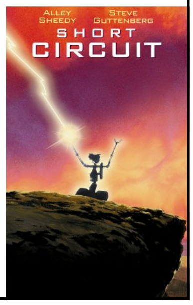
Simply-Typed Lambda Calculus

Homework Five Is Alive

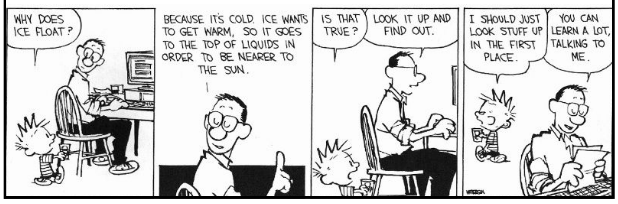
- Homework 5 has not been returned
- Waiting on a few students who want to turn it in later
- There will be no Number Six


"Number 6"



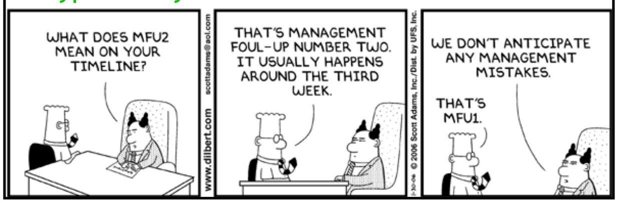
Back to School

- What is operational semantics? When would you use contextual (small-step) semantics?
- What is denotational semantics?
- What is axiomatic semantics? What is a verification condition?



Today's Cunning Plan

- Type System Overview
- First-Order Type Systems
- Typing Rules
- Typing Derivations
- Type Safety



Why Typed Languages?

- Development
  - *Type checking catches early many mistakes*
  - Reduced debugging time
  - Typed signatures are a powerful basis for design
  - Typed signatures enable separate compilation
- Maintenance
  - Types act as checked specifications
  - Types can enforce abstraction
- Execution
  - Static checking reduces the need for dynamic checking
  - Safe languages are easier to analyze statically
    - the compiler can generate better code

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Why Not Typed Languages?

- Static type checking imposes constraints on the programmer
  - Some valid programs might be rejected
  - But often they can be made well-typed easily
  - Hard to step outside the language (e.g. OO programming in a non-OO language, but cf. Ruby, OCaml, etc.)
- Dynamic safety checks can be costly
  - 50% is a possible cost of bounds-checking in a tight loop
    - In practice, the overall cost is much smaller
  - Memory management must be automatic ⇒ need a garbage collector with the associated run-time costs
  - Some applications are justified in using weakly-typed languages (e.g., by external safety proof)

#6

Properties of Type Systems

- How do types differ from other program annotations
  - Types are more precise than comments
  - Types are more easily mechanizable than program specifications
- Expected properties of type systems:
  - Types should be enforceable
  - Types should be checkable algorithmically
  - Typing rules should be transparent
    - It should be easy to see why a program is not well-typed

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Why Formal Type Systems?

- Many typed languages have informal descriptions of the type systems (e.g., in language reference manuals)
- A fair amount of careful analysis is required to avoid false claims of type safety
- A formal presentation of a type system is a precise specification of the type checker
  - And allows formal proofs of type safety
- But even informal knowledge of the principles of type systems help

#8

Formalizing a Type System

1. Syntax
   - Of expressions (programs)
   - Of types
   - Issues of binding and scoping
2. Static semantics (typing rules)
   - Define the typing judgment and its derivation rules
3. Dynamic semantics (e.g., operational)
   - Define the evaluation judgment and its derivation rules
4. Type soundness
   - Relates the static and dynamic semantics
   - State and prove the soundness theorem

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Typing Judgments

- Judgment (recall)
  - A statement J about certain formal entities
  - Has a truth value $\models$ J
  - Has a derivation $\vdash$ J (= "a proof")
- A common form of typing judgment:
  $\Gamma \vdash e : \tau$ (e is an expression and τ is a type)
- Γ (Gamma) is a set of type assignments for the free variables of e
  - Defined by the grammar $\Gamma ::= \cdot \mid \Gamma, x : \tau$
  - Type assignments for variables not free in e are not relevant
  - e.g, $x : \text{int}, y : \text{int} \vdash x + y : \text{int}$

#10

Typing rules

- Typing rules are used to derive typing judgments
- Examples:

$$\frac{}{\Gamma \vdash 1 : \texttt{int}}$$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

$$\frac{\Gamma \vdash e_1 : \texttt{int} \quad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}$$

#11

Typing Derivations

- A typing derivation is a derivation of a typing judgment (big surprise there …)
- Example:

$$\dfrac{\dfrac{}{x : \texttt{int} \vdash x : \texttt{int}} \quad \dfrac{x : \texttt{int} \vdash x : \texttt{int} \quad x : \texttt{int} \vdash 1 : \texttt{int}}{x : \texttt{int} \vdash x + 1 : \texttt{int}}}{x : \texttt{int} \vdash x + (x + 1) : \texttt{int}}$$

- We say $\Gamma \vdash e : \tau$ to mean there exists a derivation of this typing judgment (= "we can prove it")
- Type checking: given Γ, e and τ find a derivation
- Type inference: given Γ and e, find τ and a derivation

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Proving Type Soundness

- A typing judgment is either true or false
- Define what it means for a value to have a type

$$v \in \| \tau \|$$

(e.g. $5 \in \| \text{int} \|$ and $\text{true} \in \| \text{bool} \|$)

- Define what it means for an expression to have a type

$$e \in | \tau | \quad \text{iff} \quad \forall v.\ (e \Downarrow v \Rightarrow v \in \| \tau \|)$$

- Prove type soundness

If $\cdot \vdash e : \tau$ then $e \in | \tau |$

or equivalently

If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $v \in \| \tau \|$

- This implies safe execution (since the result of a unsafe execution is not in $\| \tau \|$ for any τ)

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Upcoming Exciting Episodes

- We will give formal description of first-order type systems (no type variables)
  - Function types (simply typed λ-calculus)
  - Simple types (integers and booleans)
  - Structured types (products and sums)
  - Imperative types (references and exceptions)
  - Recursive types
- The type systems of most common languages are first-order
- The we move to second-order type systems
  - Polymorphism and abstract types

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Simply-Typed Lambda Calculus

- Syntax:

Terms $e ::= x \mid \lambda x{:}\tau.\ e \mid e_1\ e_2 \mid n \mid e_1 + e_2 \mid \text{iszero}\ e \mid \text{true} \mid \text{false} \mid \text{not}\ e \mid \text{if}\ e_1\ \text{then}\ e_2\ \text{else}\ e_3$

Types $\tau ::= \text{int} \mid \text{bool} \mid \tau_1 \to \tau_2$

- $\tau_1 \to \tau_2$ is the function type
- $\to$ associates to the right
- Arguments have typing annotations
- This language is also called F_1

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Static Semantics of F_1

- The typing judgment

$$\Gamma \vdash e : \tau$$

- Some (simpler) typing rules:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : \tau \to \tau'}$$

$$\frac{\Gamma \vdash e_1 : \tau_2 \to \tau \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau}$$

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More Static Semantics of F_1

$$\frac{}{\Gamma \vdash n : \mathtt{int}} \qquad \frac{\Gamma \vdash e_1 : \mathtt{int} \qquad \Gamma \vdash e_2 : \mathtt{int}}{\Gamma \vdash e_1 + e_2 : \mathtt{int}}$$

$$\frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}} \qquad \frac{\Gamma \vdash e : \mathtt{bool}}{\Gamma \vdash \mathtt{not}\ e : \mathtt{bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathtt{bool} \qquad \Gamma \vdash e_t : \tau \qquad \Gamma \vdash e_f : \tau}{\Gamma \vdash \mathtt{if}\ e_1\ \mathtt{then}\ e_t\ \mathtt{else}\ e_f : \tau}$$

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Typing Derivation in F_1

- Consider the term

$$\lambda x : \text{int}.\ \lambda b : \text{bool}.\ \text{if } b \text{ then } f\ x \text{ else } x$$

  - With the initial typing assignment $f : \text{int} \to \text{int}$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash b : \mathtt{bool} \qquad \dfrac{\Gamma \vdash f : \mathtt{int} \to \mathtt{int} \qquad \Gamma \vdash x : \mathtt{int}}{\Gamma \vdash f\ x : \mathtt{int}} \qquad \Gamma \vdash x : \mathtt{int}}{f : \mathtt{int} \to \mathtt{int}, x : \mathtt{int}, b : \mathtt{bool} \vdash \mathtt{if}\ b\ \mathtt{then}\ f\ x\ \mathtt{else}\ x : \mathtt{int}}}{f : \mathtt{int} \to \mathtt{int}, x : \mathtt{int} \vdash \lambda b : \mathtt{bool}.\ \mathtt{if}\ b\ \mathtt{then}\ f\ x\ \mathtt{else}\ x : \mathtt{bool} \to \mathtt{int}}}{f : \mathtt{int} \to \mathtt{int} \vdash \lambda x : \mathtt{int}.\lambda b : \mathtt{bool}.\ \mathtt{if}\ b\ \mathtt{then}\ f\ x\ \mathtt{else}\ x : \mathtt{int} \to \mathtt{bool} \to \mathtt{int}}$$

Where $\Gamma = f : \text{int} \to \text{int},\ x : \text{int},\ b : \text{bool}$

#18

Type Checking in F_1

- Type checking is easy because
  - Typing rules are syntax directed
  - Typing rules are compositional (what does this mean?)
  - All local variables are annotated with types
- In fact, type inference is *also easy* for F_1
- Without type annotations an expression may have no unique type

$$\cdot \vdash \lambda x.\ x : \text{int} \rightarrow \text{int}$$
$$\cdot \vdash \lambda x.\ x : \text{bool} \rightarrow \text{bool}$$

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Operational Semantics of F_1

- Judgment:

$$e \Downarrow v$$

- Values:

$$v ::= n \mid \text{true} \mid \text{false} \mid \lambda x{:}\tau.\ e$$

- The evaluation rules ...
  - Audience participation time: raise your hand and give me an evaluation rule.

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Operational Semantics of F_1 (Cont.)

- Call-by-value evaluation rules (sample)

$$\frac{}{\lambda x : \tau.e \Downarrow \lambda x : \tau.e}$$

$$\frac{e_1 \Downarrow \lambda x : \tau.e_1' \quad e_2 \Downarrow v_2 \quad [v_2/x]e_1' \Downarrow v}{e_1\ e_2 \Downarrow v}$$

$$\frac{}{n \Downarrow n} \qquad \frac{e_1 \Downarrow n_1 \quad e_2 \Downarrow n_2 \quad n = n_1 + n_2}{e_1 + e_2 \Downarrow n}$$

$$\frac{e_1 \Downarrow \texttt{true} \quad e_t \Downarrow v}{\texttt{if } e_1 \texttt{ then } e_t \texttt{ else } e_f \Downarrow v}$$

$$\frac{e_1 \Downarrow \texttt{false} \quad e_f \Downarrow v}{\texttt{if } e_1 \texttt{ then } e_t \texttt{ else } e_f \Downarrow v}$$

Evaluation is undefined for ill-typed programs !

#21

Type Soundness for F_1

- Theorem: If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$
  - Also called, subject reduction theorem, type preservation theorem
- This is one of the most important sorts of theorems in PL
- Whenever you make up a new safe language you are expected to prove this
  - Examples: Vault, TAL, CCured, …

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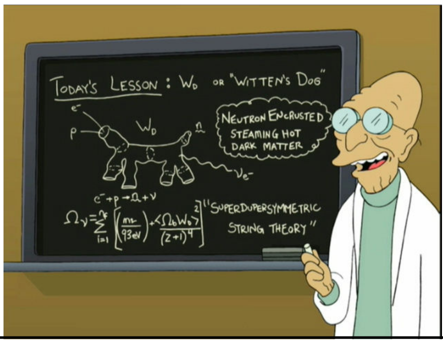
How Can We Prove It?

- Theorem: If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$



#23

Proof Approaches To Type Safety

- Theorem: If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$
- Try to prove by induction on e
  - Won't work because $[v_2/x]e'_1$ in the evaluation of $e_1\ e_2$
  - Same problem with induction on $\cdot \vdash e : \tau$
- Try to prove by induction on τ
  - Won't work because e_1 has a "bigger" type than $e_1\ e_2$
- Try to prove by induction on $e \Downarrow v$
  - To address the issue of $[v_2/x]e'_1$
  - This is it!

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Type Soundness Proof

- Consider the case

$$\mathcal{E} :: \frac{e_1 \Downarrow \lambda x : \tau_2 . e_1' \quad e_2 \Downarrow v_2 \quad [v_2/x]e_1' \Downarrow v}{e_1\ e_2 \Downarrow v}$$

and by inversion on the derivation of $e_1\ e_2 : \tau$

$$\mathcal{D} :: \frac{\cdot \vdash e_1 : \tau_2 \longrightarrow \tau \quad \cdot \vdash e_2 : \tau_2}{\cdot \vdash e_1\ e_2 : \tau}$$

- From IH on $e_1 \Downarrow$ … we have $\cdot, x : \tau_2 \vdash e_1' : \tau$
- From IH on $e_2 \Downarrow$ … we have $\cdot \vdash v_2 : \tau_2$
- Need to infer that $\cdot \vdash [v_2/x]e_1' : \tau$ and use the IH
  - We need a substitution lemma (by induction on e_1')

#25

Significance of Type Soundness

- The theorem says that the result of an evaluation has the same type as the initial expression
- The theorem does not say that
  - The evaluation *never gets stuck* (e.g., trying to apply a non-function, to add non-integers, etc.), nor that
  - The evaluation *terminates*
- Even though both of the above facts are true of F_1
- We need a small-step semantics to prove that the execution never gets stuck
- I Assert: the execution always terminates in F_1
  - When does the lambda calculus ever not terminate?

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Small-Step Contextual Semantics for F_1

- We define redexes

  $r ::= n_1 + n_2 \mid \text{if } b \text{ then } e_1 \text{ else } e_2 \mid (\lambda x{:}\tau.e_1)\ v_2$
- and contexts

  $H ::= H_1 + e_2 \mid n_1 + H_2 \mid \text{if } H \text{ then } e_1 \text{ else } e_2 \mid H_1\ e_2 \mid (\lambda x{:}\tau.\ e_1)\ H_2$
- and local reduction rules

  $n_1 + n_2 \to n_1 \text{ plus } n_2$

  $\text{if true then } e_1 \text{ else } e_2 \to e_1$

  $\text{if false then } e_1 \text{ else } e_2 \to e_2$

  $(\lambda x{:}\tau.\ e_1)\ v_2 \to [v_2/x]e_1$
- and one global reduction rule

  $H[r] \to H[e] \quad \text{iff } r \to e$

#27

Decomposition Lemmas for F_1

1. If $\cdot \vdash e : \tau$ and e is not a (final) value then there exist (unique) H and r such that $e = H[r]$
   - any well typed expression can be decomposed
   - any well-typed non-value can make progress
2. Furthermore, there exists τ' such that $\cdot \vdash r : \tau'$
   - the redex is closed and well typed
3. Furthermore, there exists e' such that $r \to e'$ and $\cdot \vdash e' : \tau'$
   - local reduction is type preserving
4. Furthermore, for any e', $\cdot \vdash e' : \tau'$ implies $\cdot \vdash H[e'] : \tau$
   - the expression preserves its type if we replace the redex with an expression of same type

#28

Type Safety of F_1

- Type preservation theorem
  - If $\cdot \vdash e : \tau$ and $e \to e'$ then $\cdot \vdash e' : \tau$
  - Follows from the decomposition lemma
- Progress theorem
  - If $\cdot \vdash e : \tau$ and e is not a value then there exists e' such that e can make progress: $e \to e'$
- Progress theorem says that execution can make progress *on a well typed expression*
- From type preservation we know the execution of well *typed expressions never gets stuck*
  - This is a (very!) common way to *state and prove type safety* of a language

#29

What's Next?

- We've got the basic simply-typed monomorphic lambda calculus
- Now let's make it more complicated …
- By adding features!

Product Types: Static Semantics

- Extend the syntax with (binary) tuples

  e ::= ... | (e_1, e_2) | fst e | snd e

  τ ::= ... | $\tau_1 \times \tau_2$

  - This language is sometimes called $F_1^\times$
- Same typing judgment $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{fst}\ e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{snd}\ e : \tau_2}$$

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Product Types: Dynamic Semantics and Soundness

- New form of values: v ::= ... | (v_1, v_2)
- New (big step) evaluation rules:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)}$$

$$\frac{e \Downarrow (v_1, v_2)}{\texttt{fst}\ e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{snd}\ e \Downarrow v_2}$$

- New contexts: H ::= ... | (H_1, e_2) | (v_1, H_2) | fst H | snd H
- New redexes:

  fst $(v_1, v_2) \to v_1$

  snd $(v_1, v_2) \to v_2$

- Type soundness holds just as before

#32

General PL Feature Plan

- The general plan for language feature design
- You invent a new feature (tuples)
- You add it to the lambda calculus
- You invent typing rules and opsem rules
- You extend the basic proof of type safety
- You declare moral victory, and milling throngs of cheering admirers wait to carry you on their shoulders to be knighted by the Queen, etc.

#33

Homework

- Read Wright and Felleisen article
- Work on your projects!
  - Status Update Due: Thursday Mar 23

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