Abstract Interpretation (Non-Standard Semantics)

a.k.a. "Picking The Right Abstraction"

Wei Hu Memorial Homework Award

- Many turned in HW3 code:

```
let rec matches re s =
match re with
| Star(r) -> union (singleton s)
                  (matches (Concat(r,Star(r))) s)
```

- Which is a direct translation of:

$$R[\![r^*]\!]s = \{s\} \cup R[\![rr^*]\!]s$$

or, equivalently:

$$R[\![r^*]\!]s = \{s\} \cup \{ y \mid \exists x \in R[\![r]\!]s \ \wedge \ y \in R[\![r^*]\!]x \}$$

- Why doesn't this work?

Why analyze programs statically?

The Problem

- It is extremely useful to predict program behavior *statically* (= without running the program)
  - For optimizing compilers, program analyses, software engineering tools, finding security flaws, etc.
- The semantics we studied so far give us the precise behavior of a program
- However, precise static predictions are impossible
  - The exact semantics is not computable
- We must settle for approximate, but correct, static analyses (e.g. VC vs. WP)

The Plan

- We will introduce abstract interpretation by example
- Starting with a miniscule language we will build up to a fairly realistic application
- Along the way we will see most of the ideas and difficulties that arise in a big class of applications

A Tiny Language

- Consider the following language of arithmetic ("shrIMP"?)

$$e ::= n \mid e_1 * e_2$$

- The denotational semantics of this language

$$[\![n]\!] = n$$

$$[\![e_1 * e_2]\!] = [\![e_1]\!] \times [\![e_2]\!]$$

- We'll take deno-sem as the "ground truth"
- For this language the precise semantics is computable (but in general it's not)

An Abstraction

- Assume that we are interested not in the value of the expression, but only in its sign:
  - positive (+), negative (-), or zero (0)
- We can define an abstract semantics that computes ***only*** the sign of the result

$$\sigma\colon \text{Exp} \to \{-, 0, +\}$$

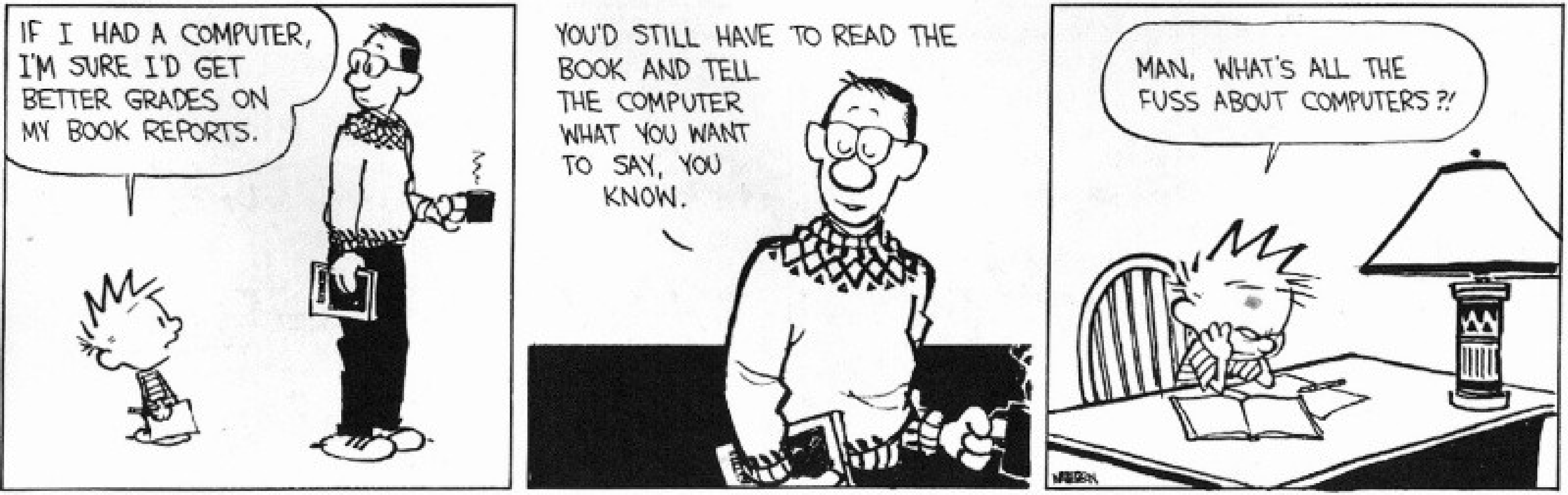
$$\sigma(n) = \text{sign}(n)$$

$$\sigma(e_1 * e_2) = \sigma(e_1) \otimes \sigma(e_2)$$

$\otimes$	-	0	+
-	+	0	-
0	0	0	0
+	-	0	+

I Saw the Sign

- Why did we want to compute the sign of an expression?
  - One reason: no one will believe you know abstract interpretation if you haven't seen the sign example :-)
- What could we be computing instead?



Correctness of Sign Abstraction

- We can show that the abstraction is correct in the sense that it predicts the sign

$$[\![e]\!] > 0 \Leftrightarrow \sigma(e) = +$$

$$[\![e]\!] = 0 \Leftrightarrow \sigma(e) = 0$$

$$[\![e]\!] < 0 \Leftrightarrow \sigma(e) = -$$

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- Our semantics is abstract but precise
- Proof is by structural induction on the expression e
  - Each case repeats similar reasoning

Another View of Soundness

- Link each concrete value to an abstract one:

$$\beta : \mathbb{Z} \to \{ -, 0, + \}$$

- This is called the abstraction function (β)
  - This three-element set is the abstract domain
- Also define the concretization function (γ):

$$\gamma : \{-, 0, +\} \to \mathcal{P}(\mathbb{Z})$$

$$\gamma(+) = \{ n \in \mathbb{Z} \mid n > 0 \}$$

$$\gamma(0) = \{ 0 \}$$

$$\gamma(-) = \{ n \in \mathbb{Z} \mid n < 0 \}$$

Another View of Soundness 2

- Soundness can be stated succinctly

$$\forall e \in \text{Exp}.\ [\![e]\!] \in \gamma(\sigma(e))$$

(the real value of the expression is among the concrete values represented by the abstract value of the expression)

- Let C be the concrete domain (e.g. $\mathbb{Z}$) and A be the abstract domain (e.g. {-, 0, +})
- Commutative diagram:

$$\begin{tikzcd} \text{Exp} \arrow[r, "\sigma"] \arrow[d, "{[\![\cdot]\!]}"'] & \text{A} \arrow[d, "\gamma"] \\ \text{C} \arrow[r, "\in"] & \mathcal{P}(\text{C}) \end{tikzcd}$$

Another View of Soundness 3

- Consider the generic abstraction of an operator

$$\sigma(e_1 \text{ op } e_2) = \sigma(e_1) \; \underline{\text{op}} \; \sigma(e_2)$$

- This is sound iff

$$\forall a_1 \forall a_2 .\ \gamma(a_1 \; \underline{\text{op}} \; a_2) \supseteq \{n_1 \text{ op } n_2 \mid n_1 \in \gamma(a_1),\ n_2 \in \gamma(a_2)\}$$

- e.g. $\gamma(a_1 \otimes a_2) \supseteq \{ n_1 * n_2 \mid n_1 \in \gamma(a_1),\ n_2 \in \gamma(a_2) \}$
- This reduces the proof of correctness to one proof for each operator

Abstract Interpretation

- This is our first example of an abstract interpretation
- We carry out computation in an abstract domain
- The abstract semantics is a sound approximation of the standard semantics
- The concretization and abstraction functions establish the connection between the two domains

Adding Unary Minus and Addition

- We extend the language to

  $e ::= n \mid e_1 * e_2 \mid - e$

- We define $\sigma(- e) = \ominus\, \sigma(e)$

	-	0	+
$\ominus$	+	0	-

- Now we add addition:

  $e ::= n \mid e_1 * e_2 \mid - e \mid e_1 + e_2$

- We define $\sigma(e_1 + e_2) = \sigma(e_1) \oplus \sigma(e_2)$

$\oplus$	-	0	+
-	-	-	?
0	-	0	+
+	?	+	+

Adding Addition

- The sign values are not closed under addition
- What should be the value of "+ $\oplus$ –"?
- Start from the soundness condition:

$$\gamma(+ \oplus -) \supseteq \{ n_1 + n_2 \mid n_1 > 0, n_2 < 0\} = \mathbb{Z}$$

- We don't have an abstract value whose concretization includes $\mathbb{Z}$, so we add one:

  $\top$ ("top" = "don't know")

$\oplus$	-	0	+	$\top$
-	-	-	$\top$	$\top$
0	-	0	+	$\top$
+	$\top$	+	+	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Loss of Precision

- Abstract computation may lose information:

$$[\![(1 + 2) + -3]\!] = 0$$

but: $\sigma((1+2) + -3) =$

$$(\sigma(1) \oplus \sigma(2)) \oplus \sigma(-3) =$$

$$(+ \oplus +) \oplus - = \top$$

- We lost some precision
- But this will simplify the computation of the abstract answer in cases when the precise answer is not computable

Adding Division

- Straightforward except for division by 0
  - We say that there is no answer in that case
  - $\gamma(+ \oslash 0) = \{ n \mid n = n_1 / 0 , n_1 > 0 \} = \emptyset$
- Introduce $\perp$ to be the abstraction of the $\emptyset$
  - We also use the same abstraction for non-termination!

    $\perp$ = "nothing"

    $\top$ = "something unknown"

$\oslash$	-	0	+	$\top$	$\perp$
-	+	0	-	$\top$	$\perp$
0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
+	-	0	+	$\top$	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Q: Books (750 / 842)

- This 1962 Newbery Medal-winning novel by Madeleine L'Engle includes Charles Wallace, Mrs. Who, Mrs. Whatsit, Mrs. Which and the space-bending Tesseract.

Computer Science

- This American Turing-award winner is known for developing Speedcoding and FORTRAN (the first two high-level languages), as well creating a way to express the formal syntax of a language and using that approach to specify ALGOL. He later focused on function-level (as opposed to value-level) programming. His first major programming project calculated the positions of the Moon. Oh, and he studied at UVA as an undergrad (but quit).

Q: Events (596 / 842)

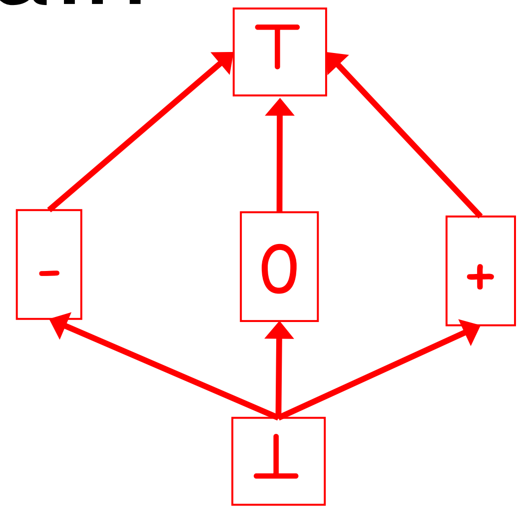
- Fill in the blanks of this 1993 joke with the name of the Prime Minister of the United Kingdom:
  - The Bosnian peace talks continued in Geneva today. The only thing that Alija Izetbegovic, Radovan Karadzic and Slobodan Milosovic could agree on was that <u>blank</u> <u>blank</u> has a funny name.

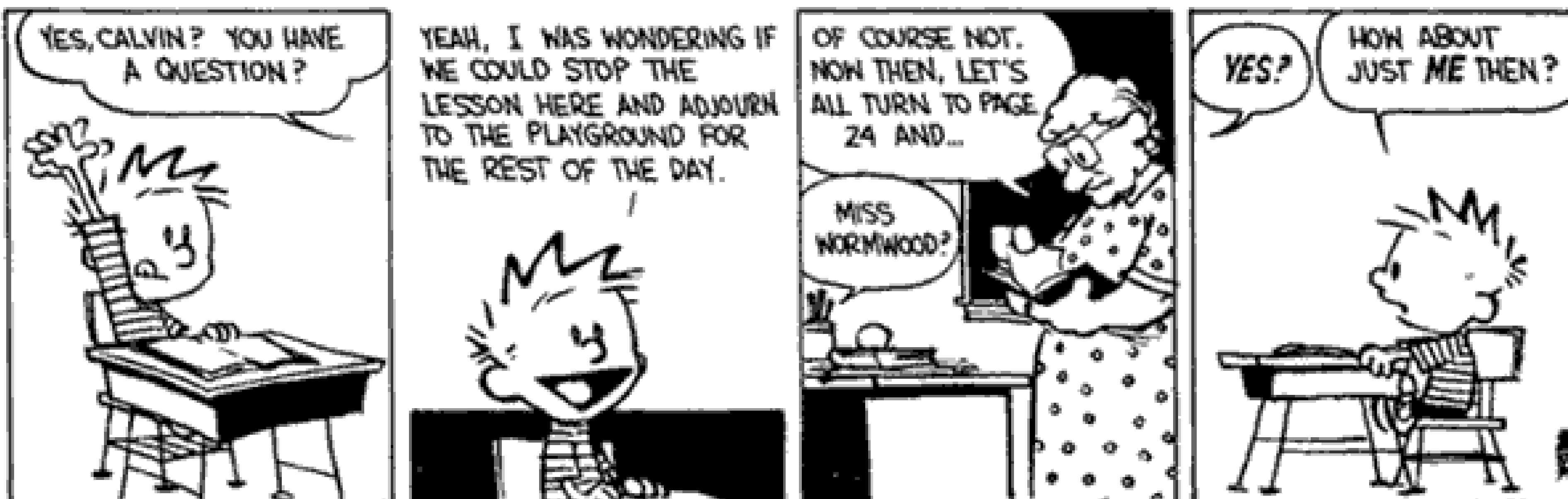
The Abstract Domain

- Our abstract domain forms a lattice
- A partial order is induced by γ

$$a_1 \leq a_2 \quad \text{iff } \gamma(a_1) \subseteq \gamma(a_2)$$

  - We say that a_1 is more precise than a_2!
- Every finite subset has a least-upper bound (lub) and a greatest-lower bound (glb)





Lattice Facts

- A lattice is complete when every subset has a lub and a gub
  - Even infinite subsets!
- Every finite lattice is (trivially) complete
- Every complete lattice is a complete partial order (recall: denotational semantics!)
  - Since a chain is a subset
- Not every CPO is a complete lattice
  - Might not even be a lattice at all

Lattice History

- Early work in denotational semantics used lattices (instead of what?)
  - But only chains need to have lubs
  - And there was no need for $\top$ and glb

Lattice History

- Early work in denotational semantics used lattices (instead of what?)
  - But only chains need to have lubs
  - And there was no need for $\top$ and glb
- In abstract interpretation we'll use $\top$ to denote *"I don't know"*.
  - Corresponds to all values in the concrete domain

From One, Many

- We can start with the abstraction function β

$$\beta : C \to A$$

(maps a concrete value to the best abstract value)

  - A must be a lattice

- We can derive the concretization function γ

$$\gamma : A \to \mathcal{P}(C)$$

$$\gamma(a) = \{ x \in C \mid \beta(x) \leq a \}$$

- And the abstraction for sets α

$$\alpha : \mathcal{P}(C) \to A$$

$$\alpha(S) = \text{lub} \{ \beta(x) \mid x \in S \}$$

Example

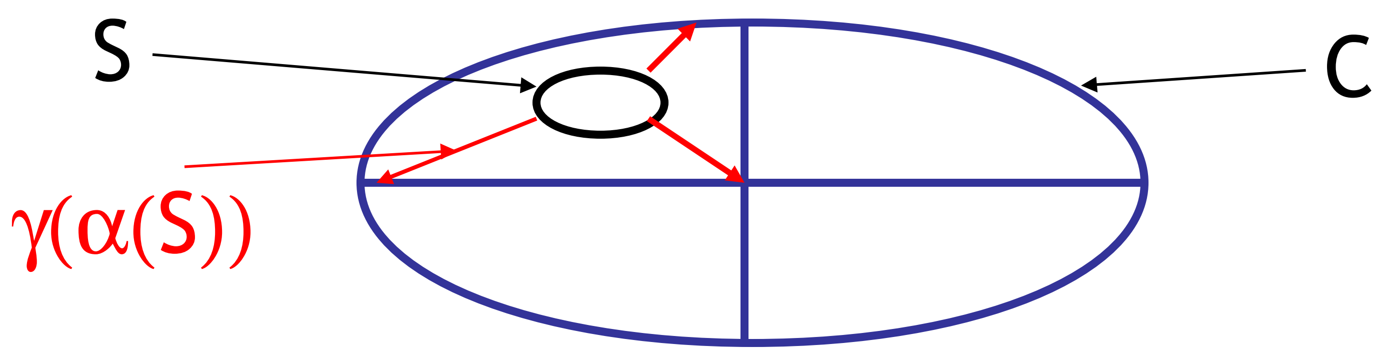
- Consider our sign lattice

$$\beta(n) = \begin{cases} + & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ - & \text{if } n < 0 \end{cases}$$

- $\alpha(S) = \text{lub} \{ \beta(x) \mid x \in S\}$
  - Example:
    $\alpha(\{1, 2\}) = \text{lub} \{ + \} = +$
    $\alpha(\{1, 0\}) = \text{lub} \{ +, 0\} = \top$
    $\alpha(\{\}) = \text{lub}\ \emptyset = \bot$
- $\gamma(a) = \{ n \mid \beta(n) \leq a \}$
  - Example:
    $\gamma(+) = \{ n \mid \beta(n) \leq +\} = \{ n \mid \beta(n) = +\} = \{ n \mid n > 0 \}$
    $\gamma(\top) = \{ n \mid \beta(n) \leq \top \} = \mathbb{Z}$
    $\gamma(\bot) = \{ n \mid \beta(n) \leq \bot\} = \emptyset$

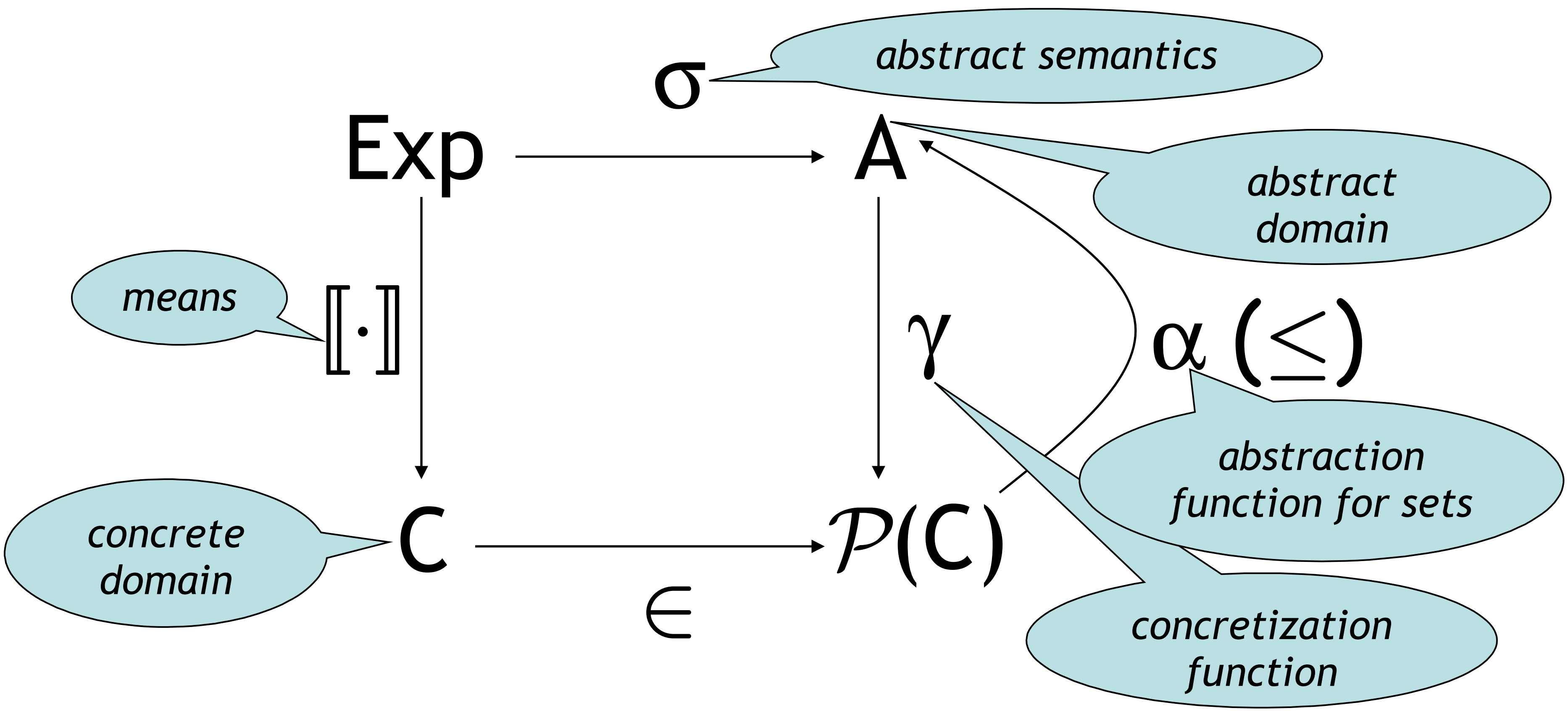
Galois Connections

- We can show that
  - γ and α are monotonic (with $\subseteq$ ordering on $\mathcal{P}$(C))
  - $\alpha(\gamma(a)) = a$ for all $a \in A$
  - $\gamma(\alpha(S)) \supseteq S$ for all $S \in \mathcal{P}(C)$
- Such a pair of functions is called a Galois connection
  - Between the lattices A and $\mathcal{P}$(C)



Correctness Condition

- In general, abstract interpretation satisfies the following (amazingly common) diagram



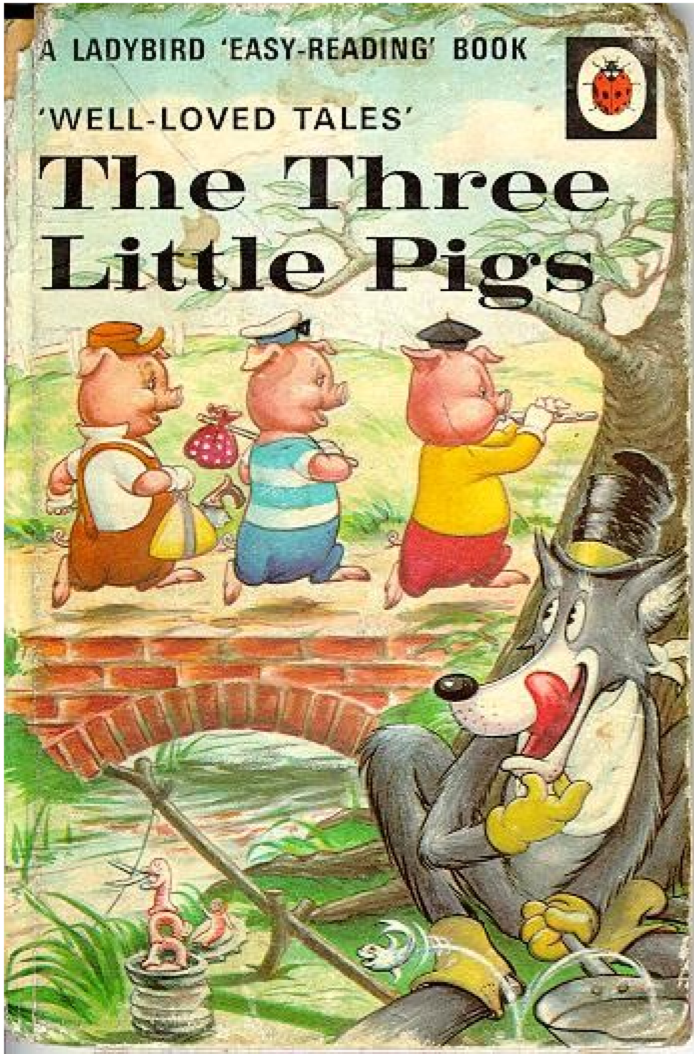
Three Little Correctness Conditions

- Three conditions define a correct abstract interpretation
- α and γ are monotonic
- α and γ form a Galois connection

  = "α and γ are almost inverses"

1. Abstraction of operations is correct

   $$a_1 \underline{\text{op}} a_2 = \alpha(\gamma(a_1) \text{ op } \gamma(a_2))$$



On The Board Questions

- What is the VC for:

$$\text{for } i = e_{low} \text{ to } e_{high} \text{ do } c \text{ done}$$

- This axiomatic rule is unsound. Why?

$$\frac{\vdash \{A \wedge p\}\ c_{then}\ \{B_{then}\} \qquad \vdash \{A \wedge \neg p\}\ c_{else}\ \{B_{else}\}}{\vdash \{A\}\ \text{if } p \text{ then } c_{then} \text{ else } c_{else}\ \{B_{then} \vee B_{else}\}}$$

Homework

- Read Cousot & Cousot Article
- Homework 4 ...