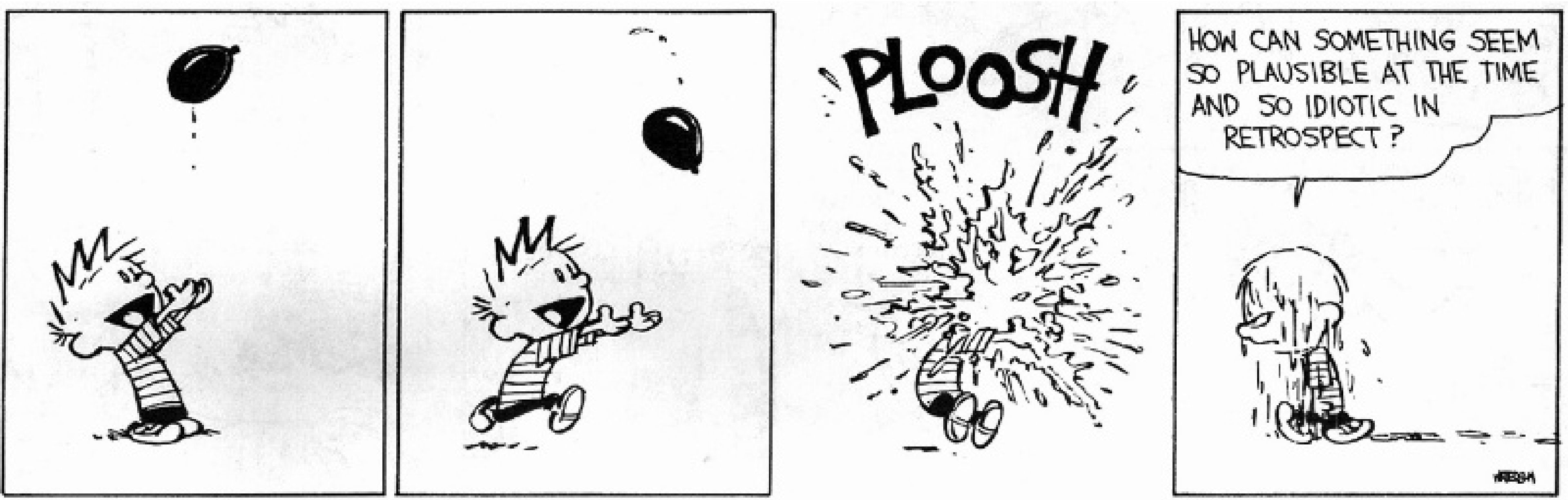
Monomorphic Type Systems



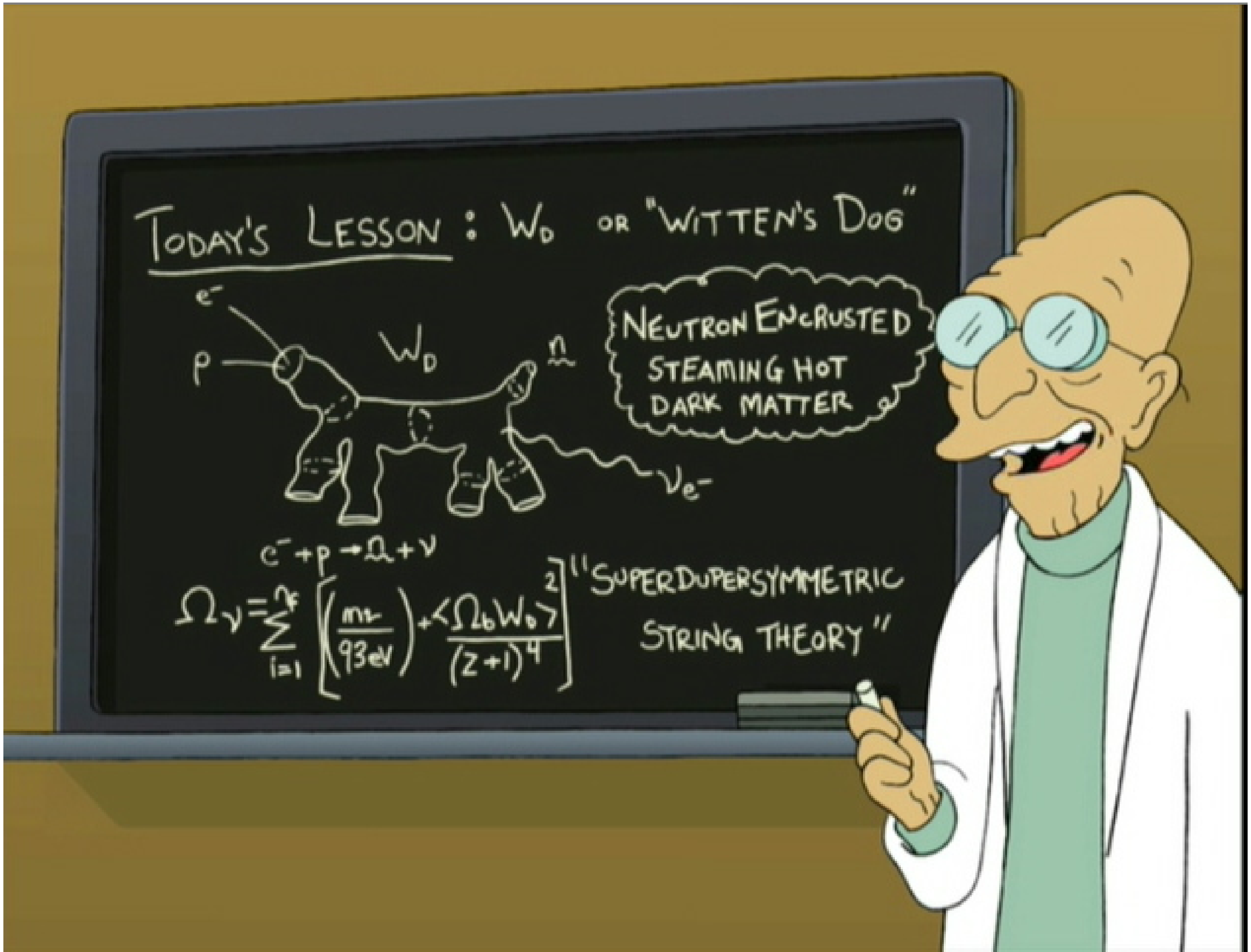
Type Soundness for F_1

What does this mean?

- Theorem: **If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$**
  - Also called, subject reduction theorem, type preservation theorem
- This is one of the most important sorts of theorems in PL
- Whenever you make up a new safe language you are expected to prove this
  - Examples: Vault, TAL, CCured, …

How Might We Prove It?

- Theorem: **If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$**



Proof Approaches To Type Safety

- Theorem: If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$
- Try to prove by induction on e
  - Won't work because $[v_2/x]e'_1$ in the evaluation of $e_1\ e_2$
  - Same problem with induction on $\cdot \vdash e : \tau$
- Try to prove by induction on τ
  - Won't work because e_1 has a "bigger" type than $e_1\ e_2$
- Try to prove by induction on $e \Downarrow v$
  - To address the issue of $[v_2/x]e'_1$
  - This is it!

Type Soundness Proof

- Consider the *function application* case

$$\mathcal{E} :: \frac{e_1 \Downarrow \lambda x : \tau_2 . e_1' \quad e_2 \Downarrow v_2 \quad [v_2/x]e_1' \Downarrow v}{e_1\ e_2 \Downarrow v}$$

and by inversion on the derivation of $e_1\ e_2 : \tau$

$$\mathcal{D} :: \frac{\cdot \vdash e_1 : \tau_2 \longrightarrow \tau \quad \cdot \vdash e_2 : \tau_2}{\cdot \vdash e_1\ e_2 : \tau}$$

- From IH on $e_1 \Downarrow$ … we have $\cdot, x : \tau_2 \vdash e_1' : \tau$
- From IH on $e_2 \Downarrow$ … we have $\cdot \vdash v_2 : \tau_2$
- Need to infer that $\cdot \vdash [v_2/x]e_1' : \tau$ and use the IH
  - We need a substitution lemma (by induction on e_1')

Significance of Type Soundness

- The theorem says that the result of an evaluation has the same type as the initial expression
- The theorem does not say that
  - The evaluation *never gets stuck* (e.g., trying to apply a non-function, to add non-integers, etc.), nor that
  - The evaluation *terminates*
- Even though both of the above facts are true of F_1
- What formal system of semantics do we use to reason about programs that might not terminate?

Significance of Type Soundness

- The theorem says that the result of an evaluation has the same type as the initial expression
- The theorem does not say that
  - The evaluation *never gets stuck* (e.g., trying to apply a non-function, to add non-integers, etc.), nor that
  - The evaluation *terminates*
- Even though both of the above facts are true of F_1
- We need a *small-step semantics* to prove that the execution never gets stuck
- I Assert: the execution always terminates in F_1
  - When does the base lambda calculus ever not terminate?

Small-Step Contextual Semantics for F_1

- We define redexes

$$r ::= n_1 + n_2 \mid \text{if } b \text{ then } e_1 \text{ else } e_2 \mid (\lambda x{:}\tau.e_1)\ v_2$$

- and contexts

$$H ::= H_1 + e_2 \mid n_1 + H_2 \mid \text{if } H \text{ then } e_1 \text{ else } e_2 \mid H_1\ e_2 \mid (\lambda x{:}\tau.\ e_1)\ H_2 \mid \bullet$$

- and local reduction rules

$n_1 + n_2 \to n_1 \text{ plus } n_2$

$\text{if true then } e_1 \text{ else } e_2 \to e_1$

$\text{if false then } e_1 \text{ else } e_2 \to e_2$

$(\lambda x{:}\tau.\ e_1)\ v_2 \to [v_2/x]e_1$

- and one global reduction rule

$H[r] \to H[e] \quad \text{iff } r \to e$

Decomposition Lemmas for F_1

- If $\cdot \vdash e : \tau$ and e is not a (final) value then there exist (unique) H and r such that e = H[r]
  - any well typed expression can be decomposed
  - any well-typed non-value can make progress
- Furthermore, there exists τ' such that $\cdot \vdash r : \tau'$
  - the redex is closed and well typed
- Furthermore, there exists e' such that $r \to e'$ and $\cdot \vdash e' : \tau'$
  - local reduction is type preserving
- Furthermore, for any e', $\cdot \vdash e' : \tau'$ implies $\cdot \vdash H[e'] : \tau$
  - the expression preserves its type if we replace the redex with an expression of same type

Type Safety of F_1

- Type preservation theorem
  - If $\cdot \vdash e : \tau$ and $e \to e'$ then $\cdot \vdash e' : \tau$
  - Follows from the decomposition lemma
- Progress theorem
  - If $\cdot \vdash e : \tau$ and e is not a value then there exists e' such that e can make progress: $e \to e'$
- Progress theorem says that execution can make progress *on a well typed expression*
- From type preservation we know the execution of well *typed expressions never gets stuck*
  - This is a (very!) common way to *state and prove type safety* of a language

What's Next?

- We've got the basic simply-typed monomorphic lambda calculus
- Now let's make it more complicated …
- By adding features!

Product Types: Static Semantics

- Extend the syntax with (binary) tuples

$$e \quad ::= \ \dots \mid (e_1, e_2) \mid \text{fst } e \mid \text{snd } e$$

$$\tau \quad ::= \ \dots \mid \tau_1 \times \tau_2$$

  - This language is sometimes called $F_1^{\times}$

- Same typing judgment $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{fst}\ e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{snd}\ e : \tau_2}$$

Dynamic Semantics and Soundness

- New form of values: v ::= ... | (v_1, v_2)
- New (big step) evaluation rules:

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)}$$

$$\frac{e \Downarrow (v_1, v_2)}{\texttt{fst}\ e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{snd}\ e \Downarrow v_2}$$

- New contexts: H ::= ... | (H_1, e_2) | (v_1, H_2) | fst H | snd H
- New redexes:

  fst $(v_1, v_2) \rightarrow v_1$

  snd $(v_1, v_2) \rightarrow v_2$

- Type soundness holds just as before

Q: General (454 / 842)

- In traditional logic this is an inference in which one proposition (the conclusion) necessarily follows from two others (the premises). An overused example is: *"All men are mortal. Socrates is a man. Therefore, Socrates is a mortal."*

Q: General (473 / 842)

- Which of the following chemical processes or reactions would be the most difficult to conduct in a high school chemistry lab?
  - Hall-Heroult (Aluminum Extraction) Process
  - Making Nitrocellulose (Guncotton)
  - Making Slime
  - Thermite Reaction (which reaches 5000(F))

Q: Games (534 / 842)

- Each face of this 1974 six-sided plastic puzzle is subdivided into nine smaller faces, each of which can be one of six colors.

Q: Games (547 / 842)

- This viscoelastic silicone plastic "clay" came out of efforts to find a rubber substitute in World War II. It is now sold in plastic eggs as a toy for children. It bounces and can absorb the ink from newsprint. It was also used by the crew of Apollo 8 to secure tools in zero gravity.

Q: Events (595 / 842)

- Identify 3 of the following 5 world leaders based on the time and place they came to power.
  - France, May 7, 1995
  - Haiti, December 16, 1990
  - Russia, July 10, 1991
  - Serbia, December 9, 1990
  - South Africa, May 9, 1994

A: Events (595 / 842)

- Jacques Chirac
- Jean-Bertrand Aristide (ending three decades of military rule)
- Boris Yeltsin (first elected president of Russia)
- Slobodan Milosevic
- Nelson Mandela (South Africa's first black president)

General PL Feature Plan

- The general plan for language feature design
- You invent a new feature (tuples)
- You add it to the lambda calculus
- You invent typing rules and opsem rules
- You extend the basic proof of type safety
- You declare moral victory, and milling throngs of cheering admirers wait to carry you on their shoulders to be knighted by the Queen, etc.

Records

- Records are like tuples with labels (w00t!)
- New form of expressions

  $$e ::= \ldots \mid \{L_1 = e_1, \ldots, L_n = e_n\} \mid e.L$$

- New form of values

  $$v ::= \{L_1 = v_1, \ldots, L_n = v_n\}$$

- New form of types

  $$\tau ::= \ldots \mid \{L_1 : \tau_1, \ldots, L_n : \tau_n\}$$

- ... follows the model of $F_1^{\times}$
  - typing rules
  - derivation rules
  - type soundness

On the board!

Sum Types

- We need disjoint union types of the form:
  - either an int or a float
  - either 0 or a pointer
  - either a (binary tree node with two children) or a (leaf)
- New expressions and types

  e ::= ... | injl e | injr e |

  case e of injl x $\to$ e_1 | injr y $\to$ e_2

  τ ::= ... | $\tau_1 + \tau_2$

  - A value of type $\tau_1 + \tau_2$ is *either* a τ_1 *or* a τ_2
  - Like union in C or Pascal, but safe
    - distinguishing between components is *under compiler control*
  - case is a *binding operator* (like "let"): x is bound in e_1 and y is bound in e_2 (like OCaml's "match … with")

Examples with Sum Types

- Consider the type unit with a single element called * or ()
- The type integer option defined as "unit + int"
  - Useful for optional arguments or return values
    - No argument: injl * (OCaml's "None")
    - Argument is 5: injr 5 (OCaml's "Some(5)")
  - To use the argument you must test the kind of argument
  - case arg of injl x ⇒ "no_arg_case" | injr y ⇒ "...y..."
  - injl and injr are tags and case is tag checking
- bool is the union type "unit + unit"
  - true is injl *
  - false is injr *
  - if e then e_1 else e_2 is case e of injl x ⇒ e_1 | injr y ⇒ e_2

Static Semantics of Sum Types

- New typing rules

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \texttt{injl}\ e : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \texttt{injr}\ e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma, x : \tau_1 \vdash e_l : \tau \qquad \Gamma, y : \tau_2 \vdash e_r : \tau}{\Gamma \vdash \texttt{case}\ e_1\ \texttt{of}\ \texttt{injl}\ x \Rightarrow e_l \mid \texttt{injr}\ y \Rightarrow e_r : \tau}$$

- Types are *not unique* anymore

  injl 1 : int + bool

  injl 1 : int + (int $\rightarrow$ int)

  - this complicates type checking, but it is still doable

Dynamic Semantics of Sum Types

- New values $\quad$ v ::= … | injl v | injr v
- New evaluation rules

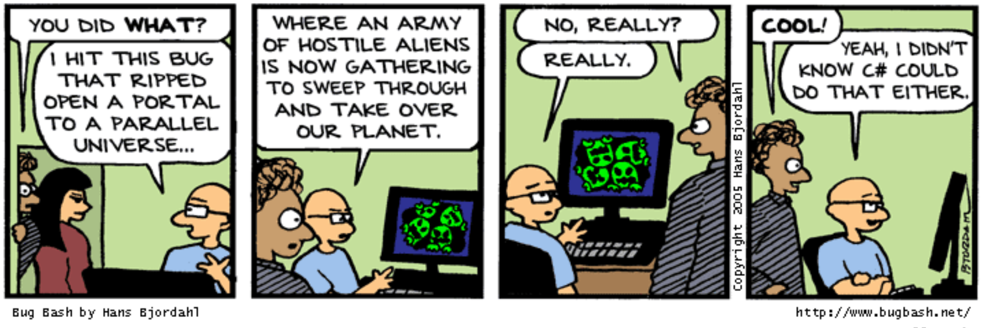
$$\frac{e \Downarrow v}{\texttt{injl}\ e \Downarrow \texttt{injl}\ v} \qquad \frac{e \Downarrow v}{\texttt{injr}\ e \Downarrow \texttt{injr}\ v}$$

$$\frac{e \Downarrow \texttt{injl}\ v \qquad [v/x]e_l \Downarrow v'}{\texttt{case}\ e\ \texttt{of}\ \texttt{injl}\ x \Rightarrow e_l \mid \texttt{injr}\ y \Rightarrow e_r \Downarrow v'}$$

$$\frac{e \Downarrow \texttt{injr}\ v \qquad [v/y]e_r \Downarrow v'}{\texttt{case}\ e\ \texttt{of}\ \texttt{injl}\ x \Rightarrow e_l \mid \texttt{injr}\ y \Rightarrow e_r \Downarrow v'}$$

Type Soundness for F_1^+

- Type soundness *still holds*
- No way to use a $\tau_1 + \tau_2$ inappropriately
- The key is that the only way to use a $\tau_1 + \tau_2$ is with case, which ensures that you are not using a τ_1 as a τ_2
- In C or Pascal checking the tag is the responsibility of the programmer!
  - Unsafe



Types for Imperative Features

- So far: types for pure functional languages
- Now: types for imperative features
- Such types are used to characterize non-local effects
  - assignments
  - exceptions
  - typestate
- Contextual semantics is useful here
  - Just when you thought it was safe to forget it …

Reference Types

- Such types are used for mutable memory cells
- Syntax (as in ML)

Why do I need : τ ?

$$e ::= \ldots \mid \text{ref } e : \tau \mid e_1 := e_2 \mid !\, e$$

$$\tau ::= \ldots \mid \tau \text{ ref}$$

  – ref e : τ - evaluates e, allocates a new memory cell, stores the value of e in it and returns the address of the memory cell
    - like malloc + initialization in C, or new in C++ and Java
  – $e_1 := e_2$, evaluates e_1 to a memory cell and updates its value with the value of e_2
  – ! e - evaluates e to a memory cell and returns its contents

Global Effects, Reference Cells

- A reference cell can <u>escape</u> the static scope where it was created

  $(\lambda f{:}int \rightarrow int\ ref.\ !(f\ 5))\quad(\lambda x{:}int.\ ref\ x : int)$

- The value stored in a reference cell *must be visible from the entire program*
- The "result" of an expression must now include the changes to the heap that it makes (cf. IMP's opsem)
- To model reference cells we must extend the evaluation model

Modeling References

- A heap is a mapping from addresses to values

$$h ::= \cdot \mid h, a \leftarrow v : \tau$$

  - $a \in$ Addresses (Addresses $\neq \mathbb{Z}$?)
  - We tag the heap cells with their types
  - Types are useful only for static semantics. They are not needed for the evaluation $\Rightarrow$ are not a part of the implementation
- We call a program an expression with a heap

p ::= heap h in e

  - The initial program is "heap $\cdot$ in e"
  - Heap addresses act as bound variables in the expression
  - This is a trick that allows easy *reuse of properties of local variables for heap addresses*
    - e.g., we can rename the address and its occurrences at will

Static Semantics of References

- Typing rules for expressions:

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash (\texttt{ref}\ e : \tau) : \tau\ \texttt{ref}} \qquad \frac{\Gamma \vdash e : \tau\ \texttt{ref}}{\Gamma \vdash !e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau\ \texttt{ref} \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 := e_2 : \texttt{unit}}$$

- and for programs

$$\frac{\Gamma \vdash v_i : \tau_i\ (i = 1 \ldots n) \qquad \Gamma \vdash e : \tau}{\vdash \texttt{heap}\ h\ \texttt{in}\ e : \tau}$$

where $\Gamma = a_1 : \tau_1\ \texttt{ref}, \ldots, a_n : \tau_n\ \texttt{ref}$

and $h = a_1 \leftarrow v_1 : \tau_1, \ldots, a_n \leftarrow v_n : \tau_n$

Contextual Semantics for References

- Addresses are values: $v ::= \ldots \mid a$
- New contexts: $H ::= \text{ref } H \mid H_1 := e_2 \mid a_1 := H_2 \mid \,!\, H$
- No new *local* reduction rules
- But some new *global* reduction rules
  - heap h in $H[\text{ref } v : \tau] \to$ heap $h, a \leftarrow v : \tau$ in $H[a]$
    - where a is fresh (this models allocation – the heap is extended)
  - heap h in $H[!\, a] \to$ heap h in $H[v]$
    - where $a \leftarrow v : \tau \in h$ (heap lookup – can we get stuck?)
  - heap h in $H[a := v] \to$ heap $h[a \leftarrow v]$ in $H[*]$
    - where $h[a \leftarrow v]$ means a heap like h except that the part "$a \leftarrow v_1 : \tau$" in h is replaced by "$a \leftarrow v : \tau$" (memory update)
- Global rules are used to propagate the effects of a write to the entire program (eval order matters!)

Example with References

- Consider these (the redex is underlined)
  - heap · in <u>(λf:int → int ref. !(f 5)) (λx:int. ref x : int)</u>
  - heap · in !<u>((λx:int. ref x : int) 5)</u>
  - <u>heap · in !(ref 5 : int)</u>
  - <u>heap a = 5 : int in !a</u>
  - heap a = 5 : int in 5
- The resulting program has a useless memory cell
- An equivalent result would be

  heap · in 5
- This is a simple way to model garbage collection

Homework

- Read Wright and Felleisen article
  - … that you didn't read on Tuesday.
  - Or that optional Goodenough one …
- Work on your projects!