A Simple Imperative Language

Operational Semantics
(= “meaning”)

Some Survey Results

12. 2. 1.	I have taken a course that covered induction. I am comfortable proving things using induction.
6. 3. 6.	I am comfortable with a functional programming language (e.g., LISP, Scheme, ML, or even Python).
6. 0. 9.	I have used an “automated” bug-finding tool (e.g., FindBugs, PREfast, ESC/Java, JLint, PMD, Fortify, LCLint, Coverity, etc.).
4. 4. 7.	I can typeset documents in LaTeX.
2. 1. 12.	I have written a compiler that had a type checker.

Survey Results: Goals

- How PL relates to security (2)
- Type systems and theory (2)
- Get the basics of PL (2)
- New languages
- Symbolic execution
- Abstract interpretation
- Theorem proving
- Find a research topic
- Understand the CQual paper
- Help with Quals
- Advanced Topics

Homework #1 Out Today

- Due Tuesday, Jan 31 (1 week from now)
- Take a look tonight
- My office hours are on Wednesday

Today's Plan

- Study a simple imperative language IMP
  - Abstract syntax
  - Operational semantics
  - Denotational semantics
  - Axiomatic semantics
  - … and relationships between various semantics (with proofs, peut-être)
  - Today: operational semantics
    - (Chapter 2 of Winskel)

Syntax of IMP

- Concrete syntax
  - The rules by which programs can be expressed as strings of characters
  - Keywords, identifiers, statement separators (terminators), comments, indentation, etc.
- Concrete syntax is important in practice
  - For readability, familiarity, parsing speed, effectiveness of error recovery, clarity of error messages
- Well understood principles
  - Use finite automata and context-free grammars
  - Automatic lexer/parser generators

(Note On Recent Research)

- If-as-and-when you find yourself making a new language, consider GLR (elkhound) instead of LALR(1) (bison)
- Scott McPeak, George G. Necula: *Elkhound: A Fast, Practical GLR Parser Generator*. CC 2004: pp. 73-88
- As fast as LALR(1), more natural, handles basically all of C++, etc.

Abstract Syntax

- We ignore parsing issues and study programs given as abstract syntax trees
- Abstract syntax tree is (a subset of) the parse tree of the program
  - Ignores issues like comment conventions
  - More convenient for formal and algorithmic manipulation

IMP Abstract Syntactic Entities

- int — integer constants ($n \in \mathbb{Z}$)
- bool — boolean constants (true, false)
- L — locations of variables (x, y)
- Aexp — arithmetic expressions (e)
- Bexp — boolean expressions (b)
- Com — commands (c)

- (these also encode the types)

Abstract Syntax (Aexp)

- **Arithmetic expressions (Aexp)**

$$
\begin{array}{lll}
e ::= & n & \text{for } n \in \mathbb{Z} \\
 & \mid x & \text{for } x \in L \\
 & \mid e_1 + e_2 & \text{for } e_1, e_2 \in \text{Aexp} \\
 & \mid e_1 - e_2 & \text{for } e_1, e_2 \in \text{Aexp} \\
 & \mid e_1 * e_2 & \text{for } e_1, e_2 \in \text{Aexp}
\end{array}
$$

- Notes:
  - Variables are not declared
  - All variables have integer type
  - No side-effects (in expressions)

Abstract Syntax (Bexp)

- **Boolean expressions (Bexp)**

$$
\begin{array}{lll}
b ::= & \text{true} & \\
 & \mid \text{false} & \\
 & \mid e_1 = e_2 & \text{for } e_1, e_2 \in \text{Aexp} \\
 & \mid e_1 \leq e_2 & \text{for } e_1, e_2 \in \text{Aexp} \\
 & \mid \neg b & \text{for } b \in \text{Bexp} \\
 & \mid b_1 \wedge b_2 & \text{for } b_1, b_2 \in \text{Bexp} \\
 & \mid b_1 \vee b_2 & \text{for } b_1, b_2 \in \text{Bexp}
\end{array}
$$

"Boolean"

- George Boole
  - 1815-1864
- I'll assume you know boolean algebra …

Abstract Syntax (Com)

- Commands (Com)

  c ::= skip

  | x := e — x∈L ∧ e∈Aexp

  | c_1 ; c_2 — c_1,c_2∈Com

  | if b then c_1 else c_2 — c_1,c_2∈Com ∧ b∈Bexp

  | while b do c — c∈Com ∧ b∈Bexp
- Notes:
  - The typing rules have been embedded in the syntax definition
  - Other parts are not context-free and need to be checked separately (e.g., all variables are declared)
  - Commands contain all the side-effects in the language
  - Missing: pointers, function calls, what else?

Popular Culture

"Ah. You seek meaning."
'Yes.'
"Then listen to the music, not the song."
-- Kosh and Talia, *Deathwalker*

"Angel... How did you get in here?"
'I was invited. The sign in front of the school... *Formatia trans sicere educatorum.*'
"Enter all ye who seek knowledge."
'What can I say? I'm a knowledge seeker.'
-- Jenny Calendar and Angelus, *Passion*

Why Study Formal Semantics?

- Language design (denotational)
- Proofs of correctness (axiomatic)
- Language implementation (operational)
- Reasoning about programs
- Providing a clear behavioral specification
- "All the cool people are doing it."
  - You need this to understand PL research
- "First one's free."

Consider This Java

```
x = 0;
try {
  x = 1;
  break mygoto;
} finally {
  x = 2;
  raise
   NullPointerException;
}
x = 3;
mygoto:
x = 4;
```

- What happens when you execute this code?
- Notably, what assignments are executed?

14.20.2 Execution of try-catch-finally

- A try statement with a finally block is executed by first executing the try block. Then there is a choice:
- If execution of the try block completes normally, then the finally block is executed, and then there is a choice:
  - If the finally block completes normally, then the try statement completes normally.
  - If the finally block completes abruptly for reason *S*, then the try statement completes abruptly for reason *S*.
- If execution of the try block completes abruptly because of a throw of a value *V*, then there is a choice:
  - If the run-time type of *V* is assignable to the parameter of any catch clause of the try statement, then the first (leftmost) such catch clause is selected. The value *V* is assigned to the parameter of the selected catch clause, and the *Block* of that catch clause is executed. Then there is a choice:
    - If the catch block completes normally, then the finally block is executed. Then there is a choice:
      - If the finally block completes normally, then the try statement completes normally.
      - If the finally block completes abruptly for any reason, then the try statement completes abruptly for the same reason.
    - If the catch block completes abruptly for reason *R*, then the finally block is executed. Then there is a choice:
      - If the finally block completes normally, then the try statement completes abruptly for reason *R*.
      - If the finally block completes abruptly for reason *S*, then the try statement completes abruptly for reason *S* (and reason *R* is discarded).
  - If the run-time type of *V* is not assignable to the parameter of any catch clause of the try statement, then the finally block is executed. Then there is a choice:
    - If the finally block completes normally, then the try statement completes abruptly because of a throw of the value *V*.
    - If the finally block completes abruptly for reason *S*, then the try statement completes abruptly for reason *S* (and the throw of value *V* is discarded and forgotten).
- If execution of the try block completes abruptly for any other reason *R*, then the finally block is executed. Then there is a choice:
  - If the finally block completes normally, then the try statement completes abruptly for reason *R*.
  - If the finally block completes abruptly for reason *S*, then the try statement completes abruptly for reason *S* (and reason *R* is discarded).

Ouch!

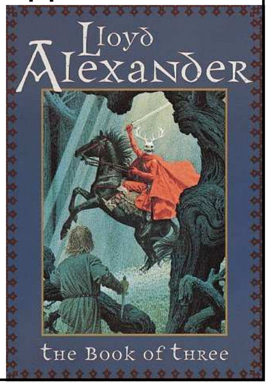
- Wouldn't it be nice if we had some way of describing what a language (feature or program) means …
  - More precisely than English
  - More compactly than English
  - So that you might build a compiler
  - So that you might prove things about programs

Analysis of IMP

- Questions to answer:
  - What is the "meaning" of a given IMP expression/command?
  - How would we go about evaluating IMP expressions and commands?
  - How are the evaluator and the meaning related?

Three Canonical Approaches

- Operational
  - How would I execute this?
  - "Symbolic Execution"
- Axiomatic
  - What is true after I execute this?
- Denotational
  - What is this trying to compute?



An Operational Semantics

- Specifies how expressions and commands should be evaluated
- Operational semantics abstracts the execution of a concrete interpreter
- Depending on the form of the expression
  - 0, 1, 2, . . . don't evaluate any further.
    - They are normal forms or values.
  - $e_1 + e_2$ is evaluated by first evaluating e_1 to n_1 , then evaluating e_2 to n_2 . (post-order traversal)
    - The result of the evaluation is the literal representing $n_1 + n_2$.
  - Similarly for $e_1 * e_2$

Semantics of IMP

- The meaning of IMP expressions depends on the values of variables
  - What does "x+5" mean? It depends on "x"!
- The value of variables at a given moment is abstracted as a function from L to $\mathbb{Z}$ (a state)
  - If $x \mapsto 8$ in our state, we expect "x+5" to mean 13
- The set of all states is $\Sigma = L \to \mathbb{Z}$
- We shall use σ to range over Σ
  - σ, a state, maps variables to values

Notation: Judgment

- We write:

$$<e, \sigma> \Downarrow n$$

- To mean that e evaluates to n in state σ.
- This is a judgment. It asserts a relation between e, σ and n.
- In this case we can view $\Downarrow$ as a function with two arguments (e and σ).

Operational Semantics

- This formulation is called natural operational semantics
  - or big-step operational semantics
  - the judgment relates the expression and its "meaning"
- How should we define

$$<e_1 + e_2, \sigma> \Downarrow \ldots ?$$

Notation: Rules of Inference

- We express the evaluation rules as rules of inference for our judgment
  - called the derivation rules for the judgment
  - also called the evaluation rules (for operational semantics)
- In general, we have one rule for each language construct:

$$\frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 + e_2, \sigma\rangle \Downarrow n_1 + n_2}$$

Rules of Inference

$$\frac{\text{Hypothesis}_1 \ \ldots \ \text{Hypothesis}_N}{\text{Conclusion}}$$

$$\frac{\Gamma \vdash b : \text{bool} \qquad \Gamma \vdash e1 : \tau \qquad \Gamma \vdash e2 : \tau}{\Gamma \vdash \text{if } b \text{ then } e1 \text{ else } e2 : \tau}$$

- For any given proof system, a finite number of rules of inference (or schema) are listed somewhere
- Rule instances should be easily checked
- What is the definition of “NP”?

Derivation

$$\dfrac{\dfrac{\dfrac{\Gamma(x) = int}{\Gamma \vdash x : int}\,\text{var} \qquad \dfrac{}{\Gamma \vdash 3 : int}\,\text{int}}{\Gamma \vdash x > 3 : bool}\,\text{gt} \qquad \dfrac{\dfrac{\Gamma(x) = int}{\Gamma \vdash x : int}\,\text{var} \qquad \dfrac{\dfrac{\Gamma(x) = int}{\Gamma \vdash x : int}\,\text{var} \qquad \dfrac{}{\Gamma \vdash 1 : int}\,\text{int}}{\Gamma \vdash x - 1 : int}\,\text{sub}}{\Gamma \vdash x := x - 1}\,\text{assign}}{\Gamma \vdash \texttt{while}\ x > 3\ \texttt{do}\ x := x - 1\ \texttt{done}}\,\text{while}$$

- Tree-structured (conclusion at bottom)
- May include multiple sorts of rules-of-inference
- Could be constructed, typically are not
- Typically verified in polynomial time

Evaluation Rules (for Aexp)

$$\frac{}{\langle n, \sigma\rangle \Downarrow n} \qquad \frac{}{\langle x, \sigma\rangle \Downarrow \sigma(x)}$$

$$\frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 + e_2, \sigma\rangle \Downarrow n_1 + n_2} \qquad \frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 - e_2, \sigma\rangle \Downarrow n_1 - n_2}$$

$$\frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 * e_2, \sigma\rangle \Downarrow n_1 * n_2}$$

- This is called structural operational semantics
  - rules defined based on the structure of the expression
- These rules do not impose an order of evaluation!

Evaluation Rules (for Bexp)

$$\frac{}{\langle \texttt{true}, \sigma\rangle \Downarrow \text{true}} \qquad \frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 \leq e_2, \sigma\rangle \Downarrow n_1 \leq n_2}$$

$$\frac{}{\langle \texttt{false}, \sigma\rangle \Downarrow \text{false}} \qquad \frac{\langle e_1, \sigma\rangle \Downarrow n_1 \qquad \langle e_2, \sigma\rangle \Downarrow n_2}{\langle e_1 = e_2, \sigma\rangle \Downarrow n_1 = n_2}$$

$$\frac{\langle b_1, \sigma\rangle \Downarrow \text{false}}{\langle b_1 \wedge b_2, \sigma\rangle \Downarrow \text{false}} \qquad \frac{\langle b_2, \sigma\rangle \Downarrow \text{false}}{\langle b_1 \wedge b_2, \sigma\rangle \Downarrow \text{false}}$$

$$\frac{\langle b_1, \sigma\rangle \Downarrow \text{true} \qquad \langle b_2, \sigma\rangle \Downarrow \text{true}}{\langle b_1 \wedge b_2, \sigma\rangle \Downarrow \text{true}}$$

(show: possible $\vee$ rule)

How to Read the Rules?

- Forward (top-down) = inference rules
  - if we know that the hypothesis judgments hold then we can infer that the conclusion judgment also holds
  - If we know that $\langle e_1, \sigma\rangle \Downarrow 5$ and $\langle e_2, \sigma\rangle \Downarrow 7$, then we can infer that $\langle e_1 + e_2, \sigma\rangle \Downarrow 12$

How to Read the Rules?

- Backward (bottom-up) = evaluation rules
  - Suppose we want to evaluate $e_1 + e_2$, i.e., find n s.t. $e_1 + e_2 \Downarrow n$ is derivable using the previous rules
  - By inspection of the rules we notice that the last step in the derivation of $e_1 + e_2 \Downarrow n$ **must be** the addition rule
    - the other rules have conclusions that would not match $e_1 + e_2 \Downarrow n$
    - this is called reasoning by inversion on the derivation rules

Evaluation By Inversion

- Thus we must find n_1 and n_2 such that $e_1 \Downarrow n_1$ and $e_2 \Downarrow n_2$ are derivable
  - This is done recursively
- If there is exactly one rule for each kind of expression we say that the rules are syntax-directed
  - At each step at most one rule applies
  - This allows a simple evaluation procedure as above
  - True for our Aexp but not Bexp. Why?

Evaluation of Commands

- The evaluation of a Com may have side effects but has no direct result
  - What is the result of evaluating a command ?
- The "result" of a Com is a new state:

$$\langle c, \sigma \rangle \Downarrow \sigma'$$

  - But the evaluation of Com might not terminate! Danger Will Robinson!

Com Evaluation Rules 1

$$\frac{}{\langle \text{skip}, \sigma \rangle \Downarrow \sigma} \qquad \frac{\langle c_1, \sigma \rangle \Downarrow \sigma' \quad \langle c_2, \sigma' \rangle \Downarrow \sigma''}{\langle c_1 ; c_2, \sigma \rangle \Downarrow \sigma''}$$

$$\frac{\langle b, \sigma \rangle \Downarrow \text{true} \quad \langle c_1, \sigma \rangle \Downarrow \sigma'}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2, \sigma \rangle \Downarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \Downarrow \text{false} \quad \langle c_2, \sigma \rangle \Downarrow \sigma'}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2, \sigma \rangle \Downarrow \sigma'}$$

Com Evaluation Rules 2

$$\frac{\langle e, \sigma \rangle \Downarrow n}{\langle x := e, \sigma \rangle \Downarrow \sigma[x := n]}$$

Def: $\sigma[x:= n](x) = n$
$\sigma[x:= n](y) = \sigma(y)$

- Let's do while together

Com Evaluation Rules 3

$$\frac{\langle e, \sigma \rangle \Downarrow n}{\langle x := e, \sigma \rangle \Downarrow \sigma[x := n]}$$

Def: $\sigma[x:= n](x) = n$
$\sigma[x:= n](y) = \sigma(y)$

$$\frac{\langle b, \sigma \rangle \Downarrow \text{false}}{\langle \text{while } b \text{ do } c, \sigma \rangle \Downarrow \sigma}$$

$$\frac{\langle b, \sigma \rangle \Downarrow \text{true} \quad \langle c; \text{while } b \text{ do } c, \sigma \rangle \Downarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma \rangle \Downarrow \sigma'}$$

Homework

- Homework 1 Out Today
  - Actually out last Friday …
  - Due Tuesday, January 31
- Read at least 1 of these 3 Articles
  - 1. Wegner's *Programming Languages - The First 25 years*
  - 2. Wirth's *On the Design of Programming Languages*
  - 3. Nauer's *Report on the algorithmic language ALGOL 60*
- Skim the optional reading – we'll discuss opsem "in the wild" next time