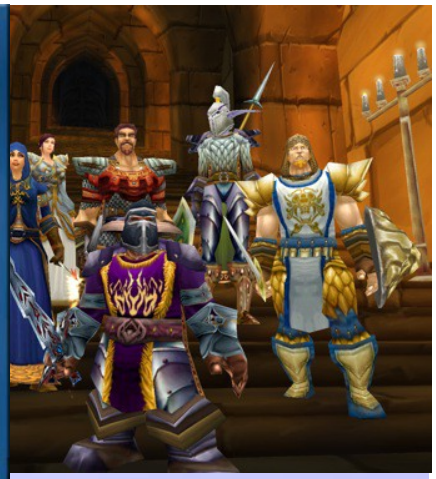
Introduction To Game Theory: Two-Person Games of Perfect Information *and* Winning Strategies

Game?



Theory?

Tetris is Hard, Even to Approximate

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The game of Tetris. In the popular computer game of Tetris, we are given an initial m-by-n *gameboard*, which is a partially filled rectangular grid. The player is given, one by one, a sequence of p *tetrominoes*; see Figure 1. Each piece begins in the middle of the top row of the gameboard, and falls downward at a constant speed. As it falls, the player can rotate the piece and slide it horizontally. If, when the piece comes to rest, all gridsquares in an entire row h of the gameboard are filled, row h is *cleared*: all rows above h fall one row lower, and the top row of the gameboard is replaced by an entirely unfilled row. As soon as a piece is fixed in place, the next piece appears at the top of the gameboard. Typically a one-piece *lookahead* is provided: when the ith piece begins falling, the identity of the $(i+1)$st piece is revealed. A player *loses* when a new piece is blocked by filled gridsquares from entirely entering the gameboard. The player's objective is to maximize his or her score (which increases as pieces are placed and as rows are cleared).

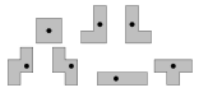

Figure 1: *The tetrominoes* Sq *("square"),* LG *("left gun"),* RG *("right gun"),* LS *("left snake"),* RS *("right snake"),* I, *and* T, *with each piece's* center *marked.*

Offline Tetris. We introduce the natural full-information (offline) version of Tetris: there is a *deterministic, finite* piece sequence, and the player knows the identity and order of all pieces that will be presented. We study the offline version because its hardness captures much of the difficulty of playing Tetris; intuitively, it is only easier to play Tetris with complete knowledge of the future, so the difficulty of playing the offline version suggests the difficulty of playing the online version. It is also natural to let m and n grow, since a relatively simple dynamic program solves the case of $m \cdot n = O(1)$ in time poly(p).

NP-hardness of maximizing rows cleared. We first show that maximizing the number of rows cleared while playing the given sequence is NP-complete. Our proof proceeds by a reduction from 3-PARTITION, in which we are given a set $S = \{a_1, \ldots, a_{3s}\}$ of integers and a bound T, and asked to partition S into s sets of three numbers each, so that the sum of the numbers in each set is exactly T. (3-PARTITION is NP-complete even when the a_i's and T are given in unary [2].) The initial gameboard is shown in Figure 2. The piece sequence consists of the following sequence of pieces for each i: one *initiator* $\langle$I, LG, Sq$\rangle$, then a_i repetitions of the *filler* $\langle$LG, LS, LG, LG, Sq$\rangle$, and then one *terminator* $\langle$Sq, Sq$\rangle$. After the pieces associated with $a_1, \ldots, a_{3s}$, we have the following additional pieces: s successive I's, one RG, and $3T/2 + 5$ successive I's. (Without loss of generality, we force T to be even by doubling all input numbers.) The last three columns of the gameboard form a *lock* which prevents any rows from being cleared using only the pieces corresponding to $a_1, \ldots, a_{3s}$. If all buckets are filled exactly to the same height, then the entire board can be cleared using the last portion of our piece sequence.

The piece sequence is chosen carefully so that all pieces corresponding to each a_i must be placed into the same bucket. The bulk of our proof of correctness is devoted to showing that, despite the decoupled nature of a sequence of Tetris pieces, the only way to possibly clear the entire gameboard is to place into a single bucket all pieces associated with each integer. We can then prove that there is a legal 3-partition for $\{a_1, \ldots, a_{3s}\}$ if and only if the gameboard can be entirely cleared using the given piece sequence. The NP-completeness of Tetris follows.

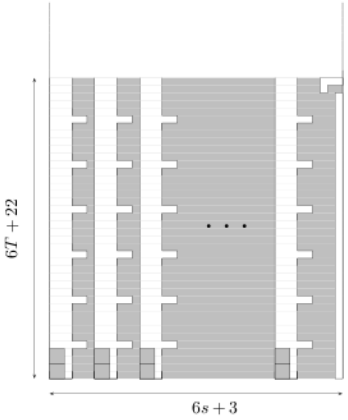

Figure 2: *The initial gameboard for a Tetris game mapped from an instance of* 3-PARTITION.

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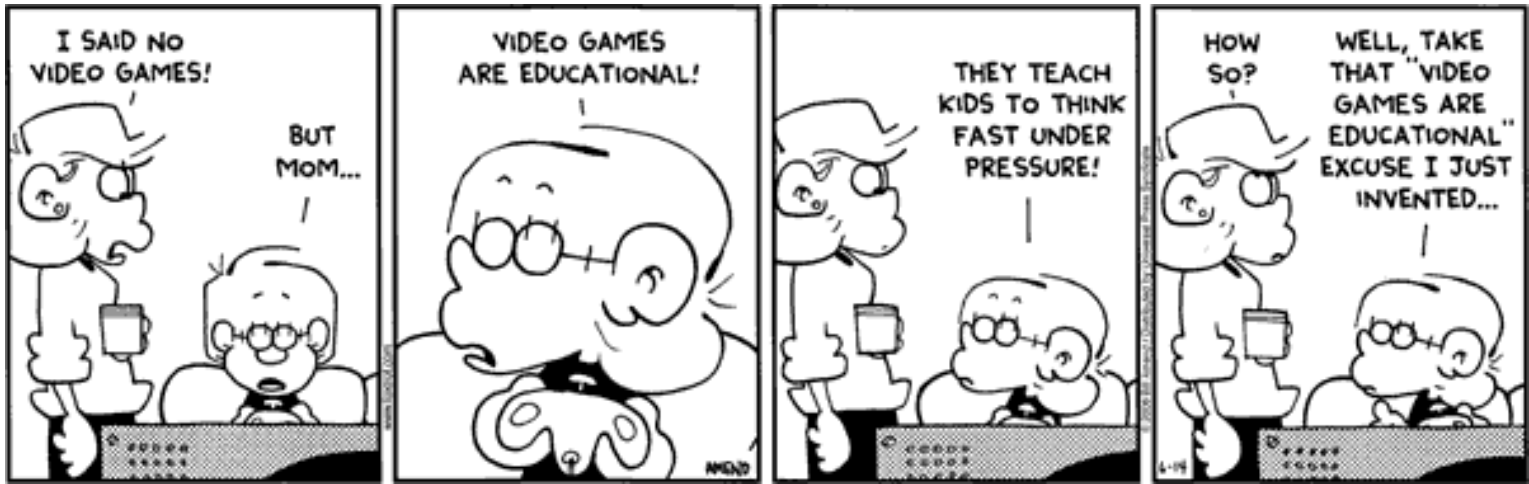
Lecture Outline

- Introduction
- Properties of Games
- Tic-Toe
- Game Trees
- Strategies
- Impartial Games
  - Nim
  - Hackenbush
- Sprague-Grundy Theorem

Game Theory

- **Game Theory** is a branch of applied math used in the social sciences (econ), biology, compsci, and philosophy. Game Theory studies *strategic* situations in which one agent's success depends on the choices of other agents.



Broad Applicability

- Finding equilibria (Nash) – sets of strategies where agents are unlikely to change behavior.
- Econ: understand and predict the behavior of firms, markets, auctions and consumers.
- Animals: (Fisher) communication, gender
- Ethics: normative, good and proper behavior
- PolySci: fair division, public choice. Players are voters, states, interest groups, politicians.
- Software: verifying interfaces can be viewed as a two-player game between the program and the environment (e.g., Henzinger, ...)

Game Properties

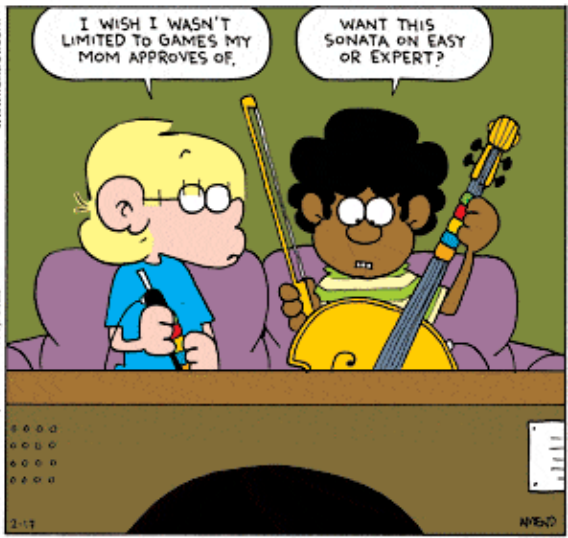
- *Cooperative* (binding contracts, coalitions) or ***non-cooperative***
- ***Symmetric*** (chess, checkers: changing identities does not change strategies) or *asymmetric* (Axis and Allies, Soulcalibur)
- *Zero-sum* (poker: your wins exactly equal my losses) or ***non-zero-sum*** (prisoner's dilemma: gain by me does not necessarily correspond to a loss by you)

Game Properties II

- *Simultaneous* (rock-paper-scissors: we all decide what to do before we see other actions resolve) or ***sequential*** (your turn, then my turn)
- ***Perfect information*** (chess, checkers, go: everyone sees everything) or *imperfect information* (poker, Catan: some hidden state)
- *Infinitely long* (relates to set theory) or ***finite*** (chess, checkers: add a “tie” condition)

Game Properties III

- ***Deterministic*** (chess, checkers, rock-paper-scissors, tic-tac-toe: the "game board" is deterministic, even if the players are not) vs *non-deterministic* (Yahtzee, Monopoly, poker: you roll dice or draw lots)
- More later ...

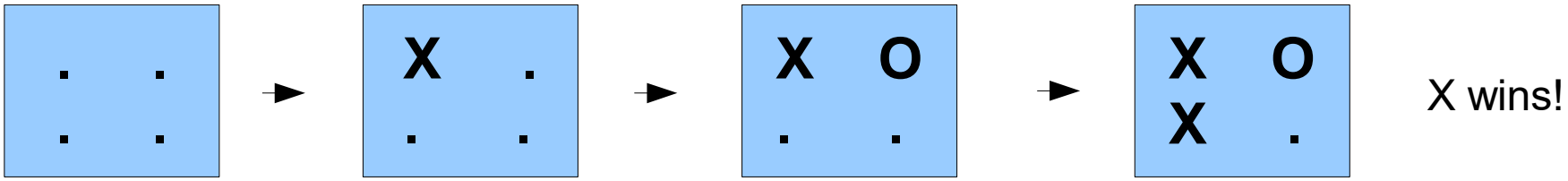


Game Representation

- We will represent games as **trees**
  - Tree of all possible game instances
- There is one **node** for every possible state of the game (e.g., every game board configuration)
  - **Initial Node:** we start here
  - **Decision Node:** I have many moves
  - **Terminal Node:** who won? what's my score?

Introducing: Tic-Toe

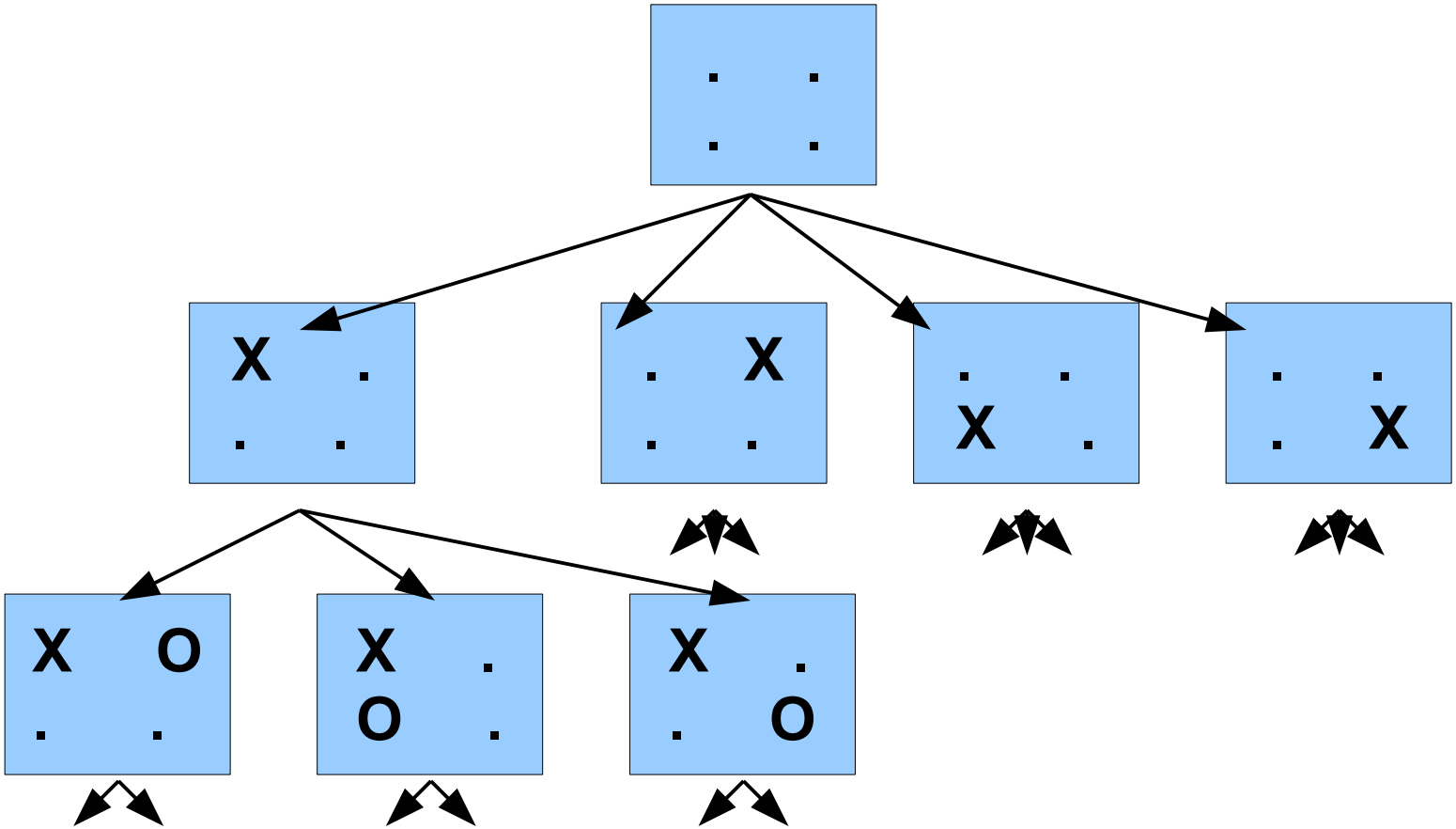
- ***Tic-Toe*** is like Tic-Tac-Toe, but on a 2x2 board where two-in-a-row wins (not diagonal).
  - X goes first
  - Resolutions: X wins, tie , X loses
- Example game:



  - Later: Can O ever win?
  - Later: Can O ever win if X is smart?

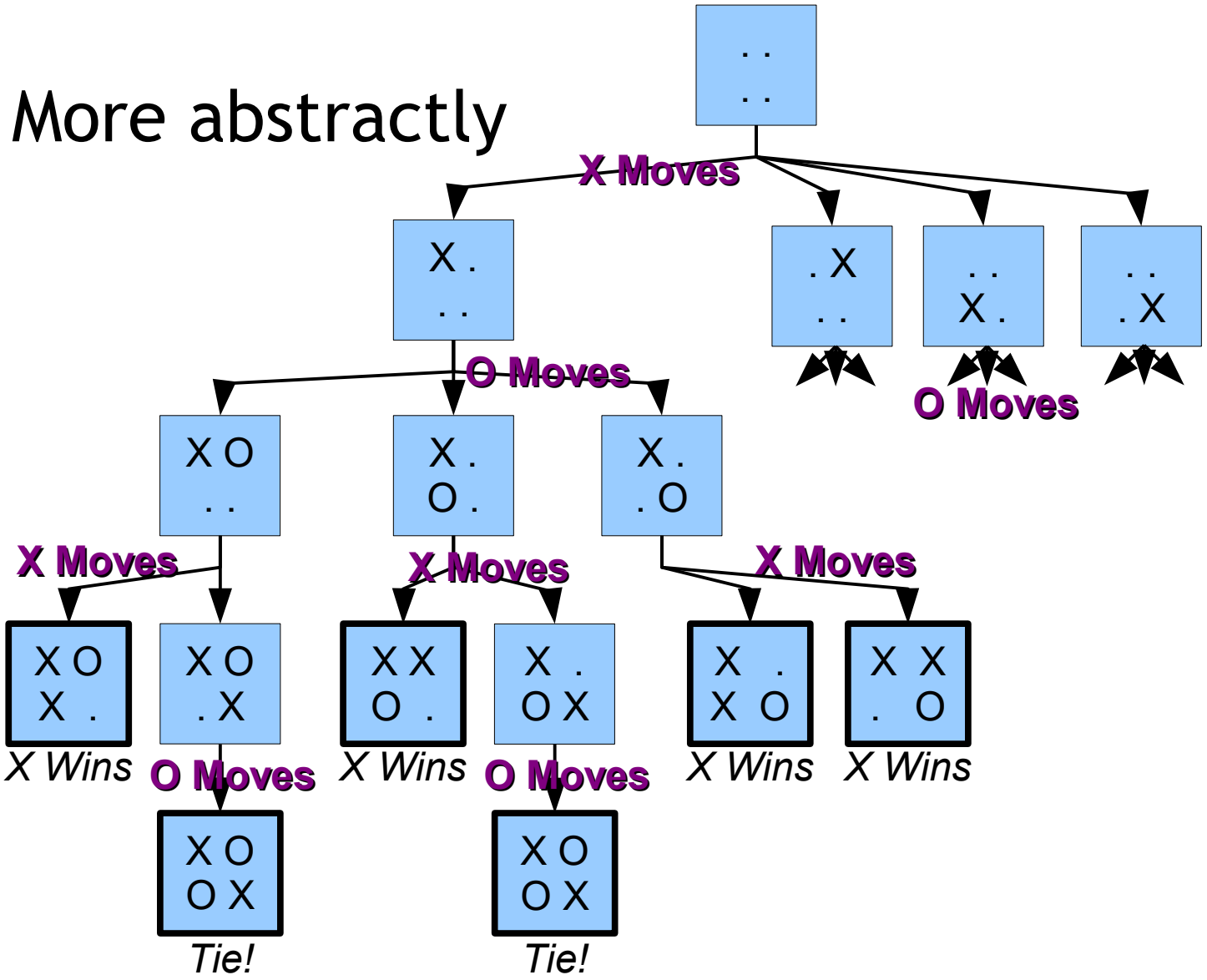
Tic-Toe Trees

- Partial game tree for Tic-Toe



Tic-Toe Trees

- More abstractly



More Definitions

- An ***instance of a game*** is a path through a game tree starting at the initial node and ending in a terminal node.
- ***X's moves*** in a game instance P are the set of edges along that path P taken from decision nodes labeled “X moves”.
- A ***strategy for X*** is a function mapping decision each node labeled “X moves” to a single outgoing edge from that node.

Still Going!

- A deterministic strategy for X, a deterministic strategy for O, and a deterministic game lead deterministically to a single game instance
  - Example: if you always play tic-tac-toe by going in the uppermost, leftmost available square, and I always play it by going in the lowermost, rightmost available square, every time we play we'll have the same result.
- Now we can study various strategies and their outcomes!

Winning Strategies

- A ***winning strategy for X*** on a game G is a strategy S1 for X on G such that, **for all** strategies S2 for O on G, the result of playing G with S1 and S2 is a win for X.
- Does X have a winning strategy for Tic-Toe?
- Does O have a winning strategy for Tic-Toe?
- **Fact:** If the first player in a turn-based deterministic game has a winning strategy, the second player cannot have a winning strategy.
  - Why?

Impartial Games

- An ***impartial*** game has (1) allowable moves that depend only on the position and not on which player is currently moving, and (2) symmetric win conditions (payoffs).
  - Only difference between Player1 and Player2 is that Player1 goes first.
- This is not the case for Chess: White cannot move Black's pieces
  - So I need to know which turn it is to categorize the allowable moves.
- A game that is not impartial is *partisan*.

Nim

- ***Nim*** is a two-player game in which players take turns removing objects from distinct heaps.
  - Non-cooperative, symmetric, sequential, perfect information, finite, **impartial**
- One each turn, a player **must remove** at least one object, and may remove **any number** of objects provided they all come **from the same heap**.
- If you cannot take an object, **you lose**.
- Similar to Chinese game “Jian-shizi” (“picking stones” 捡石子); European refs in 16th century

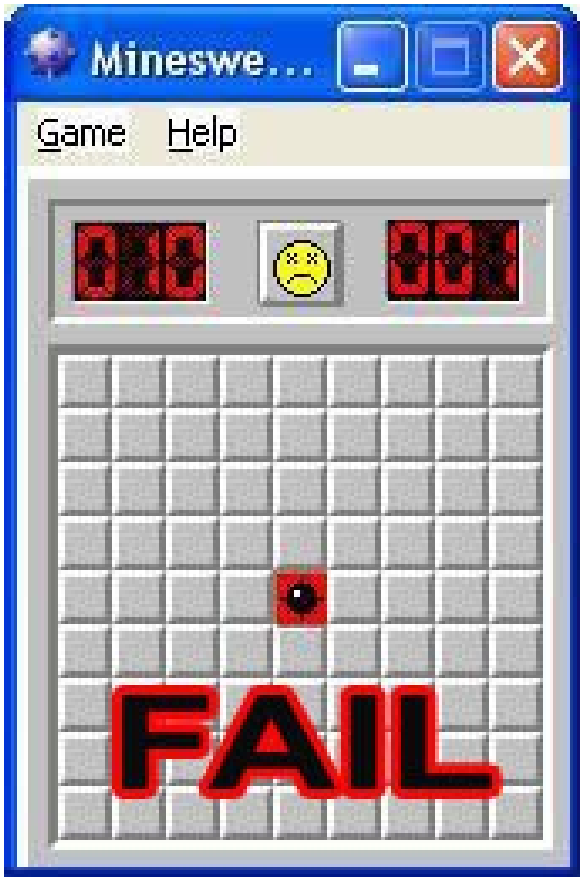
Example Nim

- Start with heaps of 3, 4 and 5 objects:
  - AAA, BBBB, CCCCC
- Here's a game:

-	AAA	BBBB	CCCCC	I take 2 from A
-	A	BBBB	CCCCC	You take 3 from C
-	A	BBBB	CC	I take 1 from B
-	A	BBB	CC	You take 1 from B
-	A	BB	CC	I take all of A
-		BB	CC	You take 1 from C
-		BB	C	I take 1 from B
-		B	C	You take all of C
-		B		I take all of B
-				**You lose!** (you cannot go)

Real-Life Nim Demo

- I will now play Nim against audience members.
- Starting Board: 3, 4, 7
  - AAA, BBBB, CCCCCCC
- You go first ...



The Rats of NIM

- How did I win every time?
  - Did I win every time? If not, pick on me mercilessly.
- Nim can be mathematically solved for any number of initial heaps and objects.
- There is an easy way to determine which player will win and what winning moves are available.
  - Essentially, a way to evaluate a board and determine its payoff / goodness / winning-ness.

Analysis

- You lose on the empty board.
- Working backwards, you also lose on two identical singleton heaps (A, B)
  - You take one, I take the other, you're left with the empty board.
- By **induction**, you lose on two identical heaps *of any size* (A^n, B^n)
  - You take *x* from heap A. I also take *x* from heap B, reducing it to a smaller instance of “two identical heaps”.

Analysis II

- On the other hand, you win on a board with a singleton heap (C).
  - You take C, leaving me with the empty board.
- You win with a single heap of any size (C^n).
- What if we add these insights together?
  - (AA, BB) is a loss for the current player
  - (C) is a win for the current player
  - (AA, BB, C) is what?

Analysis III

- (AA, BB, C) is a win for the current player.
  - You take C, leaving me with (AA, BB) – which is just as bad as leaving me with the empty board.
- When you take a turn, it becomes my turn
  - So leaving me with a board that would be a loss for you, if it were your turn
  - ... becomes a win for you!
- (AAA, BBB, C) – also a win for Player1.
- (AAAA, BBBB, CCCC) – also a win for Player1.

Generalize

- We want a way of evaluating nim heaps to see who is going to win (if you play optimally).
- Intuitively ...
- Two equal subparts cancel each other out
  - (AA, BB) is the same as the empty board (,)
- Win plus Loss is Win
  - (CC) is a win for me, (A,B) is a loss for me, (A,B,CC) is a win for me.
- What do we know that's kind of like addition but cancels out equal numbers?

The Trick!

- ***Exclusive Or***
  - XOR, $\oplus$, vector addition over GF(2), or ***nim-sum***
- If the XOR of all of the heaps is 0, you lose!
  - empty board = 0 = lose
  - (AAA,BBB) = $3 \oplus 3 = 0$ = lose
- Otherwise, goal is to leave opponent with a board that XORs to zero
  - (AAA,BBB,C) = $3 \oplus 3 \oplus 1 = 1$, so move to
    - (AAA,BBB) or (AA,BBB,C) or (AAA,BB,C)

Real-Life Nim Demo II

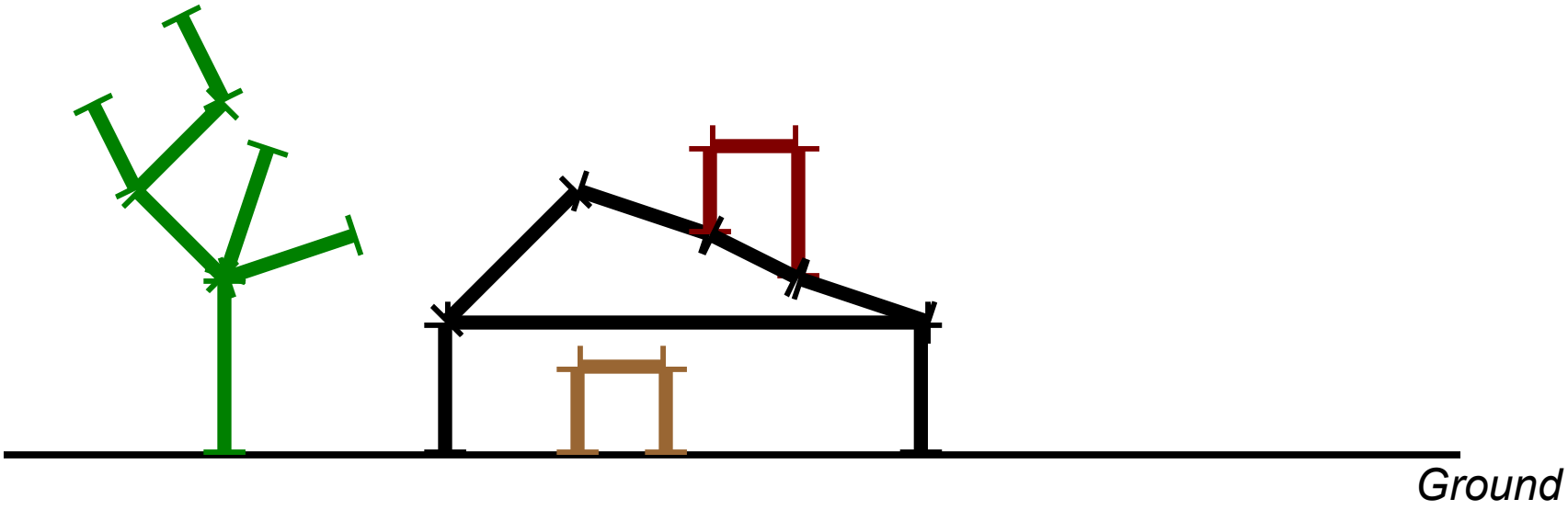
- I played Nim against audience members.
- Starting Board: 3, 4, 7
  - AAA, BBBB, CCCCCCC
- The nim sum is $3 \oplus 4 \oplus 7 = 0$
  - A loss for the first player!
- This time, I'll go first.
- You, the audience, must beat me. Muahaha!

Hackenbush

- ***Hackenbush*** is a two-player impartial game played on any configuration of line segments connected to one another by their endpoints and to a **ground**.
- On your turn, you "cut" **(erase) a line segment** of your choice. Line segments no longer connected to the ground are erased.
- If you cannot cut anything (empty board) **you lose**.

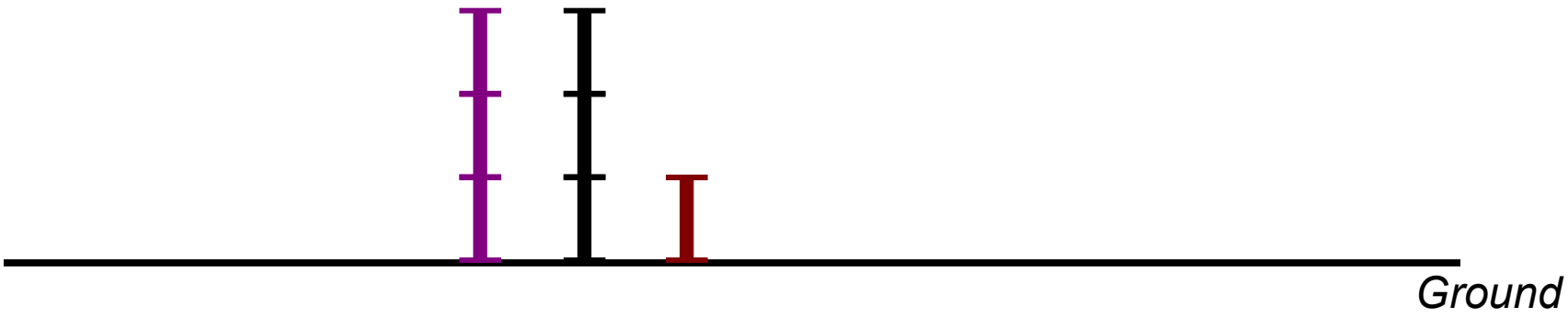
Hackenbush Example

- Each ⟝ is a line segment. Ignore color.
- Let's play! I'll go first.



Hackenbush Subsumes Nim

- Consider (AAA, BBB, C) = (3,3,1) in Nim
- Who wins this *completely unrelated* Hackenbush game?



A Thorny Problem

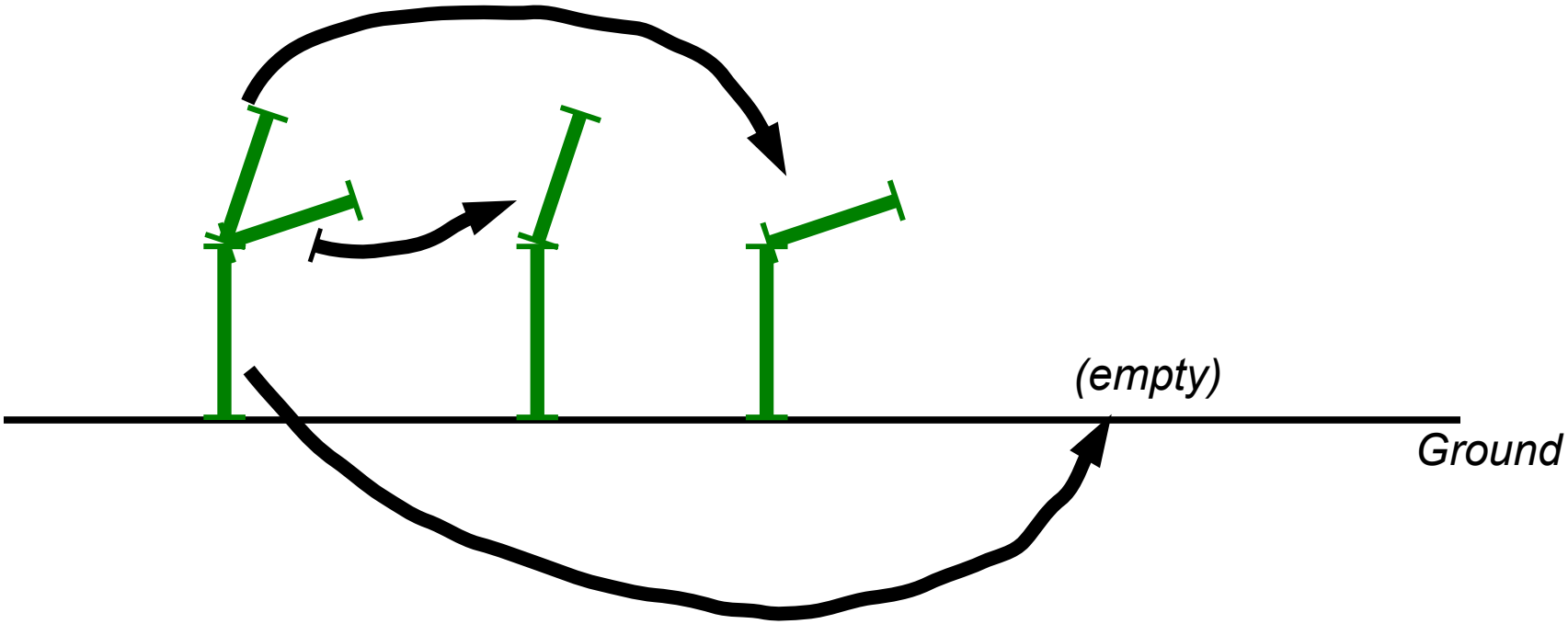
- What about that Hackenbush tree?
- What value (nim-sum) does it have? Who wins?

A Simple Twig

- Consider a simpler tree ...
- What moves do you have?

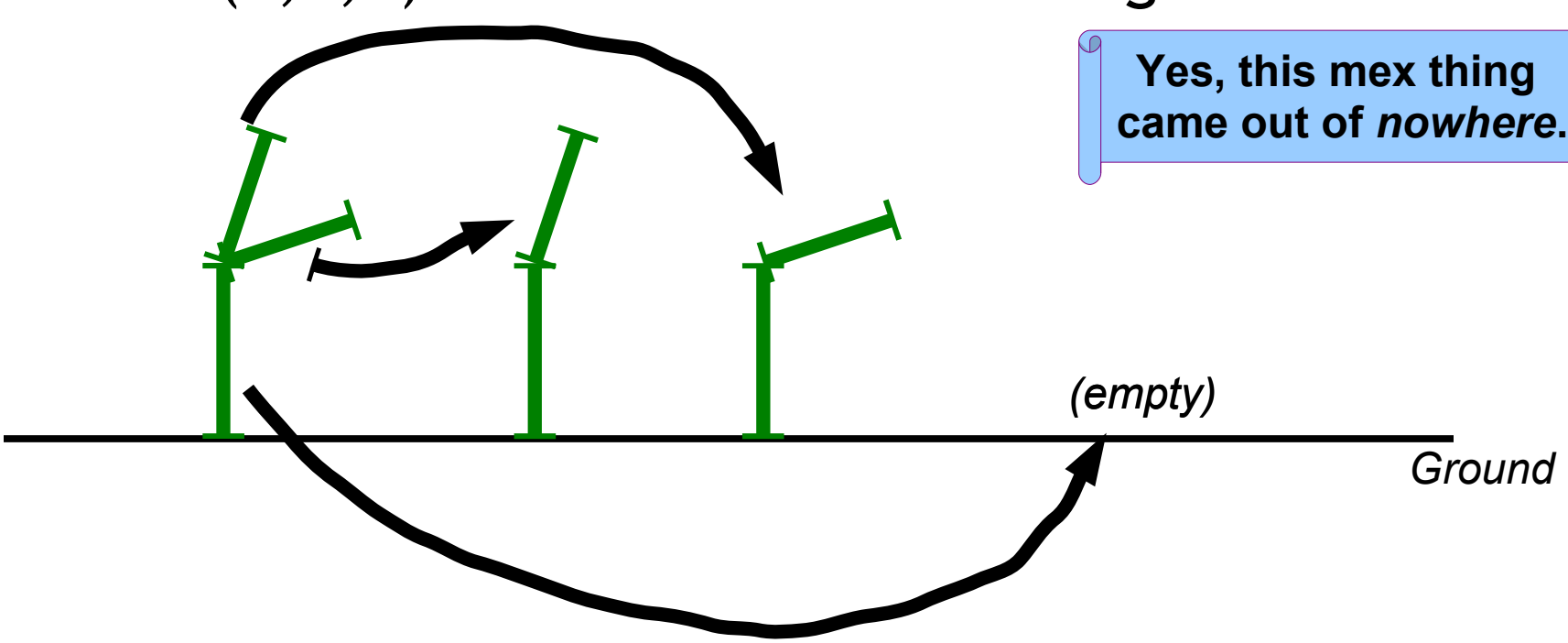
Twig Analysis

- Consider a simpler tree ...
- What moves do you have?



Maximum Excluded

- You can move to “2”, “2” or “0”.
- The ***minimal excluded*** of (2,2,0) is 1
  - mex(2,2,0) = 1 = value of that twig



Game Equivalence

- I've claimed that the twig has nim-sum 1
- How to prove that? When are games equal?
- We write $G \approx G'$ when **G is equivalent to G'**.
- **Lemma 1.** Iff $G \approx G'$ then for all H, $G \oplus H \approx G' \oplus H$.
- **Lemma 2.** $G \oplus G \approx 0$.
- **Lemma 3.** $G \approx G'$ if and only if $G \oplus G' \approx 0$.
  - Restated: $G \approx G'$ iff $G \oplus G'$ is a loss for Player 1.
  - If $G \approx G'$, then $G \oplus G \approx G \oplus G'$ (by Lemma 1).
  - Since $G \oplus G \approx 0$ (by Lemma 2), we have $0 \approx G \oplus G'$.

Trivia: Games

- In 1995 Klaus Teuber published this multiplayer game of trading, planning and resource management. The first German-style designer board game to obtain mainstream international popularity, it popularized notions such as simple rules, indirect player interaction, abstract components, strategy over luck, economy over military, and keeping all players in the game the whole time.

Trivia: Games

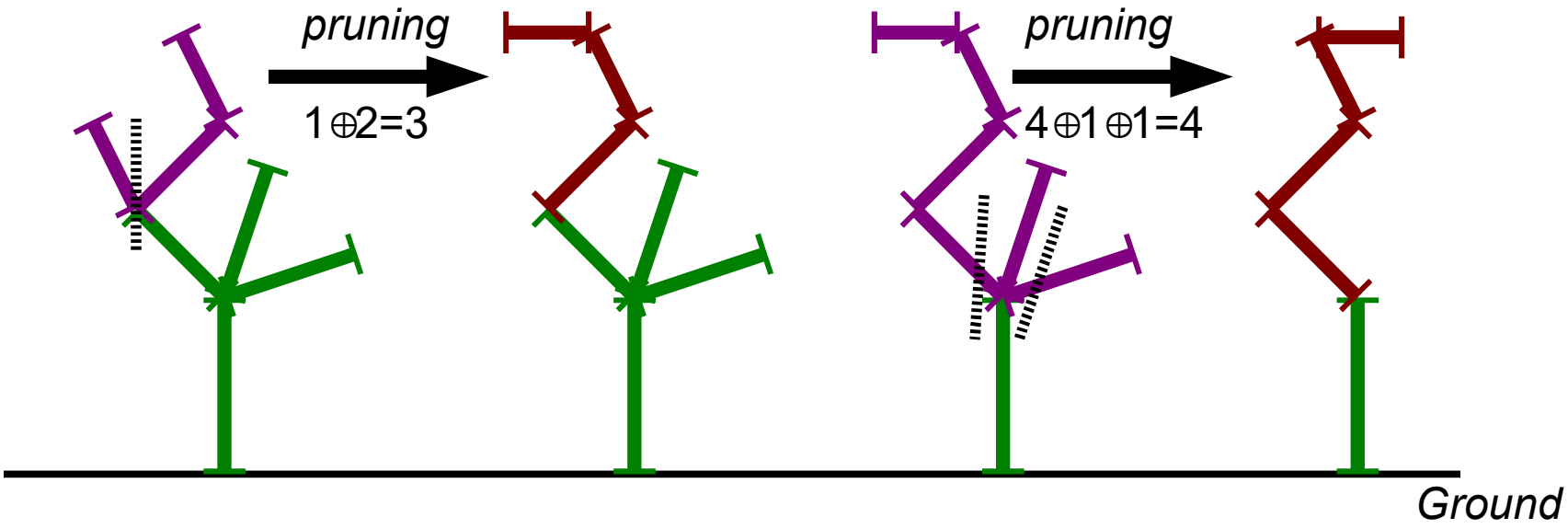
- In 1993 Richard Garfield earned his PhD in combinatorial mathematics from Penn, becoming a mathematics professor in Washington. In the same year, he released *this* game, which was inducted in to the Games Magazine hall of fame in 2003 as soon as the 10 year limit passed. Garfield was awarded a patent for “a novel method of game play and game components that in one embodiment are in the form of trading cards" including concepts such as changing the orientation of a game component to indicate use – a patent which is not enforced against other companies.

A Simple Twig

- So twig≈1 if twig⊕1≈0
- twig⊕1≈0 means twig⊕1 is a first-player loss
  - You go first; two trials against me to verify ...

Generalized Pruning

- Can replace any subtree above a single branch point with the XOR of those branches
  - Via similar game-equivalence argument



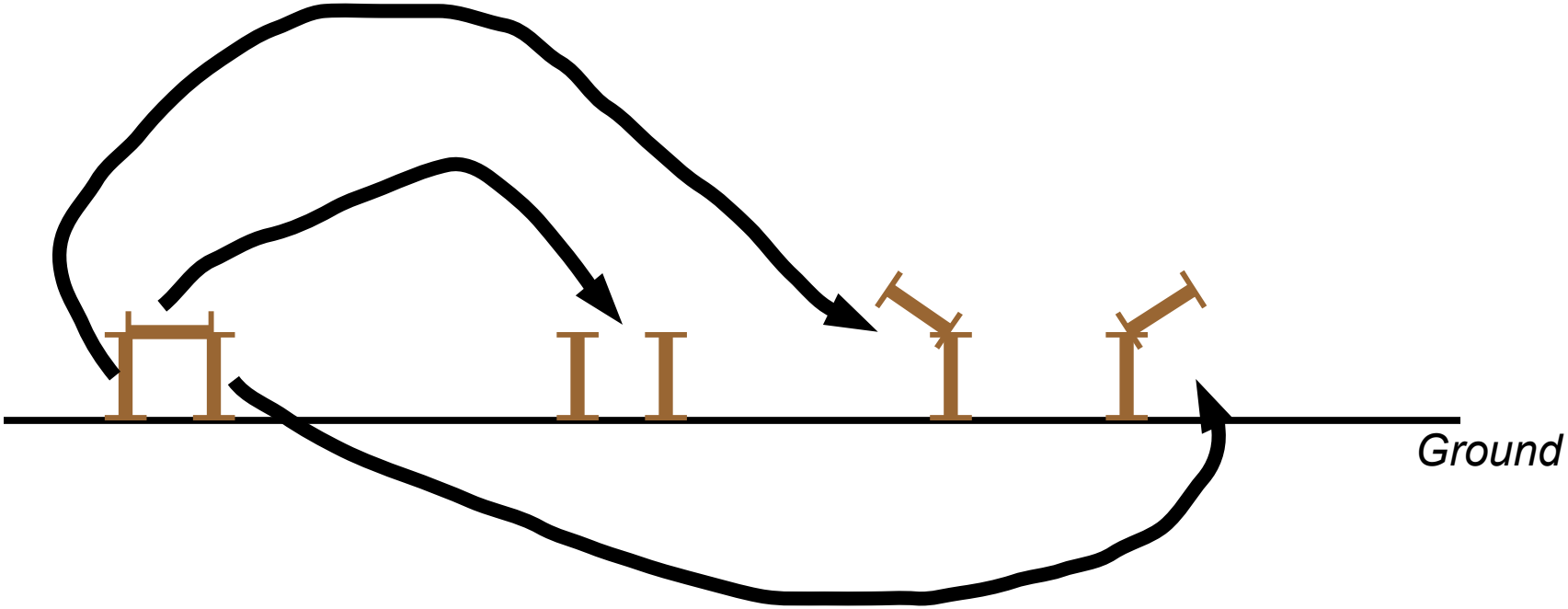
The whole tree has value "5".

Door Analysis

- What about cycles?
- What is the value (nim-sum) of this door?

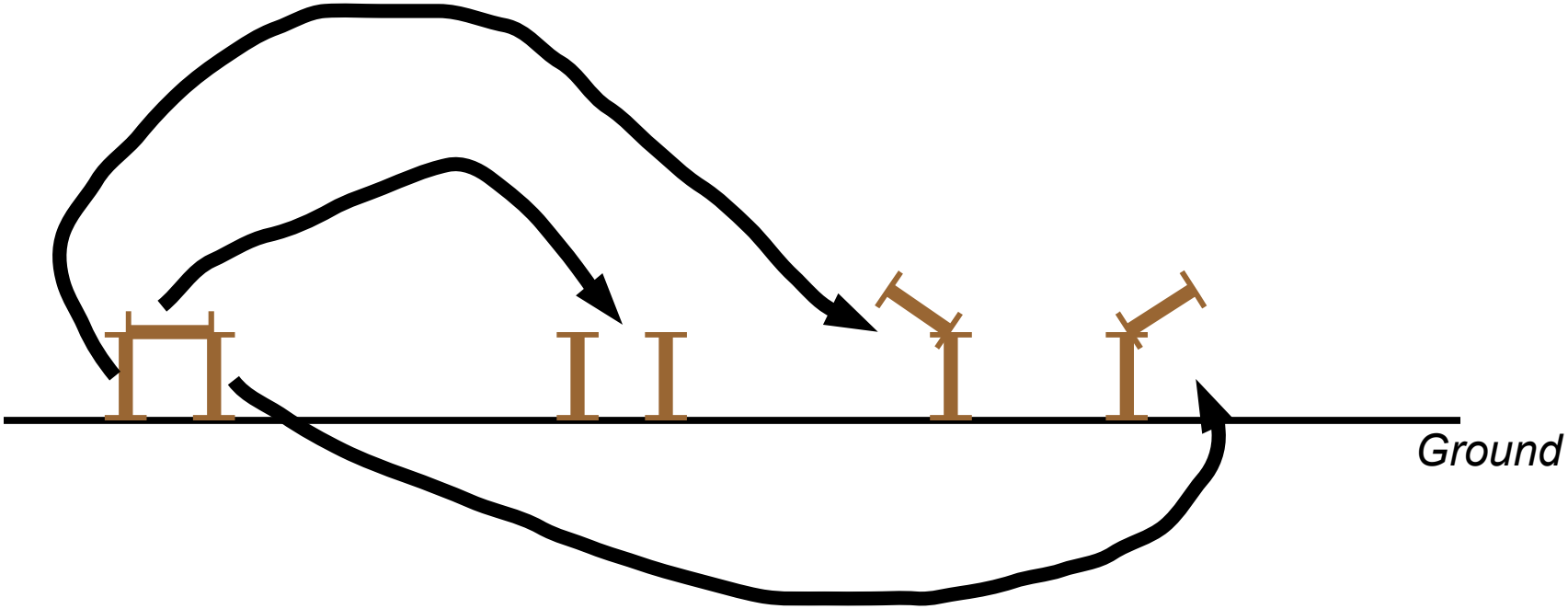
Door Analysis

- Well, what moves can you take from here?



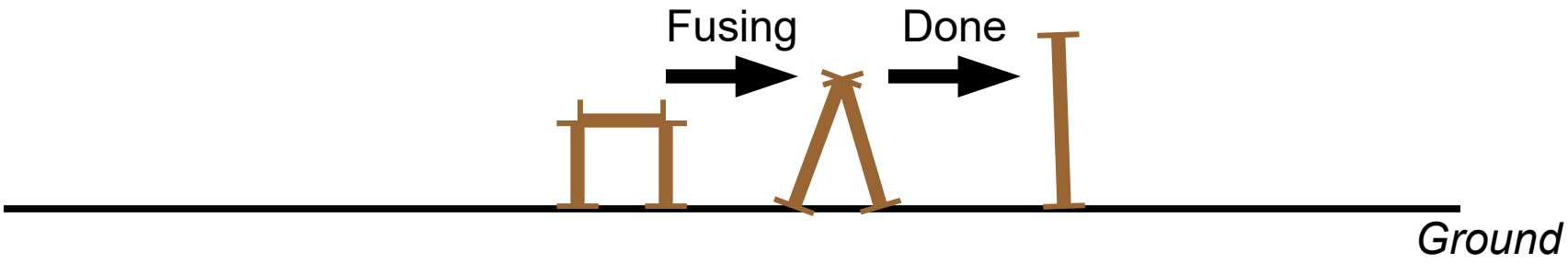
Door Analysis

- You can move to “0”, “2” or “2”.
  - mex(2,2,0) = 1 (recall: minimal excluded)
  - Value of door = 1



Fusion Principle

- We may replace any cycle with an equivalent subgraph where all of the non-ground vertices of that cycle are fused into one vertex and all of the ground vertices of that cycle are fused into another vertex.



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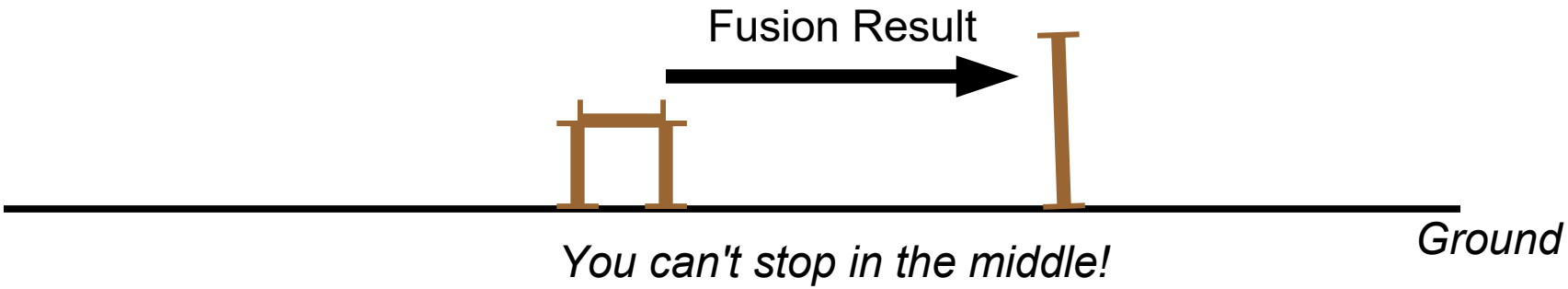
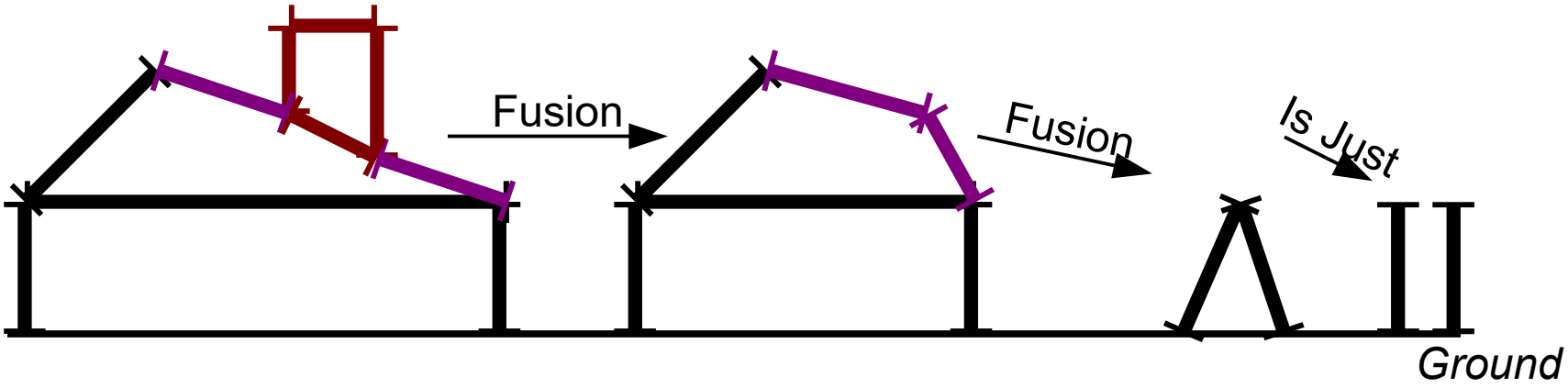


Cold Fusion

- Let's boil the house down to something simple!

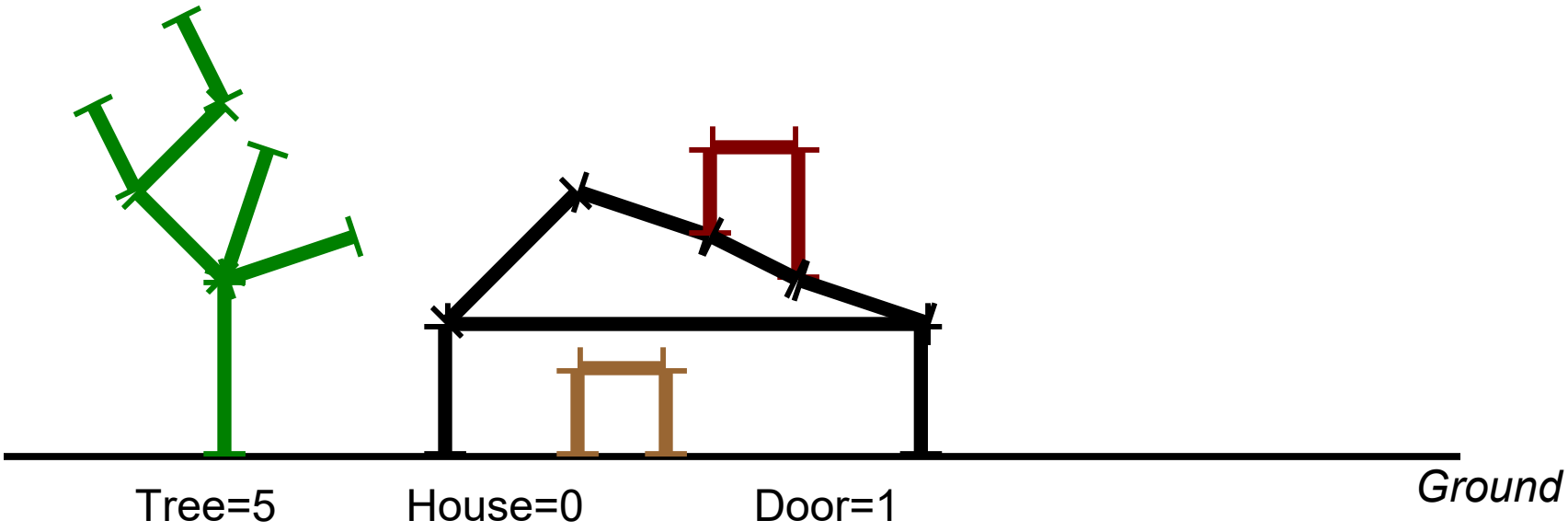


The whole house has value $1 \oplus 1 = 0$.

How would I check that?

Hackenbush Example

- This board has value $5 \oplus 0 \oplus 1 = 4$.
- You go first. Beat me. (Time permitting.)



Why Do We Care?

- ... about Nim and Hackenbush?
- **Theorem (Sprague-Grundy, '35-'39). *Every* impartial game is equivalent to a nim sum.**
- Proof: How?
  - Hint: what is the most important proof technique in computer science?

Why Do We Care?

- ... about Nim and Hackenbush?
- **Theorem (Sprague-Grundy, '35-'39). *Every* impartial game is equivalent to a nim sum.**
- Proof: By structural induction on the set (tree) representing the game.
  - Let $G = \{G_1, G_2, \ldots, G_k\}$. G_i is the game resulting if the current player takes move i.
  - By IH, each G_i is a nim sum, $G_i \approx N_i$.
  - Let $m = \text{mex}(N_1, N_2, \ldots, N_k)$. We'll show: $G \approx m$.

Sprague-Grundy Proof

- Let G' = $\{N_1, N_2, \ldots, N_k\}$. Then G ≈ G'. Why?
  - Player 1 makes a move i in G to $G_i \approx N_i$. Then

    Player 2 can make a move equivalent to N_i in G'.

    So the resulting game is a first-player loss, so by Lemma 3, G ≈ G'.
- To show G≈m, we'll show G+m is a first-player loss.
- We'll give an explicit strategy for the second player in the *equivalent* G'+m.

Sprague-Grundy Proof II

- To Show: P2 Wins in G'+m
- Suppose P1 moves in the m subpart to some option q with q<m. But since m was the minimal excluded number, P2 can move in G' to q as well.
- Suppose instead P1 moves in the G' subpart to the option N_i.
  - If N_i < m then P2 moves in the m subpart from m to N_i.
  - If N_i > m then P2, using the IH, moves to m in the G' subpart (which has been reduced to the smaller game N_i by P1's move).

    There must be such a move since N_i is the mex of options in N_i.

    If m<N_i were not a suboption, the mex would be m!
- Therefore, G'+m is a first-player loss. By Lemma 1, G+m is a first-player loss. So G≈m. QED.

Old-School CS Work

- Explore a new formalism
- Define properties and categories
- Investigate a few popular instances
- Show that many interesting instances are in fact in the ***same equivalence class***
- ... and thus that your results about that equivalence class have broad applicability.
- **Today: all impartial games are just nim!**