Fix $(\omega, u) \in G^\gamma$. The first agent to act under ψ is agent $s_1 = T_1^N(\psi(\omega, u))$. Since $T_0^1(s_1) = \emptyset$, property C* implies that

$$\mathcal{J}^{\gamma^{s_1}} \bigcap [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset([T_1^N \circ \psi]^{-1}(s_1))) \subset \mathcal{F}(\emptyset) \bigcap [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset(G^\gamma)). \tag{B.13}$$

Because

$$\{\omega\} \times U \in [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset([T_1^N \circ \psi]^{-1}(s_1))) \subset [\mathcal{P}_\emptyset]^{-1}(\mathcal{P}_\emptyset(G^\gamma)), \tag{B.14}$$

and

$$\begin{aligned} \mathcal{F}(\emptyset) \bigcap (\{\omega\} \times U) &= (\mathcal{B} \otimes \{\emptyset, U\}) \bigcap (\{\omega\} \times U) \\ &= \{\emptyset, \{\omega\} \times U\}, \end{aligned} \tag{B.15}$$

the restriction of (B.13) to $\{\omega\} \times U$ can be rewritten as

$$\mathcal{J}^{\gamma^{s_1}} \bigcap (\{\omega\} \times U) \subset \{\emptyset, \{\omega\} \times U\}. \tag{B.16}$$

But (B.16) implies that at the point (ω, u), γ^{s_1} does not depend on u (recall that $\mathcal{J}^{\gamma^{s_1}} := [\gamma^{s_1}]^{-1}(\mathcal{U}^{s_1})$); consequently, given ω, agent s_1 acts without precognition.

Now, suppose that $k-1$ agents (agents $s_1, s_2, \ldots, s_{k-1}$) have acted without precognition and in accordance with ψ (i.e., $s = T_{k-1}^N(\psi(\omega, u))$). The kth agent to act under ψ is agent $s_k = (T_k^N(\psi(\omega, u)))_k$. Since $T_{k-1}^k(s, s_k) = s$, property C* implies that

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_s]^{-1}(\mathcal{P}_s([T_k^N \circ \psi]^{-1}(s, s_k))) \subset \mathcal{F}(s) \bigcap [\mathcal{P}_s]^{-1}(\mathcal{P}_s(G^\gamma)). \tag{B.17}$$

Because

$$\begin{aligned} [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) &\in [\mathcal{P}_s]^{-1}(\mathcal{P}_s([T_k^N \circ \psi]^{-1}(s, s_k))) \\ &\subset [\mathcal{P}_s]^{-1}(\mathcal{P}_s(G^\gamma)) \end{aligned} \tag{B.18}$$

and

$$\begin{aligned} &\mathcal{F}(s) \bigcap [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) \\ &\quad = [\mathcal{P}_s]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right) \bigcap [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) \\ &\quad = \{\emptyset, [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}})\}, \end{aligned} \tag{B.19}$$

the restriction of (B.17) to $[\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}})$ can be rewritten as

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) \subset \{\emptyset, [\mathcal{P}_s]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}})\}. \tag{B.20}$$

But (B.20) implies that at the point (ω, u), γ^{s_k} does not depend on the s_kth, s_{k+1}th, $\ldots$, or s_Nth components of u; consequently, when nature and agents $s_1, s_2, \ldots, s_{k-1}$'s actions are $(\omega, u^{s_1}, \ldots, u^{s_{k-1}})$, agent s_k acts without precognition. It follows, by induction, that all agents act without precognition. Thus γ possesses property DF* and the theorem is proved. $\square$

Appendix C.

Proof of Theorem 2. (i). Fix $\gamma \in \Gamma$, let $\mathrm{G}^\gamma := \{(\omega, u) \in \Omega \times U : \gamma(\omega, u) = u\}$, and suppose that ψ is an order function such that γ possesses property CI*. By assumption, the closed-loop equation $\gamma(\omega, u) = u$ admits at least one solution $u_\omega^\gamma \in \mathrm{G}^\gamma|_\omega := \{u \in U : \gamma(\omega, u) = u\}$ for all $\omega \in \Omega$ (i.e., $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$); hence, to prove that γ possesses property SM*, it suffices to show, for each $\omega \in \Omega$, that this solution is unique, and that the mapping $\Sigma^\gamma : \Omega \to U$ induced by these solutions (i.e., $\Sigma^\gamma(\omega) = u_\omega^\gamma$) is $\mathcal{B}/\mathcal{U}$-measurable (cf. Definitions 2 and 3).

Uniqueness. Fix $\omega \in \Omega$ and $u_\omega^\gamma \in \mathrm{G}^\gamma|_\omega$, and let $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u_\omega^\gamma)$. Let π_U denote the canonical projection of $\Omega \times U$ onto U, let $L^\gamma : \Omega \times U \to \Omega \times U$ be defined as in (3.19) and (3.20). Clearly, $u = \pi_U(L_N^\gamma(\omega, u))$ for all $u \in \mathrm{G}^\gamma|_\omega$, including u_ω^γ. Accordingly, to establish the uniqueness of u_ω^γ, it suffices to show that

$$(\omega, u_\omega^\gamma) = L_N^\gamma(\omega, r) \tag{C.1}$$

for all $r \in U$ (since $\mathrm{G}^\gamma|_\omega \subset U$), or equivalently, that

$$\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma) = \mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega, r)) \tag{C.2}$$

when $k = N + 1$.

Fix $r \in U$. When $k = 1$, $T_{k-1}^N(s) = \emptyset$ and

$$\begin{aligned} \mathcal{P}_\emptyset(\omega, u_\omega^\gamma) &= (\omega) \\ &= \mathcal{P}_\emptyset(\omega, \gamma(L_{N-1}^\gamma(\omega, r))) \\ &= \mathcal{P}_\emptyset(L_N^\gamma(\omega, r)). \end{aligned} \tag{C.3}$$

Suppose, for $k > 1$, that (C.2) holds. Since property CI* holds with order function ψ,

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma)) \subset \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma))\}. \tag{C.4}$$

Equation (C.4) and the fact that $\mathcal{U}^{s_k}$ contains the singletons of U^{s_k} (§2, 1(c)), implies that at the point $(\omega, u_\omega^\gamma) \in \Omega \times U$, γ^{s_k} does not depend on the s_kth, s_{k+1}th, $\ldots$, or s_Nth components of u_ω^γ (recall that $\mathcal{J}^{\gamma^{s_k}} := [\gamma^{s_k}]^{-1}(U^{s_k})$). Accordingly, (C.2) implies that

$$\gamma^{s_k}(\omega, u_\omega^\gamma) = \gamma^{s_k}(L_N^\gamma(\omega, r)), \tag{C.5}$$

and consequently, that

$$\begin{aligned} \mathcal{P}_{T_k^N(s)}(\omega, u_\omega^\gamma) &= (\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma), \gamma^{s_k}(\omega, u_\omega^\gamma)) \\ &= (\mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega, r)), \gamma^{s_k}(L_N^\gamma(\omega, r))) \\ &= \mathcal{P}_{T_k^N(s)}(L_N^\gamma(\omega, r)). \end{aligned} \tag{C.6}$$

It follows, by induction, that (C.2) holds for all $k = 1, 2, \ldots, N+1$; hence, $(\omega, u_\omega^\gamma) = L_N^\gamma(\omega, r)$ for all $r \in U$, and consequently, the unique solution u_ω^γ to the closed-loop equation $u = \gamma(\omega, u)$ is $\pi_U(L_N^\gamma(\omega, r))$, where $r \in U$ is the (arbitrary) "seed" that starts the recursive solution process.

Measurability. Fix $r \in U$ and let π_U and π_Ω denote, respectively, the canonical projections of $\Omega \times U$ onto U and Ω. To establish the $\mathcal{B}/\mathcal{U}$-measurability of the induced closed-loop solution map $\Sigma^\gamma : \Omega \to U$, it suffices to show that the u-section of $\pi_U \circ L_N^\gamma$, $\pi_U \circ L_N^\gamma|_r$, is $\mathcal{B}/\mathcal{U}$-measurable because, for fixed r,

$$\Sigma^\gamma(\omega) = (\pi_U \circ L_N^\gamma|_r)(\omega) := (\pi_U \circ L_N^\gamma)(\omega, r). \tag{C.7}$$

To begin, note that (3.19) implies that

$$L^\gamma(\omega, r) = (\pi_\Omega(\omega, u), \gamma(\omega, r)). \tag{C.8}$$

By definition, π_Ω and π_U are, respectively, $\mathcal{B}\otimes\mathcal{U}/B$- and $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable. Likewise, γ^k, $k = 1, 2, \dots, N$, is $\mathcal{J}^k/\mathcal{U}^k$-measurable. Accordingly, $\gamma := (\gamma^1, \gamma^2, \dots, \gamma^N)$ is $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable (since $\mathcal{J}^k \subset \mathcal{B} \otimes \mathcal{U}$ for all k). It follows that L^γ and, by composition [4, Thm. 13.1], L_k^γ and $\pi_U \circ L_N^\gamma$ are, respectively, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, and $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable. But all u-sections of $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable functions are $\mathcal{B}/\mathcal{U}$-measurable [4, Thm. 18.1]; consequently, $\Sigma^\gamma = \pi_U \circ L_N^\gamma|_r$ is $\mathcal{B}/\mathcal{U}$-measurable.

(ii). *Sufficiency.* Fix $\gamma \in \Gamma$, and suppose that ψ is an order function such that γ possesses property CI*. To prove that γ is deadlock-free it suffices to show that for each $\omega \in \Omega$, the agents can be ordered such that no agent's action depends on itself, or actions of its successors.

Fix $\omega \in \Omega$. By (i), the closed-loop equation $u = \gamma(\omega, u)$ possesses a unique solution $u_\omega^\gamma \in U$. Let

$$s := (s_1, s_2, \dots, s_N) = \psi(\omega, u_\omega^\gamma). \tag{C.9}$$

Since property CI* holds with order function ψ, for all $k = 1, 2, \dots N$,

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma)) \subset \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma))\}. \tag{C.10}$$

But (C.10) implies that at the point $(\omega, u_\omega^\gamma) \in \mathrm{G}^\gamma$, γ^{s_k} does not depend on the s_kth, s_{k+1}th, $\dots$, or s_Nth components of $(\omega, u_\omega^\gamma)$ (recall that $\mathcal{J}^{\gamma^{s_k}} := [\gamma^{s_k}]^{-1}(U^{s_k})$); consequently, for all $k = 1, 2, \dots, N$, the s_kth agent's action does not depend on the actions of agents $s_k, s_{k+1}, \dots$, and s_N. This proves sufficiency.

Necessity. Fix $\gamma \in \Gamma$, and suppose that γ does not possess property CI* for any order function ψ. Then $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) \neq \Omega$, or there exists at least one outcome in G^γ, say (ω^*, u^*), such that for all N-agent orderings $s := (s_1, s_2, \dots, s_N) \in S_N$, the inclusion

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega^*, u^*)) \subset \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega^*, u^*))\} \tag{C.11}$$

fails for at least one $k \in \{1, 2, \dots, N\}$. To prove necessity, it suffices to demonstrate that γ is not deadlock-free in either case.

When $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) \neq \Omega$, for some $\omega \in \Omega$, the closed-loop equation $\gamma(\omega, u) = u$ has no solution; consequently, for that ω, γ has no implementation (let alone a deadlock-free implementation). When there exists an outcome $(\omega^*, u^*) \in \mathrm{G}^\gamma$ such that for every N-agent ordering $s \in S_N$, (C.11) fails for at least one $k \in \{1, 2, \dots, N\}$ for fixed $s \in S_N$,

$$\begin{aligned} &\mathcal{J}^{\gamma^{s_{k^*}}} \bigcap [\mathcal{P}_{T_{k^*-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k^*-1}^N(s)}(\omega^*, u^*)) \\ &\qquad \not\subset \{\emptyset, [\mathcal{P}_{T_{k^*-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k^*-1}^N(s)}(\omega^*, u^*))\} \end{aligned} \tag{C.12}$$

for some $k^* \in \{1, 2, \ldots, N\}$. But (C.12) implies that at the point (ω^*, u^*), $\gamma^{s_{k^*}}$ depends on the actions of agents that have yet to act under s; consequently, agent s_{k^*} cannot act without precognition under s. Since the same argument applies for all $s \in S_N$, γ must deadlock. This proves necessity. $\square$

Appendix D.

Proof of Corollary 1. Although this corollary is an immediate consequence of Theorems 1(ii) and 2(ii) (property C* $\Rightarrow$ property DF* $\Rightarrow$ property CI*), it is instructive to prove it directly.

Fix $\gamma \in \Gamma$ and suppose that ψ is an order function such that γ possesses property C*. Since property C* ensures that $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$, it suffices to show that ψ is also an order function such that γ possesses property CI*—i.e., that (3.11) of property C* (with $s = T_k^N(\psi(\omega, u)) \in S_k$), implies (3.12) of property CI* (with $s = \psi(\omega, u) \in S_N$) for all $(\omega, u) \in \mathrm{G}^\gamma$ and $k = 1, 2, \ldots, N$.

Fix $(\omega, u) \in \mathrm{G}^\gamma$ and $k \in \{1, 2, \ldots, N\}$, and let

$$s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u). \tag{D.1}$$

Since $T_k^N(s) \in S_k$ and $T_{k-1}^N = T_{k-1}^k \circ T_k^N$, (3.11) of property C* implies that

$$\begin{aligned} &\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}([T_k^N \circ \psi]^{-1}(T_k^N(s)))) \\ &\quad \subset \mathcal{F}(T_{k-1}^N(s)) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\mathrm{G}^\gamma)). \end{aligned} \tag{D.2}$$

Restricting both sides of (D.2) to

$$[\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \tag{D.3}$$

yields the desired result—(3.12) of property CI*—if

$$\begin{aligned} &[\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}([T_k^N \circ \psi]^{-1}(T_k^N(s)))) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \\ &\quad = [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \end{aligned} \tag{D.4}$$

and

$$\begin{aligned} &\mathcal{F}(T_{k-1}^N(s)) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\mathrm{G}^\gamma)) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \\ &\quad = \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u))\}. \end{aligned} \tag{D.5}$$

Equation (D.5) follows from the definition of $\mathcal{F}(T_{k-1}^N(s))$,

$$\mathcal{F}(T_{k-1}^N(s)) := [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right), \tag{D.6}$$

the fact that inverse images preserve intersections—i.e.,

$$\begin{aligned} &[\mathcal{P}_{T_{k-1}^N(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \\ &\quad = \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u))\} \end{aligned} \tag{D.7}$$

—and the fact that

$$\begin{aligned}&[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\mathrm{G}^\gamma)) \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u))\\ &\qquad = [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u))\end{aligned} \tag{D.8}$$

since

$$[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \in [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\mathrm{G}^\gamma)) \tag{D.9}$$

when $(\omega, u) \in \mathrm{G}^\gamma$.

Equation (D.4) follows from the observation that

$$\begin{aligned}&[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(w, u))\\ &\qquad \subset [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}([T^N_k \circ \psi]^{-1}(T^N_k(s))))\end{aligned} \tag{D.10}$$

by (D.1). □

Appendix E.

Proof of Theorem 3. Fix $\gamma \in \Gamma$ and suppose that γ is sequential. Then there exists a *constant* order function ψ such that γ possesses property CI*. Since property CI* ensures that $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$, it suffices to show that ψ is also an order function such that γ possesses property C*—i.e., that for all $k = 1, 2, \ldots, N$, the fact that (3.12) of property CI* holds for all $(\omega, u) \in \mathrm{G}^\gamma$ with $s = s^* \in S_N$ constant implies that (3.11) of property C* holds for all $s \in S_k$.

Fix $k \in \{1, 2, \ldots, N\}$ and let

$$s^* := (s^*_1, s^*_2, \ldots, s^*_N) \tag{E.1}$$

denote the constant order induced by ψ. Since $T^N_{k-1} = T^k_{k-1} \circ T^N_k$, and since for all $s \in S_k$,

$$\begin{aligned}&[\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)))\\ &\qquad = \begin{cases} [\mathcal{P}_{T^N_{k-1}(s*)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s*)}(\mathrm{G}^\gamma)) & \text{when } s = T^N_k(s^*),\\ \emptyset & \text{else,}\end{cases}\end{aligned} \tag{E.2}$$

it suffices to show that

$$\mathcal{J}^{\gamma^{s^*_k}} \subset \mathcal{F}(T^N_{k-1}(s^*)) \tag{E.3}$$

for all $k = 1, 2, \ldots, N$.

By definition (§2, 1(d)), $\mathcal{J}^{\gamma^{s^*_k}} \subset \mathcal{J}^{s^*_k}$ is a subset of

$$\mathcal{B} \otimes \mathcal{U} = [\mathcal{P}_{T^N_N(s^*)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^N \mathcal{U}^{s^*_i}\right)\right). \tag{E.4}$$

Since (3.12) holds for all $(\omega, u) \in \mathrm{G}^\gamma$ when $s = s^*$, all events in $\mathcal{J}^{\gamma^{s^*_k}}$ must be of the form

$$[\mathcal{P}_{T^N_N(s^*)}]^{-1}\left(A \times \left(\prod_{i=k}^N U^{s^*_i}\right)\right), \tag{E.5}$$

where $A \subset \Omega \times (\Pi_{i=1}^{k-1} U^{s_i^*})$; accordingly, $\mathcal{J}^{\gamma^{s_k^*}}$ is also a subset of

$$\mathcal{C}_{s^*} := \sigma\left([\mathcal{P}_{T_N^N(s^*)}]^{-1}\left(A \times \left(\prod_{i=k}^{N} U^{s_i^*}\right)\right) : A \subset \Omega \times \left(\prod_{i=1}^{k-1} U^{s_i^*}\right)\right)$$
$$= \sigma\left([\mathcal{P}_{T_{k-1}^N(s^*)}]^{-1}(A) : A \subset \Omega \times \left(\prod_{i=1}^{k-1} U^{s_i^*}\right)\right) \tag{E.6}$$

—the cylindrical extension of the power set of $\Omega \times (\prod_{i=1}^{k-1} U^{s_i^*})$ to $\Omega \times U$. But

$$(\mathcal{B} \otimes \mathcal{U}) \bigcap \mathcal{C}_{s^*} = [\mathcal{P}_{T_{k-1}^N(s^*)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i^*}\right)\right)$$
$$:= \mathcal{F}(T_{k-1}^N(s^*)). \tag{E.7}$$

Consequently, $\mathcal{J}^{\gamma^{s_k^*}} \subset \mathcal{F}(T_{k-1}^N(s^*))$. $\square$

Appendix F.

Proof of Theorem 4. Fix $\gamma \in \Gamma$ and suppose that ψ is an order function such that γ possesses property CI*. Since property CI* ensures that $\mathcal{P}_{\emptyset}(\mathrm{G}^\gamma) = \Omega$, it suffices to show that ψ is also an order function such that γ possesses property C*—i.e., that for all $k = 1, 2, \ldots, N$, the fact that (3.12) holds for all $(\omega, u) \in \mathrm{G}^\gamma$ with order function ψ implies that (3.11) holds for all $s \in S_k$ with order function ψ.

By assumption, the σ-fields $\mathcal{B}$ and $\mathcal{U}^k$, $k = 1, 2, \ldots, N$, contain, respectively, the singletons of the countable sets Ω and U^k, $k = 1, 2, \ldots, N$ ($\mathcal{U}^k$ contains the singletons of U^k due to (§2, 1(c)). Accordingly, for all $s := (s_1, s_2, \ldots, s_k) \in S_k$, $k = 1, 2, \ldots, N$, the product field $\mathcal{B} \otimes (\bigotimes_{i=1}^{k} \mathcal{U}^{s_i})$ contains the singletons of the countable set $\Omega \times (\Pi_{i=1}^{k} U^{s_i})$, implying that $\mathcal{B} \otimes (\bigotimes_{i=1}^{k} \mathcal{U}^{s_1})$ is the power set of $\Omega \times (\Pi_{i=1}^{k} U^{s_i})$. It follows, for all $s \in S_k$, $k = 1, 2, \ldots, N$, that

$$\mathcal{F}(T_{k-1}^k(s)) := [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right)$$
$$= \sigma\left([\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(A) : A \subset \Omega \times \left(\prod_{i=1}^{k-1} U^{s_i}\right)\right) \tag{F.1}$$

—i.e., that $\mathcal{F}(T_{k-1}^k(s))$ is the cylindrical extension of the power set of $\Omega \times (\Pi_{i=1}^{k-1} U^{s_i})$ to $\Omega \times U$.

Fix $k \in \{1, 2, \ldots, N\}$ and $s \in S_k$. Since property CI* holds with order function ψ, (3.12) and (F.1) imply that for all $(\omega, u) \in [T_k^N \circ \psi]^{-1}(s)$ and $A \in \mathcal{J}^{\gamma^{s_k}}$,

$$A \bigcap [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\omega, u)) \in \{\emptyset, [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\omega, u))\}$$
$$\subset \mathcal{F}(T_{k-1}^k(s)). \tag{F.2}$$

Since $[T_k^N \circ \psi]^{-1}(s) \in \mathrm{G}^\gamma$ is a countable set, and since inverse and direct images preserve unions, it follows by (F.2) that

$$A \bigcap [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s)))$$

$$= A \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}\left(\mathcal{P}_{T^k_{k-1}(s)}\left(\bigcup_{(\omega,u)\in[T^N_k\circ\psi]^{-1}(s)}(\omega,u)\right)\right)$$

$$= \bigcup_{(\omega,u)\in[T^N_k\circ\psi]^{-1}(s)} (A \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\omega,u))) \tag{F.3}$$

$$\in \mathcal{F}(T^k_{k-1}(s)).$$

But because $[T^N_k \circ \psi]^{-1}(s) \subset \mathrm{G}^\gamma$,

$$[\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) \subset [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\mathrm{G}^\gamma)); \tag{F.4}$$

hence (F.3) implies that

$$A \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)))$$
$$\in \mathcal{F}(T^k_{k-1}(s)) \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\mathrm{G}^\gamma)). \tag{F.5}$$

Since (F.5) holds for all $A \in \mathcal{J}^{\gamma^{s_k}}$,

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)))$$
$$\subset \mathcal{F}(T^k_{k-1}(s)) \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\mathrm{G}^\gamma)). \tag{F.6}$$

This proves the theorem. □

Appendix G.

Proof of Theorem 5. (i) and (ii). Properties S and SM are, by definition, specializations of properties S* and SM* to all $\gamma \in \Gamma$ (cf. Definitions 1 and 2, and [9, §4, Definitions]). Accordingly, all $\gamma \in \Gamma$ possess property S* (respectively SM*) if and only if $\mathcal{I}$ possesses property S (respectively SM).

(iii). By Theorem 1(ii) of [2], $\mathcal{I}$'s possession of property CI is a necessary and sufficient condition for all $\gamma \in \Gamma$ to possess property DF*. By Theorem 2(ii), $\gamma \in \Gamma$ possesses property DF* if and only if γ possesses property CI*. Accordingly, all $\gamma \in \Gamma$ possess property CI* if and only if $\mathcal{I}$ possesses property CI.

(iv). To prove that all $\gamma \in \Gamma$ possess property C* when $\mathcal{I}$ possesses property C, let $\psi : \Omega \times U \to S_N$ be an order function for which $\mathcal{I}$ possesses property C, fix $\gamma \in \Gamma$, and let ψ^γ denote the restriction of ψ to G^γ. Since γ induces a unique $\mathcal{B}/\mathcal{U}$-measurable mapping $\Sigma^\gamma : \Omega \to U$ with graph G^γ [9, Thm. 1], $\mathcal{P}_\emptyset(\mathrm{G}^\gamma) = \Omega$; accordingly, to establish that γ possesses property C*, it suffices to show that (3.11) of property C* holds with order function ψ^γ for all $s := (s_1, s_2, \ldots, s_k) \in S_k$ and $k = 1, 2, \ldots, N$.

Since $\Omega \times U \in \mathcal{J}^{s_k}$, property C [9, Lem. 1] implies that

$$[T^N_k \circ \psi]^{-1}(s) \in \mathcal{F}(T^k_{k-1}(s)) := [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right). \tag{G.1}$$

Consequently,

$$[\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s))) = [T^N_k \circ \psi]^{-1}(s). \tag{G.2}$$

Since $\mathcal{J}^{\gamma^{s_k}} \subset \mathcal{J}^{s_k}$, property C [9] also implies that

$$\mathcal{J}^{\gamma^{s_k}} \bigcap [T^N_k \circ \psi]^{-1}(s) \subset \mathcal{F}(T^k_{k-1}(s)). \tag{G.3}$$

Substitute (G.2) into (G.3), and restrict both sides of the result to

$$[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\mathrm{G}^\gamma)). \tag{G.4}$$

The desired result—(3.11) of property C*—follows if

$$\begin{aligned}&[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))) \bigcap [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\mathrm{G}^\gamma))\\ &\qquad = [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi^\gamma]^{-1}(s))).\end{aligned} \tag{G.5}$$

To verify (G.5), note that

$$\begin{aligned}[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\mathrm{G}^\gamma)) &= \bigcup_{(\omega,u)\in \mathrm{G}^\gamma} [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}(\omega, u))\\ &= \bigcup_{s'\in S_k} [\mathcal{P}_{T_{k-1}^k(s')}]^{-1}(\mathcal{P}_{T_{k-1}^k(s')}([T_k^N \circ \psi^\gamma]^{-1}(s'))).\end{aligned} \tag{G.6}$$

Since ψ^γ is the restriction of ψ to G^γ, and since direct and inverse images preserve inclusions,

$$\begin{aligned}&[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi^\gamma]^{-1}(s)))\\ &\qquad \subset [\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))).\end{aligned} \tag{G.7}$$

But

$$\begin{aligned}&\{[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))) : s \in S_k\}\\ &\qquad = \{[T_k^N \circ \psi]^{-1}(s) : s \in S_k\}\end{aligned} \tag{G.8}$$

partitions $\Omega \times U$ (cf. (G.2)). Hence, by (G.7), the restriction of (G.6) to

$$[\mathcal{P}_{T_{k-1}^k(s)}]^{-1}(\mathcal{P}_{T_{k-1}^k(s)}([T_k^N \circ \psi]^{-1}(s))) \tag{G.9}$$

is (G.5), and thus γ possesses property C*. $\square$

Acknowledgments. The authors thank the referees and editors for their helpful comments.

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