

Dependent
Type Systems
(saying what you are)

Data
Abstraction
(hiding what you are)

Review

- We studied a variety of type systems
- We repeatedly made the type system **more expressive** to enable the type checker to catch more errors
- But we have steered clear of **undecidable** systems
 - Thus there must still be **many errors that are not caught**
- Now we explore more **complex type systems** that bring type checking closer to **program verification**

One-Slide Summary

- A **dependent type** is a type that depends on the (exact) value of the expression. They are powerful but also hard to implement in practice. **Proof checking** (cf. Certificates in complexity class NP) is useful, decidable application.
- **Existential types** support abstraction and modularity, hiding the implementation while exporting the interface.

Proximate Cause

- Theorem proving is quite useful and can determine if things are true or false: “your file system can segfault” or “this formula is satisfiable”
- However, we also want theorem provers to provide **checkable** proofs to back up what they decide
- Fortunately, “**proof checking is equivalent to type checking in a dependent type system**”

A **dependent type** is a type that depends on a value.

Dependent Types

- Say that we have the functions
 - `zero : nat → vector` (creates vector of requested length)
 - `dotprod : vector → vector → real` (dot product)
- The types do not prevent using dotprod on **vectors of different length**

```
let v1 = zero 5 in
```

```
let v2 = zero 15 in
```

```
print (dotprod v1 v2)
```

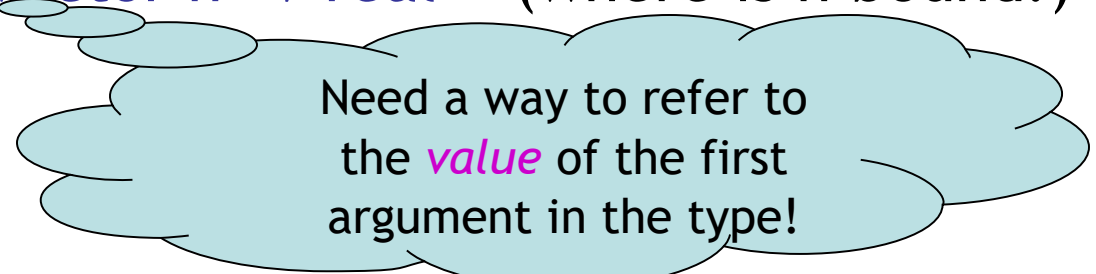
Dependent Types

- Say that we have the functions
 - $\text{zero} : \text{nat} \rightarrow \text{vector}$ (creates vector of requested length)
 - $\text{dotprod} : \text{vector} \rightarrow \text{vector} \rightarrow \text{real}$ (dot product)
- The types do not prevent using dotprod on **vectors of different length**
 - If they did, we could catch more bugs!
- Idea: Make “vector” a type family annotated by a natural number

“vector n” is the type of vectors of length n

$\text{dotprod} : \text{vector } n \rightarrow \text{vector } n \rightarrow \text{real}$ (where is n bound?)

$\text{zero} : \text{nat} \rightarrow \text{vector} ?$



Need a way to refer to the *value* of the first argument in the type!

Dependent Type Notation

- How to write the type of $\text{zero} : \text{nat} \rightarrow \text{vector}$?
- Given two sets A and B verify the isomorphism

$$A \rightarrow B \simeq \prod_{x \in A} B$$

- The latter is the cartesian product of B with itself as many times as there are elements in A
- Also written as $\prod x:A. B$ (x plays no role so far!)
- But now we can make B depend on x!
- **Definition:** $\prod x:A. B$ is the type of functions with argument in A and with the result type B (possibly depending on the value of the argument x in A)
 - We write “ $\text{zero} : \prod x:\text{nat}. \text{vector } x$ ”
 - Special case when $x \notin B$ we abbreviate as $A \rightarrow B$
 - We play “fast and loose” with the binding of \prod

Dependent Typing Rules

$$\frac{\Gamma, x : \tau_2 \vdash e : \tau}{\Gamma \vdash \lambda x : \tau_2. e : \Pi x : \tau_2. \tau} \quad \frac{\Gamma \vdash e_1 : \Pi x : \tau_2. \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : [e_2/x]\tau}$$

- Note that **expressions are now part of types**
- Have types like “vector 5” and “vector (2 + 3)”
- We need **type equivalence**

$$\frac{\Gamma \vdash e : \tau \quad \Gamma \vdash \tau \equiv \tau'}{\Gamma \vdash e : \tau'}$$
$$\frac{\Gamma \vdash e_1 \equiv e_2}{\Gamma \vdash \text{vector } e_1 \equiv \text{vector } e_2}$$

Dependent Types and Program Specifications

- **Types act as specifications**
- With dependent types we can specify *any property!*
- For example, define the following types:
 - “eq e” - the type of values equal to “e”.
Also named “sng e” (the singleton type)
 - “ge e” - the type of values larger or equal to “e”
 - “lt e” - the type of values smaller than “e”
 - “and $\tau_1 \tau_2$ ” - the type of values having both type τ_1 and τ_2
- Need appropriate typing rules for the new types
- The precondition for vector-accessing (cf. HW5)
 - read: $\prod n:\text{nat}.\text{vector } n \rightarrow (\text{and } (\text{ge } 0) (\text{lt } n)) \rightarrow \text{int}$
- The **type checker must do program verification**

Dependent Type Commentary

- Type checking with Π types can be *as hard as full program verification*
- Type equivalence can be **undecidable**
 - If types are dependent on expressions drawn from a powerful language (“powerful” = “arithmetic”)
 - Then even **type checking will be undecidable**
- Dependent types play an important role in the **formalization of logics**
 - Started with Per Martin-Lof
 - **Proof checking via type checking**
 - Proof-carrying code uses a dependent type checker to check proofs
 - There are program specification tools based on Π types

Dependent Sum Types

- We want to pack a vector with its length
 - $e = (n, v)$ where “ $v : \text{vector } n$ ”
 - The type of an element of a pair depends on the *value* of another element
 - This is another form of dependency
 - The type of e is “ $\text{nat} \times \text{vector} ?$ ”
- Given two sets A and B verify the isomorphism

$$A \times B \simeq \sum_{x \in A} B$$

- The latter is the *disjoint union* of B with itself as many times as there are elements in A
- Also written as $\sum x:A. B$ (x here plays no role)
- But now we can make B depend on x !

Dependent Sum Types

- Definition: $\Sigma x:A.B$ is the type of **pairs** with first element of type A and second element of type B (*possibly depending on the value of first element x*)
 - Now we can write $e : \Sigma x:\text{nat}. \text{vector } x$
- Old functions that compute the length of a vector
 - $\text{vlength} : \Pi n:\text{nat}. \text{vector } n \rightarrow \text{nat}$
 - (the result is not constrained)
 - $\text{length} : \Pi n:\text{nat}. \text{vector } n \rightarrow \text{sng } n$
 - “sng n” is a dependent type that contains only n
 - called the singleton type (recall from 3 slides ago ...)
- What if the vector is packed with its length?
 - $\text{pvlength} : \Sigma n:\text{nat}. \text{vector } n \rightarrow \text{nat}$
 - $\text{pslength} : \Sigma n:\text{nat}. \text{vector } n \rightarrow \text{sng } n$

Dependent Sum Types

Static Semantics

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : [e_1/x]\tau_2}{\Gamma \vdash (e_1, e_2) : \Sigma x : \tau_1. \tau_2}$$

$$\frac{\Gamma \vdash e : \Sigma x : \tau_1. \tau_2}{\Gamma \vdash \text{snd } e : [\text{fst } e/x]\tau_2}$$

- Note how this rule **reduces to the usual rules** for tuples when there is no dependency
- The evaluation rules are **unchanged**

Weimeric Commentary

- Dependant types seem obscure: why care?
- Grand Unified Theory
 - Type Checking = Verification (= Model Checking = Proof Checking = Abstract Interpretation ...)
- Example: CCured Project
 - Rumor has it this project was successful
 - The whole thing is dependant sum types
 - SEQ = (pointer + lower bound + upper bound)
 - FSEQ = (pointer + upper bound)
 - WILD = (pointer + lower bound + upper bound + rtti)

Proof Generation

- We want our theorem prover to **emit proofs**
 - **No need to trust the prover**
 - Can find bugs in the prover
 - Can be used for proof-carrying code
 - Can be used to extract invariants
 - Can be used to extract models (e.g., in SLAM)
- Implements the soundness argument
 - On every run, **a soundness proof is constructed**

Proof Representation

- Proofs are trees

- Leaves are hypotheses/axioms
- Internal nodes are inference rules

- Axiom: “true introduction”

- Constant: $truei : pf$
- pf is the type of proofs

$$\frac{}{\vdash true} \text{truei}$$

- Inference: “conjunction introduction”

- Constant: $andi : pf \rightarrow pf \rightarrow pf$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \text{andi}$$

- Inference: “conjunction elimination”

- Constant: $andel : pf \rightarrow Pf$

$$\frac{\vdash A \wedge B}{\vdash A} \text{andel}$$

- Problem:

- “ $andel \ truei : pf$ ” but does not represent a valid proof
- Need a more powerful *type system that checks content*

Proofs and Dependent Types

- Make **pf** a family of types indexed by formulas
 - $f : \text{Type}$ (type of encodings of formulas)
 - $e : \text{Type}$ (type of encodings of expressions)
 - $\text{pf} : f \rightarrow \text{Type}$ (the type of proofs indexed by formulas: it is a proof *that f is true*)
- Examples:
 - $\text{true} : f$
 - $\text{and} : f \rightarrow f \rightarrow f$
 - $\text{truei} : \text{pf true}$
 - $\text{andi} : \text{pf } A \rightarrow \text{pf } B \rightarrow \text{pf (and } A \ B)$
 - **$\text{andi} : \Pi A:f. \Pi B:f. \text{pf } A \rightarrow \text{pf } B \rightarrow \text{pf (and } A \ B)$**
 - ($\Pi A:f.X$ means “forall A of type f , dependent type X uses value A ”)

Proof Checking

- Validate proof trees by **type-checking** them
- Given a proof tree X claiming to prove $A \wedge B$
- Must check $X : \text{pf}$ (and $A \wedge B$)
- We use “**expression tree equality**”, so
 - andel (andi “ $1+2=3$ ” “ $x=y$ ”) does ***not*** have type **pf** ($3=3$)
 - This is already a proof system! If the proof-supplier wants to use the fact that $1+2=3 \Leftrightarrow 3=3$, she can **include a proof of it** somewhere!
- Thus **Type Checking = Proof Checking**
 - And it’s quite easily ***decidable!*** \square

Types for Data Abstraction

What's inside the implementation?
We don't know!

QUESTIONS NOT EVEN 5+ YEARS OF GRAD SCHOOL WILL HELP YOU ANSWER



PHD IN ENVIRONMENTAL ENGINEERING



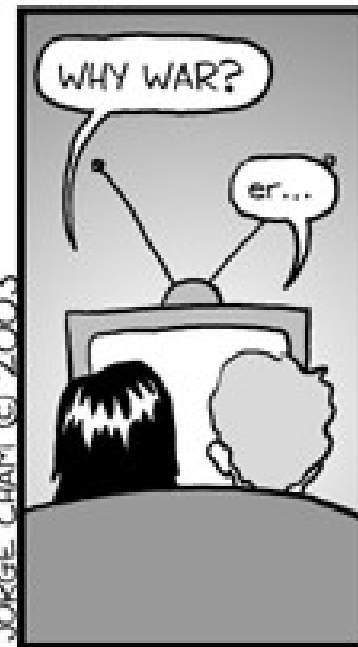
PHD IN PHYSICS



PHD IN BIOLOGY



PHD IN MECHANICAL ENGINEERING



PHD IN POLITICAL SCIENCE

Q: Games (540 / 842)

- This seminal 1991 turn-based strategy computer game by Sid Meier of Microprose spawned an entire genre about micromanaging exploration, expansion and conflict.

Q: Yanqi Wang

- This counting frame was used as early as 2500 BCE in Sumeria (before Arabic numerals and “zero”!) to aid with arithmetic. The Chinese *suanpan* 算盤 (top) and Japanese *soroban* そろばん (middle), and Korean *jupan* 주판 are examples.



Q: Books (754 / 842)

- Name the factory owner, the workers, and the newly-developed form of unending suckable candy in the 1964 children's book that features the title character finding a golden ticket and visiting the title chocolate factory.

Turing Award Winners

- This Ron-bearing golden trio is known best not for defeating Voldemort but for making one of the first practical public-key cryptosystems. A user publishes a public key based on two large, secret prime numbers. Anyone can use the public key to encrypt a message, but (ideally) only someone who knows the two secret prime numbers can decrypt it.

An **abstract data type** has a public name, a hidden representation, and operations to create, combine, and observe values of the abstraction.

Data Abstraction

- Ability to **hide (abstract) concrete implementation details**
- **Modularity** builds on data abstraction
- Improves program structure and minimizes dependencies
- One of the most influential developments of the 1970's
- Key element for much of the success of object orientation in the 1980's

Example of Abstraction

- Cartesian points (gotta love it!)
- Introduce the “**abstype**” language construct:
 - abstype** point **implements**
 - mk : real × real → point
 - xc : point → real
 - yc : point → real
 - is**
 - < point = real × real,
 - mk = λx. x,
 - xc = fst,
 - yc = snd >
- Shows a concrete implementation
- Allows the rest of the program to access the implementation *through an abstract interface*
- Only the interface need to be publicized
- Allows **separate compilation**

Data Abstraction

- It is useful to **separate the creation of the abstract type and its use** (newsflash ...)
- Extend the syntax ($t = \text{imp}$, $\sigma = \text{interface}$):
Terms ::= ... | $\langle t = \tau, e : \sigma \rangle$ | open e_a as t , $x : \sigma$ in e_b
Types ::= ... | $\exists t. \sigma$
- The expression $\langle t = \tau, e : \sigma \rangle$ takes the concrete implementation e and “**packs it**” as a value of an abstract type
 - Alternative notation: “pack e as $\exists t. \sigma$ with $t = \tau$ ”
 - “existential types” - used to model the stack, etc.
- The “**open**” expression allows e_b to access the abstract type expression e_a using the name x , the unknown type of the **concrete implementation** “ t ” and the **interface** σ

Example with Abstraction

- $C = \{mk = \lambda x.x, xc = fst, yc = snd\}$ is a *concrete implementation* of points as $real \times real$
- We want to hide the type of the representation σ is the following type:
 $\{ mk : real \times real \rightarrow point,$
 $xc : point \rightarrow real, yc : point \rightarrow real\}$
- Note that $C : [real \times real / point] \sigma$
- $A = \langle point = real \times real, C : \sigma \rangle$ is an expression of the abstract type $\exists point. \sigma$
- We want clients to access only the second component of A and just use the abstract name “point” for the first component:
 $open A as point, P : \sigma in \dots P.xc(P.mk(1.0, 2.0)) \dots$

Typing Rules for Existential Types

- We add the following typing rules:

$$\frac{\Gamma \vdash [\tau/t]e : [\tau/t]\sigma}{\Gamma \vdash \langle t = \tau, e : \sigma \rangle : \exists t.\sigma}$$

$$\frac{\Gamma \vdash e_a : \exists t.\sigma \quad \Gamma, t, p : \sigma \vdash e_b : \tau}{\Gamma \vdash \text{open } e_a \text{ as } t, p : \sigma \text{ in } e_b : \tau} \quad t \notin FV(\Gamma \cup \tau)$$

- The restriction in the rule for “open” ensures that **t does not escape its scope**

Evaluation Rules for Abstract Types

- We add a new form of value

$$v ::= \dots \mid \langle t = \tau, v : \sigma \rangle$$

- This is **just like** v but with some type decorations that make it have an existential type

$$\frac{e_a \Downarrow \langle t = \tau, v : \sigma \rangle \quad [v/x][\tau/t]e_b \Downarrow v'}{\text{open } e_a \text{ as } t, x : \sigma \text{ in } e_b \Downarrow v'}$$

- At the time e_b is evaluated, abstract-type variables are replaced with concrete values
 - If we ignore the type issues “open e_a as $t, x : \sigma$ in e_b ” is like “let $x : \sigma = e_a$ in e_b ”
 - Difference: e_b *cannot know statically* what is the concrete type of x so it cannot take advantage of it

Abstract Types as a Specification Mechanism

- Just like polymorphism, **existential types are mostly a type checking mechanism**
- A function of type $\forall t. t \text{ List} \rightarrow \text{int}$ does not know *statically* what is the type of the list elements. Therefore no operations are allowed on them
 - But it will have at run-time the actual value of t
 - “There are no type variables at run-time”
- Same goes for existentials
- These type mechanisms are a very powerful (and widely used!) **form of static checking**
 - Recall Wadler’s “Theorems for Free”

Data Abstraction and the Real World

- Example: **file descriptors**
- Solution 1:
 - Represent file descriptors as “**int**” and export the interface **{open:string→int, read:int→ data}**
- An untrusted client of the interface calls “read”
- How can we know that “read” is invoked with a file descriptor that was obtained from “open”? Anyone?

Data Abstraction and the Real World

- Example: **file descriptors**
- Solution 1:
 - Represent file descriptors as “**int**” and export the interface `{open:string→int, read:int→data}`
- An untrusted client of the interface calls “read”
- How can we know that “read” is invoked with a file descriptor that was obtained from “open”?
 - We must **keep track of all integers that represent file descriptors**
 - We design the interface such that all such integers are small integers and we can essentially keep a bitmap
 - This becomes expensive with more complex (e.g. pointer-based) representations

Data Abstraction, Static Checking

- Solution 2: Use the same representation but *export an abstraction* of it.
 - $\exists fd. \text{File}$ or
 - $\exists fd. \{ \text{open} : \text{string} \rightarrow fd, \text{read} : fd \rightarrow \text{data} \}$
 - A possible value:
 - $\text{Fd} = \langle fd = \text{int}, \{ \text{open} = \dots, \text{read} = \dots \} : \text{File} \rangle : \exists fd. \text{File}$
- Now the *untrusted* client e
 - $\text{open Fd as fd, x : File in } e$
- At run-time “ e ” can see that file descriptors are integers
 - But cannot cast 187 as a file descriptor.
 - Static checking with no run-time costs!
 - Catch: you must be able to type check e !

Modularity

- A module is a program fragment along with *visibility constraints*
- Visibility of functions and data
 - Specify the function interface but hide its implementation
- Visibility of type definitions
 - More complicated because the type might appear in specifications of the visible functions and data
 - Can use data abstraction to handle this
- A module is represented as a **type component** and an **implementation component**
 - $\langle t = \tau, e : \sigma \rangle$ (where t can occur in e and σ)
 - even though the specification (σ) refers to the implementation type we can still hide the latter
 - But there are problems with \exists ...

Problems with Existentialists

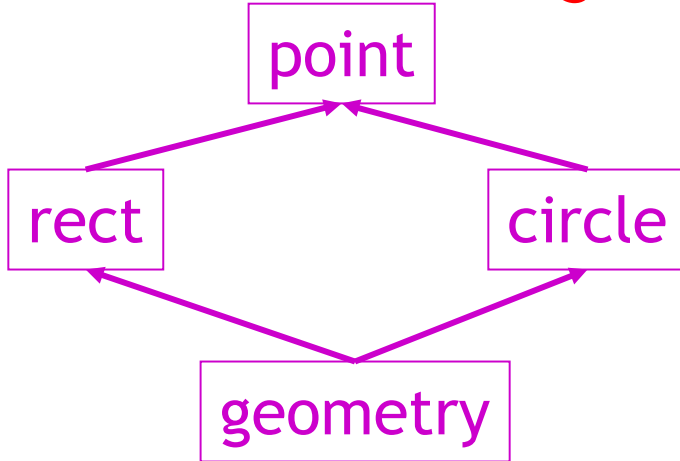
- Existentialist types
 - Assert that **truth is subjectivity**
 - **Oppose the rational tradition** and positivism
 - Are subject to an “**absurd**” universe
- Problems:
 - "In so far as Existentialism is a philosophical doctrine, it remains an idealistic doctrine: it hypothesizes specific historical conditions of human existence into ontological and metaphysical characteristics. **Existentialism thus becomes part of the very ideology which it attacks**, and its radicalism is illusory." (Herbert Marcuse, "Sartre's Existentialism", p. 161)

Problems with Existentials

- Existential types
 - Allow **representation (type) hiding**
 - Allow **separate compilation**. Need to know only the type of a module to compile its client
 - **First-class modules**. They can be selected at run-time. (cf. OO interface subtyping)
- Problems:
 - **Closed scope**. Must open an existential before using it!
 - Poor support for **module hierarchies**

Problems with Existentials (Cont.)

- There is an inherent tension between **handling modules in isolation** (good for **separate compilation**, interchangeability) and the need to **integrate them**



(the arrow means “depends on”)

- Solution 1: **open “point” at top level**
 - Inversion of program structure
 - The most basic construct has the widest scope

Give Up Abstraction?

- Solution 2: incorporate point in rect and circle
$$R = \langle \text{point} = \dots, \langle \text{rect} = \text{point} \times \text{point}, \dots \rangle \dots \rangle$$
$$C = \langle \text{point} = \dots, \langle \text{circle} = \text{point} \times \text{real}, \dots \rangle \dots \rangle$$
- When we open R and C we get *two distinct notions of point!*
 - And we will *not be able to combine them*
- Another option is to allow the type checker to see the representation type
 - and thus give up representation hiding

Strong Sums

- New way to open a package

Terms $e ::= \dots \mid \text{Ops}(e)$

Types $\tau ::= \dots \mid \Sigma t. \tau \mid \text{Typ}(e)$

- Use **Typ** and **Ops** to decompose the module
- Operationally, they are just like “fst” and “snd”
- $\Sigma t. \tau$ is the **dependent sum type**
- It is like $\exists t. \tau$ except we can look at the type

$$\frac{\Gamma \vdash e : \Sigma t. \tau}{\Gamma \vdash \text{Ops}(e) : \tau[\text{Typ}(e)/t]}$$

Modularity with Strong Sums

- Consider the R and C defined as before:

Pt = $\langle \text{point} = \text{real} \times \text{real}, \dots \rangle : \Sigma \text{point}. \tau_p$

R = $\langle \text{point} = \text{Typ}(\text{Pt}),$

$\langle \text{rect} = \text{point} \times \text{point}, \dots \rangle : \Sigma \text{rect}. \tau_R$

C = $\langle \text{point} = \text{Typ}(\text{Pt}),$

$\langle \text{circle} = \text{point} \times \text{real}, \dots \rangle : \Sigma \text{circle}. \tau_C$

- Since we use strong-sums the **type checker sees that the two point types are the same**

Modules with Strong Sums

- ML's module system is based on strong sums

Problems:

- **Poorer data abstraction**
- Expressions appear in types ($\text{Typ}(e)$)
 - Types might not be known until at run time
 - **Lost separate compilation**
 - Trouble if e has side-effects (but we can use a value restriction - e.g., "IntSet.t")
- **Second-class modules** (because of value restriction)
- We can combine existentials with strong sums
 - Translucent sums: partially visible

Homework

- HW6

- Need help?
Send email,
post on the
forum, etc.

