Estimation and Control with Quantized Sensors

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Abstract

This paper treats the problem of state observation and control for linear plants with quantized sensors. The treatment utilizes the method of stochastic linearization, whereby the nonlinearity is replaced by an equivalent gain that depends on the standard deviation of the signal at its input. An asymptotic observer is designed for the plant with nonlinear sensor, and its behavior is analyzed using stochastic linearization. The performance of the observer is predicted in both open- and closed-loop environments. Simulation results suggest that stochastic linearization may yield inaccurate predictions in some situations. An analysis of this phenomenon, using the Cramer expansion of the autocorrelation function of the output of the nonlinear sensor, is provided, which leads to an assessment of the accuracy of the method.

1. Introduction

1.1 Motivation

Quantized actuators and sensors often appear in modern control systems. Control systems with quantized actuators have been considered in a number of publications [1], [2], [3]. This work is devoted to quantized sensors.

There are several situations in which engineers have to deal with quantized sensors. In a system with distributed sensors and actuators, the output of the sensor might be sent to the controller via a digital communication channel, which has limits on the information (bit) rate [4], [5]. Hence, the output of the sensor is quantized before being sent through the communication channel. In the automobile industry, an inexpensive lambda Exhaust Gas Oxygen sensor might be used in air/fuel control of engines. This sensor has a relay type of input/output relation which is the simplest kind of quantization having only two quantization levels [6], [7]. The quantization of the output might come from the resolution of the sensor itself. For example, the wire-wound potentiometer sensor has a uniform quantized input/output relation [8]. Another example is the shaft encoder used to measure position and velocity in a motor, which inherently has quantized outputs [9], [10]. In the situations mentioned above, it is not possible to make the quantization step arbitrarily small since the quantization step was introduced by system constraints or sensor characteristics, which can not be changed. So, quantization of the sensor output needs to be considered.

1.2 Literature Survey

The focus of this paper is on using stochastic linearization to predict the behavior of systems with quantized sensors in both the open- and closed-loop environments, and on discussing the accuracy of this prediction by using the Cramer expansion of the autocorrelation function of the sensor output.

There are several ways to deal with quantized sensors proposed in the literature. A common way is to model the quantized measurement by a signal-plus-noise model [11], [12], [13]. In [14], the Bayesian approach is used to obtain maximum likelihood estimates for states of discrete-time dynamic systems with quantized measurements. The estimates could be calculated by a Kalman filter. However, the

Kalman filter must be re-run from the beginning every time a new measurement is made. In [15], an information theoretic approach is used to deal with state observation with quantized measurements in discrete-time feedback systems. The information of the current states of the system can be obtained by manipulating the input to the system. In [16], an approach assuming that the quantization step of the quantized sensor can be changed as the system evolves, which results in a hybrid system, is used to stabilize feedback systems with quantized measurements. In [17], the equilibrium point and stability analysis of a discrete-time linear feedback system with both quantized measurement and quantized actuation are considered. A graphical construction is used to predict the number of equilibrium points and a stability criterion was derived by using Hitz and Anderson's Theorem [18]. In [19], a qualitative control approach where the controller is viewed as a discrete event system is used to stabilize a feedback system with an arbitrary coarse quantized measurement.

To the author's best knowledge, no paper discusses the accuracy of using stochastic linearization to predict the behavior of an open- or closed-loop system with quantized measurements using the Cramer expansion of the autocorrelation of the sensor output.

2. Modeling and Problem Formulation

2.1 Model of the Open- and Closed-Loop Systems

The open-loop environment considered in this paper is depicted in Fig. 1, where P represents a stable, SISO, LTI plant, O represents an asymptotic state observer, $\psi(\bullet)$ represents the quantized sensor, r represents the reference input, which is assumed to be 0, v represents a stationary, zero mean, Gaussian white process disturbance, z represents the plant output, which is also the input to the quantized sensor, y represents the quantized sensor output, and $\hat{x}$ represents the estimated states. The estimation error is the main figure of merit to be considered.

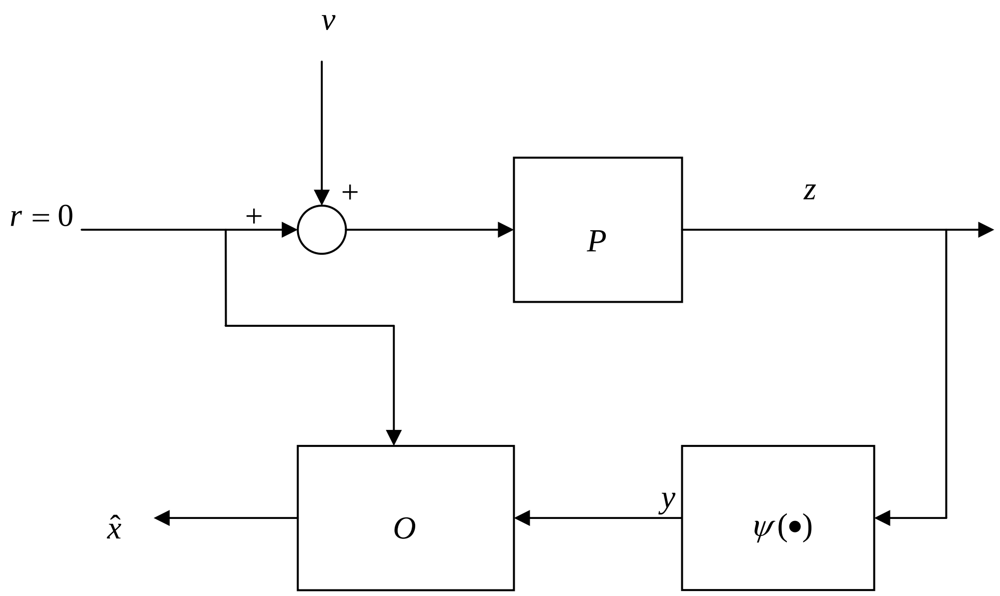

Figure 1: Open-Loop Environment Considered in This Paper

For the open loop system, the dynamics for the plant P written in state space is

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}, \qquad (2.1)$$

where (A,C) is observable, and the autocorrelation function of v is

$$\varphi_{vv}(\tau) = V^2\delta(\tau)\,. \qquad (2.2)$$

The dynamics of the state observer O will be discussed in further detail in section 4.1 and the input-output behavior of the nonlinear sensor, $\psi(\bullet)$, will be discussed later in this section.

The closed-loop environment considered in this paper is depicted in Fig. 2, where the blocks and variables in the figure have the same meanings as in Fig. 1. The estimated states from the observer are fed back to the plant input with feedback gain K. The objective of the control is disturbance rejection and the main figure of merit to be considered is the standard deviation of the plant output due to the zero mean, stationary, Gaussian white process disturbance v at the plant input.

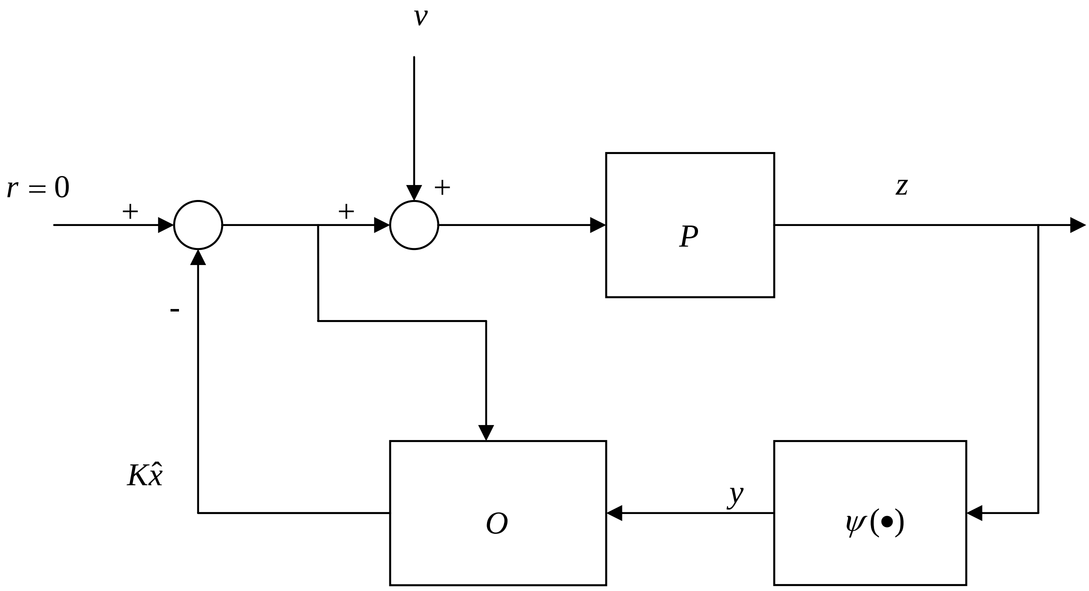

Figure 2: Closed-Loop Environment Considered in This Paper

For the closed-loop environment, the dynamics of the plant P written in state space is

$$\begin{cases} \dot{x} = Ax + Bv - BK\hat{x} \\ z = Cx \end{cases}, \tag{2.3}$$

where (A, C) is observable, (A, B) is controllable, and the autocorrelation function of v is given in (2.2). The dynamics of the state observer O will be discussed in detail in section 5.1. The quantized sensor is discussed in the following.

2.2 Model of Quantized Sensors

There are two types of quantization [1], [20] for quantized sensors. One is the midthread quantization and the other is the midriser quantization. The input-output relations of these two types of quantization are illustrated in Fig. 3, where z represents the input, y represents the output, and the quantization level, Δ, is assumed to be 1. The midthread quantization has a dead-zone at the origin whereas the midriser quantization has a discontinuity at the origin.

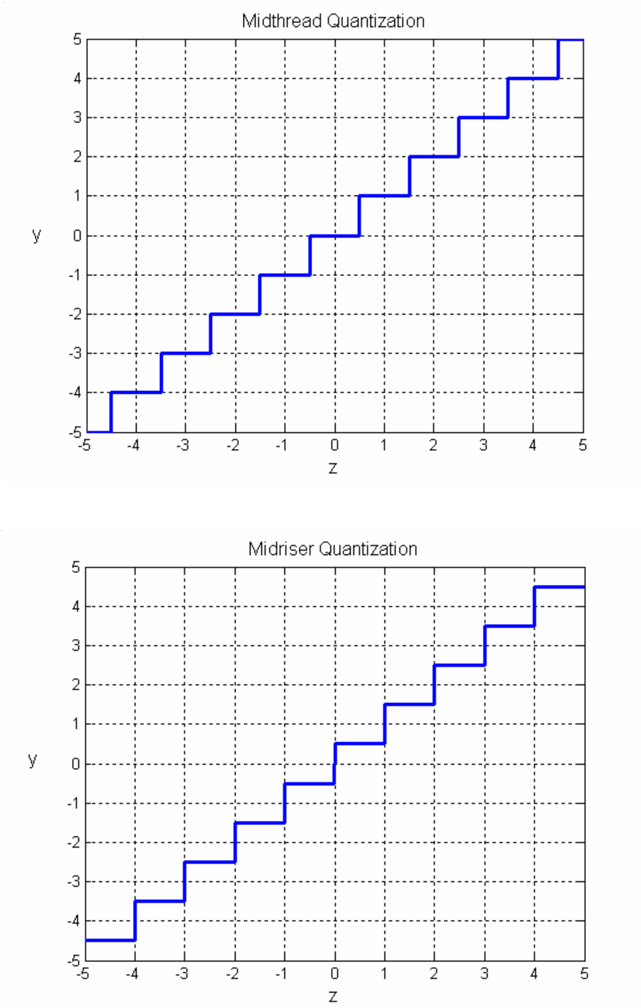

Figure 3: Input-Output Relations of Midthread and Midriser Quantizations

The input-output relation for a midthread quantization can be written as [16]

$$y = \psi(z) = \frac{\Delta}{2}\sum_{k=1}^{\infty}\left[sign(2z + \Delta(2k-1)) + sign(2z - \Delta(2k-1))\right], \qquad (2.4)$$

and if, in addition to quantization, the sensor is saturating with $2m+1$ quantization levels, the input-output relation becomes

$$y = \psi(z) = \frac{\Delta}{2}\sum_{k=1}^{m}\left[sign(2z + \Delta(2k-1)) + sign(2z - \Delta(2k-1))\right]. \qquad (2.5)$$

The input-output relation for a midriser quantization can be written as

$$y = \psi(z) = \frac{\Delta}{2}\left\{sign(z) + \sum_{k=1}^{\infty}\left[sign(2z + \Delta 2k) + sign(2z - \Delta 2k)\right]\right\}, \qquad (2.6)$$

and if the midriser quantization is saturating with $2m$ quantization levels, the input-output relation becomes

$$y = \psi(z) = \frac{\Delta}{2}\left\{ sign(z) + \sum_{k=1}^{m-1}\left[sign(2z + \Delta 2k) + sign(2z - \Delta 2k)\right]\right\}, \qquad (2.7)$$

where Δ in the above equations is the quantization step.

In a control system, a small plant output is desired. The plant output is also the input to the quantized sensor. If the sensor is a midthread quantization, it will give no information on the plant output when the plant output is small due to its dead-zone at the origin. On the other hand, if the sensor is a midriser quantization, it will still give information on the plant output when the plant output is small due to its discontinuity at the origin. For this reason, in this paper, only sensors with midriser quantization are considered.

2.3 Problem Statement

In order to choose parameters for the systems considered, information on the estimation error and the plant output must be known. Linear systems driven by zero mean, stationary, Gaussian white processes can be analyzed by solving a Lyapunov equation [21], but the systems considered in this paper have a quantized sensor and are therefore nonlinear. Analyzing such systems usually involves solving a Fokker-Plank equation which can be solved analytically in only a few special cases [22]. Therefore, stochastic linearization is used to substitute the quantized sensor by an equivalent gain, which depends on the standard deviation of the input to the quantized sensor. This is an approximation to the original nonlinear system by a linear system coupled with a nonlinear equation for calculating the equivalent gain. This approximation is used to predict the estimation error and plant output of the nonlinear system. The accuracy of this prediction is important for stochastic linearization to help select the parameters of the system.

This paper addresses the following:

- Predict the estimation error of the observer in the open-loop environment using stochastic linearization. This will be addressed in section 4.
- Predict the estimation error of the observer and the system output in the closed-loop environment using stochastic linearization. This will be addressed in section 5.
- Discuss the accuracy of using stochastic linearization for predicting the system in both the open- and closed-loop environments. This will be addressed in both sections 4 and 5 after a comparison between the results given by stochastic linearization and results from simulation.

3. Stochastic Linearization

3.1 The Method of Stochastic Linearization

Consider the block diagram in Fig. 4, where z represents a random process input to the nonlinearity, $\psi(\bullet)$, and y is the output. Stochastic linearization is a method to substitute this nonlinear element by an equivalent linear system, as depicted in Fig. 5, so that the difference between the output of the nonlinear element and the output of the equivalent linear system is minimized according to some criterion.

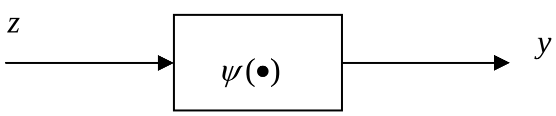

Figure 4: Input to the Nonlinear Sensor Is a Random Process

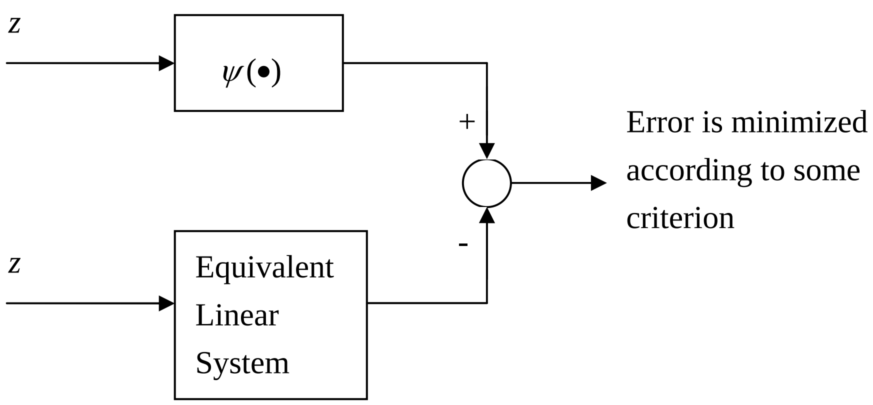

Figure 5: Concept of Substituting the Nonlinear Sensor with an Equivalent Linear System

There are many criteria proposed in the literature for deriving the equivalent linear system and the most often employed criterion is to minimize the mean squared error between the output of the nonlinear element and the output of the equivalent linear system [23]. If the input to the nonlinear element is assumed to be a zero mean, stationary, Gaussian random process and the nonlinearity is memoryless, it can be shown that the equivalent linear system is actually a linear gain (equivalent gain), which depends on the standard deviation of the input to the nonlinearity [24], [25]. The formula for calculating the equivalent gain, N, for a nonlinearity, $\psi(\bullet)$, is then,

$$N = E[\psi'(z)]. \tag{3.1}$$

Note that the midriser quantization, which is considered in this paper, exhibits discontinuities. By using delta functions for derivatives at the discontinuity points, the resulting value for the equivalent gain, N, will still give a minimum mean squared error between the output of the nonlinearity and the output of the equivalent gain.

Consider calculating the equivalent gain of a midriser quantization nonlinearity with quantization step Δ. The input is assumed to be a zero mean, stationary, Gaussian process with variance σ_z^2. The input-output relation of a midrise quantization is shown in Fig. 3, and this relation is given in equation (2.6), so

$$\psi'(z) = \Delta\left\{\delta(z) + 2\sum_{k=1}^{\infty}\left[\delta(2z + \Delta 2k) + \delta(2z - \Delta 2k)\right]\right\}, \tag{3.2}$$

and hence equivalent gain, N, is

$$N = E[\psi'(z)] = \frac{1}{\sqrt{2\pi}\sigma_z}\int_{-\infty}^{\infty}\left[\Delta\delta(z) + \Delta\sum_{k=1}^{\infty}(\delta(z + \Delta k) + \delta(z - \Delta k))\right]e^{-\frac{1}{2}\left(\frac{z^2}{\sigma_z^2}\right)}dz. \tag{3.3}$$

Utilizing the properties of the delta function yields,

$$N = \Delta\left(\frac{1}{\sqrt{2\pi}\sigma_z} + \frac{\sqrt{2}}{\sqrt{\pi}\sigma_z}\sum_{k=1}^{\infty}e^{-\frac{1}{2}\left(\frac{(k\Delta)^2}{\sigma_z^2}\right)}\right). \tag{3.4}$$

Note that if the midriser quantization is saturating with $2m$ quantization levels, the upper limit in the summation in equation (3.4) would be $m-1$ instead of ∞. Also note that is a function of the dimensionless parameter $\frac{\sigma_z}{\Delta}$. The function

$$Q(x) = \left(\frac{1}{\sqrt{2\pi}x} + \frac{\sqrt{2}}{\sqrt{\pi}\sigma_z x}\sum_{k=1}^{\infty}e^{-\frac{1}{2}\left(\frac{k^2}{x^2}\right)}\right) \tag{3.5}$$

is introduced so that

$$N = Q\left(\frac{\sigma_z}{\Delta}\right) \tag{3.6}$$

for the midriser quantization. A plot of the equivalent gain as a function of $\frac{\sigma_z}{\Delta}$ for non-saturating and saturating midriser quantizations is given in Fig. 6.

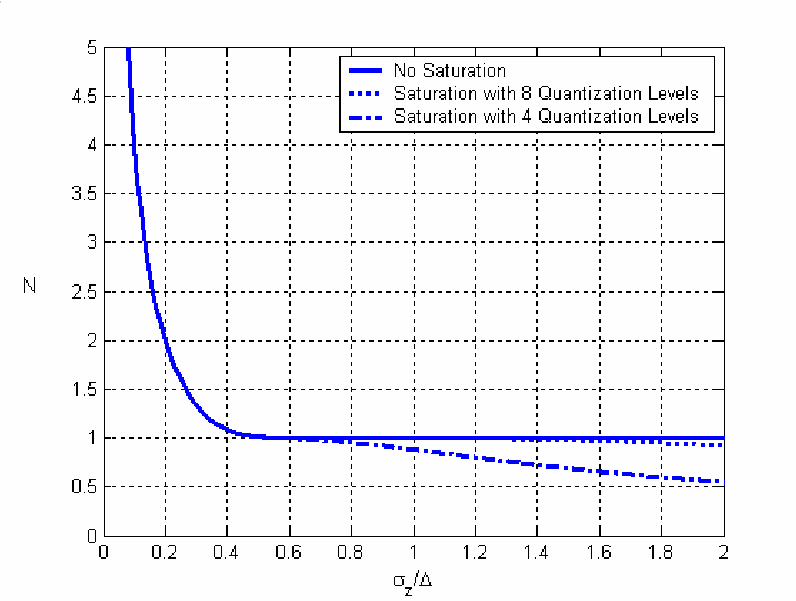

Figure 6: Equivalent Gain as a Function of $\frac{\sigma_z}{\Delta}$ for Non-Saturating and Saturating Midriser Quantizations

The equivalent gain is large when $\frac{\sigma_z}{\Delta}$ is small since the discontinuity at the origin in the input-output relation of midriser quantization will cause small inputs to take on large gains. The saturation affects the equivalent gain at larger $\frac{\sigma_z}{\Delta}$. This is reasonable since as $\frac{\sigma_z}{\Delta}$ gets larger the saturation characteristic of the saturating midriser quantization would be more actuated.

In this paper, it is assumed that the input to the nonlinearity is reasonably close to a zero mean, stationary, Gaussian process and hence equation (3.1) can be used to calculate the equivalent gain.

3.2 Accuracy of Stochastic Linearization

Substituting the nonlinearity with an equivalent gain generally does not give a good approximation of the probability density function or the autocorrelation function at the output of the nonlinearity [23]. This is because the criterion used was to minimize the mean squared error between the output of the equivalent gain and the output of the nonlinear element. To use stochastic linearization to predict the variance or standard deviation of a signal in the system, there should be an adequate amount of filtering from the output of the nonlinearity to the signal. This can be explained by expanding the output autocorrelation function of the nonlinearity as a power series expansion of the input autocorrelation function of the nonlinearity as is discussed in the following.

Suppose the output autocorrelation function of a memoryless nonlinearity with a zero mean, stationary, Gaussian random process as its input is to be calculated. Assume that the autocorrelation function of the input is $\varphi_{zz}(\tau)$. The output is also a stationary random process since the input is stationary and the nonlinearity is memoryless. By definition, the autocorrelation function of the output is

$$\varphi_{yy}(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(z_1)\psi(z_2)\frac{1}{2\pi\sigma_z^2\sqrt{1-\rho^2}}\exp\left[-\frac{z_1^2-2\rho z_1 z_2+z_2^2}{2\sigma_z^2(1-\rho^2)}\right]dz_1 dz_2\,, \tag{3.7}$$

where $\rho=\dfrac{\varphi_{zz}(\tau)}{\sigma_z^2}$.

Using Hermite polynomials [23] to expand the probability density function of the input would simplify the expression of the above double integral to

$$\varphi_{yy}(\tau)=\sum_{k=1,odd}^{\infty}a_k^2\left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2}\right)^k\,, \tag{3.8}$$

where $a_k=\dfrac{1}{\sqrt{2\pi k!\sigma_z^2}}\displaystyle\int_{-\infty}^{\infty}\psi(z)\exp\left(-\frac{z^2}{2\sigma_z^2}\right)H_k\left(\frac{z}{\sigma_z}\right)dz$,

and the $H_k(\bullet)$'s are the Hermite polynomials. The value of a_k depends on σ_z and the nonlinearity. Inspecting the coefficients, it is easy to verify that $N^2=\dfrac{a_1^2}{\sigma_z^2}$ and equation (3.8) can be written as

$$\varphi_{yy}(\tau)=N^2\varphi_{zz}(\tau)+\sum_{k=3,odd}^{\infty}a_k^2\left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2}\right)^k \tag{3.9}$$

This expression is referred to as the Cramer's expansion. Note that the first term, $N^2\varphi_{zz}(\tau)$, is what one would obtain for the output autocorrelation function if the nonlinearity were substituted by the equivalent gain, and the higher order terms starting from $k = 3$ are the terms that stochastic linearization did not account for. Typically, the Fourier transforms of these terms have wider bandwidth [23] and hence the power spectrum for this part of the output has wider bandwidth and a larger portion of the power is contained in the tail of this spectrum.

From the above analysis, the terms of the output that stochastic linearization did not account for would be attenuated and the prediction of the standard deviation would improve when adequate low pass filtering is introduced between the output of the nonlinearity and the point where the standard deviation is predicted. This agrees with arguments stating that the accuracy of stochastic linearization is improved when the linear part of the system become more low pass [23], [24], [25].

From equation (3.9), the power in the higher order terms is

$$P = \sum_{k=3,odd}^{\infty} a_k^2 \,. \tag{3.10}$$

Calculation shows that, in general, the coefficient a_3 is the largest coefficient, though not dominating, of the coefficients in the higher order terms. By calculating the value a_3^2, a measure of how large the power of the higher order terms might be can be assessed and a rough assessment of the accuracy of stochastic linearization in predicting the standard deviations of signals in the system may be made. The values of N as a function of σ_z and a_3^2 as a function of σ_z are plotted together for the saturation nonlinearity with slope 1 and saturation level 1, midthread quantization with quantization step 1, and midriser quantization with quantization step 1 in Fig. 7, Fig. 8 and Fig. 9 respectively.

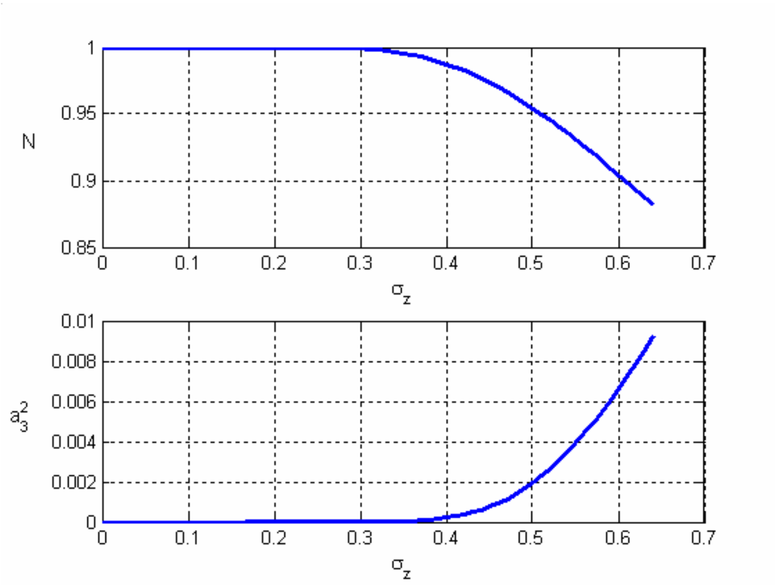

Figure 7: N and a_3^2 as a Function of σ_z for Saturation Nonlinearity with Slope 1 and Saturation Level 1.

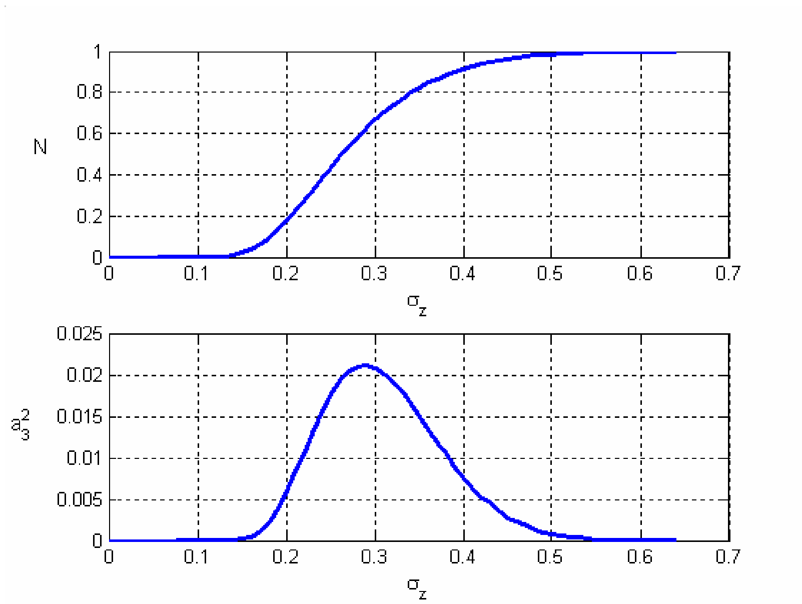

Figure 8: N and a_3^2 as a Function of σ_z for Midthread Quantization with Quantization Step $\Delta = 1$

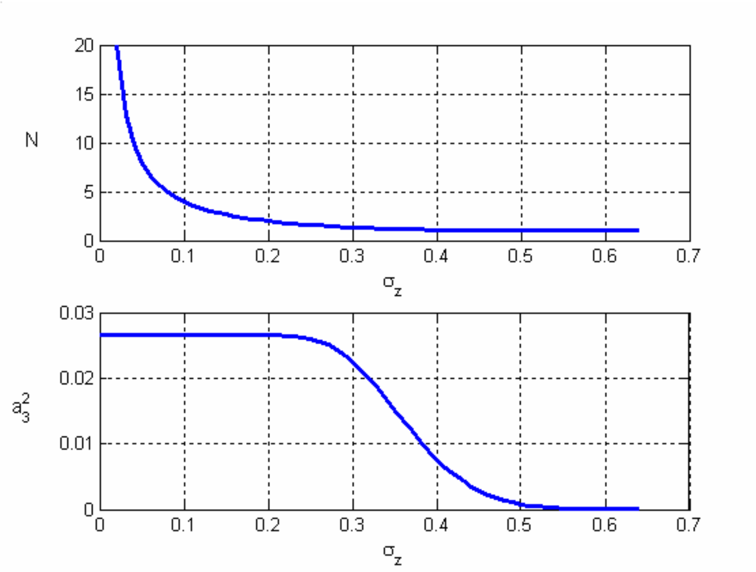

Figure 9: N and a_3^2 as a Function of σ_z for Midriser Quantization with Quantization Step $\Delta = 1$

From Figs. 7, 8 and 9, one can see that, for the saturation nonlinearity and midthread quantization, as σ_z approaches zero, a_3^2 also approaches zero, meaning that the higher order terms might approach zero and that the nonlinearity might not be actuated very much when the input power is small. On the other hand, for the midriser quantization, as σ_z approaches zero, a_3^2 approaches a constant, meaning that the higher order terms remain roughly constant and hence the nonlinearity is still actuated when the input power is small. From this observation, it can be anticipated that the accuracy of stochastic linearization in predicting standard deviations of signals in a system would not be very good if the nonlinearity is a midriser quantization and the input power is small. In section 5, an assessment of the accuracy of stochastic linearization using a_3^2 will be given.

4. Estimation

4.1 Structure of the Observer

In this section, state estimation in the open-loop environment, Fig. 1, is considered. Since there is no feedback to the plant P, the standard deviation of the plant output z, σ_z, can be calculated by solving the Lyapunov equation,

$$AR + RA^T + BV^2B^T = 0, \tag{4.1}$$

for R, and then by utilizing the formula,

$$\sigma_z = \sqrt{CRC^T}, \tag{4.2}$$

where the matrices $A, B,$ and C are the matrices governing the dynamics of the plant P in (2.1). The structure of the observer, O, is first discussed. Since the sensor used to measure the plant output has midriser quantization, it would seem reasonable to include this sensor nonlinearity into the observer. Recall from section 2.1 that the dynamics of the stable SISO LTI plant, P, is

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}, \tag{4.3}$$

that the output of the nonlinear sensor is

$$y = \psi(z), \tag{4.4}$$

and that
(A, C) is observable. An observer including this nonlinearity is

$$\dot{\hat{x}} = A\hat{x} + L(\psi(z) - \psi(\hat{z})), \hat{z} = C\hat{x}, \tag{4.5}$$

where $\hat{x}$ is the estimation of the plant states. Problems will occur in using this observer when the quantization becomes coarse.

For example, let the matrices in (4.3) be

$$A = \begin{bmatrix} -1.414 & -1 \\ 1 & 0 \end{bmatrix}, \tag{4.6}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{4.7}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \tag{4.8}$$

and let the autocorrelation function of the zero mean, stationary, Gaussian white process disturbance v be

$$\varphi_{vv}(\tau) = V^2\delta(\tau), \tag{4.9}$$

where $V = 0.5916$. Assume that the quantization step, Δ, of the sensor is 1. By solving (4.1) and using (4.2), the standard deviation of the plant output can be calculated as,

$$\sigma_z = 0.5916. \tag{4.10}$$

This will result in a rather coarse quantization since the quantization step size is assumed to be 1. Simulation of the open-loop environment using this observer is performed with the initial conditions of the plant states and their estimates at 0. A plot of the two states of the plant, x_1 and x_2, and their estimates, $\hat{x}_1$ and $\hat{x}_2$, as a function of time is in Fig. 10. The observer is not doing a good estimation since the estimates are smaller than the states by a factor of more than 100. The reason for this is the nature of the midriser quantization and is explained as follows. Whenever the observer starts sensing that there is a difference between $\psi(z)$ and $\psi(\hat{z})$, it will tend to drive $\hat{z}$ in the direction such that the magnitude of $\psi(z)-\psi(\hat{z})$ is smaller in magnitude. It turns out that because of the discontinuity at the origin of the midriser quantization, $\hat{z}$ can just move a little towards the desired direction to make this difference 0. Hence the estimates of the states of the plant will be very small. The fact that the output of the plant takes on values somewhere in the interval $(0,1)$ that on the average might be far away from 0 can not be captured. Some modification of the observer (4.3) is needed so that this deficiency can be corrected.

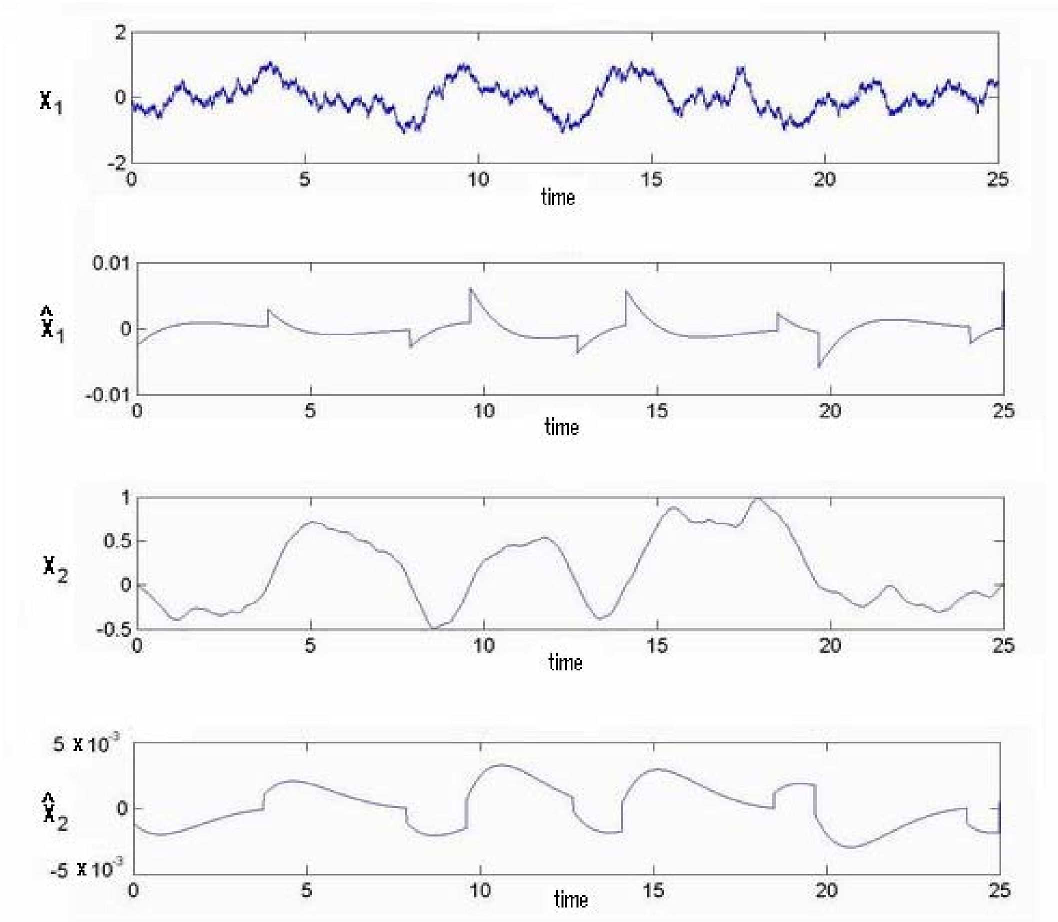

Figure 10: The Two States of the Plant and Their Estimates

Since, in the open-loop environment, by solving (4.1) and using (4.2), the standard deviation of the plant output, which is also the sensor input, can be calculated, the equivalent gain of the quantized sensor can then be calculated by (3.4). Using this equivalent gain, N, instead of the nonlinearity in the observer will yield a modified observer

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) \\ \hat{z} = C\hat{x} \end{cases}. \qquad (4.11)$$

The equivalent gain in (4.11) represents an approximated gain of the sensor driven by a signal with the given standard deviation. This modification will result in better estimates than the observer in (4.5) as verified in simulations. Using this modification in the simulation in the previous example, the two states of the plant, x_1 and x_2, and their estimates, $\hat{x}_1$ and $\hat{x}_2$, are plotted as a function of time in Fig. 11. In this paper, observer (4.11) will be used for block O in Figs. 1 and 2 throughout. Note that observer (4.11), though it gives better estimations than observer (4.5), still does not give a very good estimation of the states. The information on how the plant output actually varies is lost due to the discontinuity and step characteristics of the midriser quantization sensor.

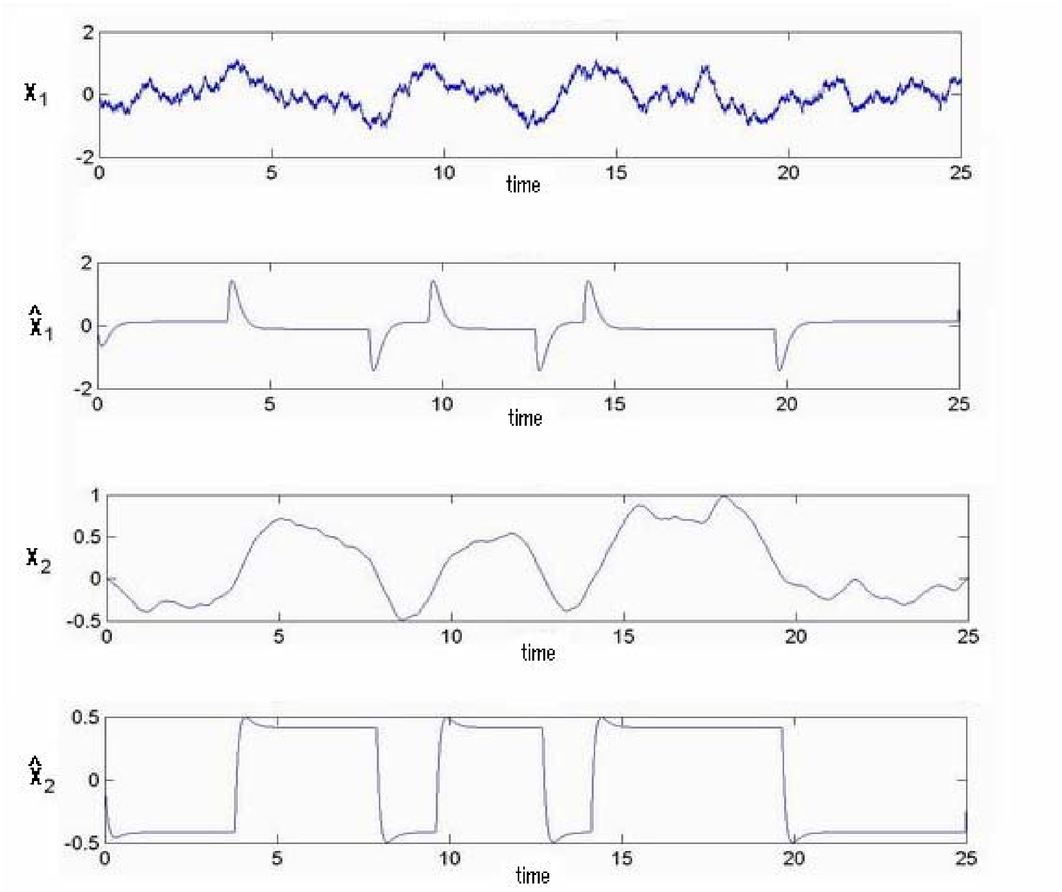

Figure 11: The Two States of the Plant and Their Estimates Using Observer (4.11)

4.2 Equations for Predicting the Estimation Error

It is desirable to know how the observer gain L in (4.11) affects the estimation error and how to select this parameter to give good estimates of the plant states. In this section, $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$, where $\tilde{x} = x - \hat{x}$, is used as a measure of the estimation error.

The dynamics of the plant with the observer is

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \\ \dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) \\ \hat{z} = C\hat{x} \\ N = Q\left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}, \tag{4.12}$$

where the autocorrelation function of v is given in (2.2) and σ_z is calculated from (4.1) and (4.2). By subtraction, the plant dynamics and the dynamics of $\tilde{x}$ can be written as

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \\ \dot{\tilde{x}} = (A - LNC)\tilde{x} + LNz - L\psi(z) + Bv \\ N = Q\left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}. \tag{4.13}$$

Since the system involves a quantized sensor and it is hard to calculate $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ analytically, stochastic linearization is used to substitute the sensor with its equivalent gain to obtain a linear approximation to this system so that a prediction of $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ can be calculated analytically. If the prediction of $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ from the stochastic linearized approximation is close to $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ of the original nonlinear system, stochastic linearization would then be helpful in investigating how the observer gain L affects the estimation error and how to select this parameter. The equations for approximating the nonlinear system are as follows.

Since the equivalent gain of the sensor was calculated to be N when designing the observer, the stochastic linearized approximation of (4.13) becomes,

$$\begin{cases} \dot{x}_{SL} = Ax_{SL} + Bv, z_{SL} = Cx_{SL} \\ \dot{\tilde{x}}_{SL} = (A - LNC)\tilde{x}_{SL} + Bv \\ N = Q\left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}, \tag{4.14}$$

where σ_z is calculated from (4.1) and (4.2) and the subscript SL reminds that the variables in (4.14) are obtained by the stochastic linearized approximation of the system. Assuming that $A - LNC$ is stable, the equations for calculating the prediction of $\sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$, which is $\sqrt{E[\tilde{x}_{SL}^T \tilde{x}_{SL}]}$, are[5]

$$\begin{cases} (A - LNC)R + R(A - LNC)^T + BV^2B^T = 0 \\ \sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]} = \sqrt{tr(R)} \end{cases}, \tag{4.15}$$

A comparison between the values of $\sqrt{E[\tilde{x}_{SL}^T \tilde{x}_{SL}]}$ and the values of $\sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$ will be presented in the next subsection.

4.3 Simulation Results

To see whether the stochastic linearized approximation of the system can give an accurate prediction of the estimation error, comparison is made between the prediction of estimation error, $\sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]}$ (calculated analytically from (4.15)), and the values of $\sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$ (from pure simulation of the nonlinear system (4.13)). In the comparison, it is assumed that the stable SISO LTI plant P is a first order dynamic system and that the matrices in (4.13) are

$$\begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases}, \tag{4.16}$$

and the autocorrelation function of v is assumed to be

$$\varphi_{vv}(\tau) = V^2\delta(\tau), \tag{4.17}$$

where $V = 1$. Three different values of L are used in the comparison, and for each L, various quantization step sizes are also used. A measure of the accuracy of predicting the estimation error is calculated by

$$estimation \quad prediction \quad error = \frac{\sigma_{\tilde{x}_{SL}} - \sigma_{\tilde{x}}}{\sigma_{\tilde{x}}} \times 100\%, \tag{4.18}$$

where

$$\sigma_{\tilde{x}_{SL}} = \sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]}, \tag{4.19}$$

which is calculated analytically by (4.14) and (4.15), and

$$\sigma_{\tilde{x}} = \sqrt{E\left[\tilde{x}^T \tilde{x}\right]}, \tag{4.20}$$

which is recorded from simulation data. Note that since the equivalent gain, N, is a function of $\frac{\sigma_z}{\Delta}$, the value of N can serve as a measure of how large the input to the sensor is compared to the quantization step size. The prediction error and the value of N are recorded in Tables 4.1, 4.2 and 4.3 for the three different values of L.

	$\Delta=1$	$\Delta=1.2$	$\Delta=1.4$	$\Delta=1.6$	$\Delta=1.8$	$\Delta=2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	4.74%	7.68%	11.35%	15.89%	20.34%	24.33%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

Table 4.1: Simulation Results for $L = 4$

	$\Delta=1$	$\Delta=1.2$	$\Delta=1.4$	$\Delta=1.6$	$\Delta=1.8$	$\Delta=2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	15.08%	22.46%	29.64%	36.45%	42.16%	46.72%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

Table 4.2: Simulation Results for $L = 9$

	$\Delta=1$	$\Delta=1.2$	$\Delta=1.4$	$\Delta=1.6$	$\Delta=1.8$	$\Delta=2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	34.04%	43.45%	50.19%	56.89%	61.43%	64.84%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

Table 4.3: Simulation Results for $L = 19$

One can see that for a fixed value of L, as the quantization step Δ gets larger, the estimation prediction error gets worse. This is reasonable since, as Δ gets larger, the system becomes more nonlinear and a linear approximation would not be very accurate. Also, from the tables, for a fixed Δ, as the parameter L gets larger, the estimation prediction error gets larger. This is a rather discouraging phenomenon since if $\sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]}$ is to be used to investigate how the parameter L affects the estimation error, the estimation prediction error should be kept small and roughly constant for varying L. Since the estimation prediction error is large in most of the situations considered, and, for a given quantization step, it is increasing for increasing L, the conclusion is that stochastic linearization does not work in predicting the estimation error in the open-loop environment. It would be interesting to find out the reason why stochastic linearized approximation does not work in predicting the estimation error and the next subsection is devoted to this.

4.4 Discussion of Simulation Results

The focus of this subsection is on explaining why, for a given quantization step Δ, the estimation prediction error gets worse as the observer gain L gets larger.

First, consider the dynamics of $\tilde{x}$ in equation (4.13), which is re-written as

$$\dot{\tilde{x}} = (A - LCN)\tilde{x} + Bv + L(Nz - \psi(z)). \tag{4.22}$$

Note that by the method of stochastic linearization, the nonlinear sensor is substituted by its equivalent gain, N, and this results in essentially ignoring the term $L(Nz - \psi(z))$ in (4.22), which gives

$$\dot{\tilde{x}}_{SL} = (A - LCN)\tilde{x}_{SL} + Bv. \tag{4.23}$$

Again, the subscript SL is used to denote that (4.23) is a stochastic linearization of (4.22). Since z is the output of a LTI plant driven by Gaussian white noise and hence Gaussian, from section 3.2, the autocorrelation of $y = \psi(z)$ is

$$\varphi_{yy}(\tau) = N^2 \varphi_{zz}(\tau) + \sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k, \tag{4.24}$$

where the first term, $N^2\varphi_{zz}(\tau)$, is just the autocorrelation function of Nz and that the remaining higher order terms have a wide bandwidth in its power spectrum. Note that the power spectrum of $Nz - \psi(z)$ is not exactly the same as that of $\sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$ since Nz is correlated with $\psi(z)$. However, the power spectrum of $Nz - \psi(z)$ still represents part of the spectrum of $\sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$.

Ignoring $L(Nz-\psi(z))$ will cause a term that has a wide bandwidth power spectrum unaccounted for.

Second, consider what happens when the observer gain L gets larger. As L gets larger, $E[\tilde{x}_{SL}^T\tilde{x}_{SL}]$, from equation (4.23), will be smaller. On the other hand, the term $L(Nz-\psi(z))$ in (4.22) will become more important and more of its power is passed through since larger L causes the dynamics of (4.22) to be less low pass. The term $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ from the true nonlinear dynamics will not be as small as stochastic linearization predicts and hence the estimation prediction error will become larger. For smaller L, the difference between $\sqrt{E\left[\tilde{x}_{SL}^T\tilde{x}_{SL}\right]}$ and $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ will become smaller because the dynamics in (4.22) is more low pass in this case and a larger portion of the higher order terms unaccounted for will be filtered out. Hence one could expect that the estimation prediction error would be larger if a larger observer gain were used.

Stochastic linearization does not give a good prediction of the estimation error in the open-loop environment but it is still interesting to see if it works in the closed-loop environment, since in [1] and [24], stochastic linearization works well in predicting the system output. This is considered in the next section.

5. Estimation and Control

5.1 Equations for Predicting the Plant Output and Estimation Error

In this section, estimation/control in the closed-loop environment, Fig. 2, is considered. In this environment the estimated states are feedback to the plant input to improve disturbance rejection. For the observer block O, the structure in (4.11) is used. However, the equivalent gain, N, of the nonlinear sensor can not be calculated as in section 4.1 since the estimated states are fed back to the plant input and the standard deviation of the input to the nonlinear sensor can not be calculated by simply solving (4.1) and using (4.2). The value of the equivalent gain, N, is now first assumed to be given. Later, N is computed to satisfy a set of equations simultaneously.

Since N is assumed to be given for the time being, the dynamics of the plant and observer is then,

$$\begin{cases} \dot{x} = Ax + Bu + Bv \\ z = Cx \\ \dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) + Bu \\ \hat{z} = C\hat{x} \\ u = -K\hat{x} \end{cases} . \tag{5.1}$$

Recall that the autocorrelation function of v is given in (2.2). By subtraction, the dynamics of the plant and $\tilde{x}$, ($\tilde{x} = x - \hat{x}$), can be written together as

$$\begin{cases} \dot{x} = (A - BK)x + BK\tilde{x} + Bv \\ z = Cx \\ \dot{\tilde{x}} = (A - LNC)\tilde{x} + Bv - L(\psi(z) - Nz) \end{cases} . \tag{5.2}$$

To select the feedback gain, K, and the observer gain, L, in order to improve the disturbance rejection of the closed-loop system, how σ_z changes under various K and L must be known. However, since the dynamics of (5.2) involves a nonlinear sensor and the value of N is unknown, σ_z can not be calculated from (5.2) analytically. Stochastic linearization is used to obtain an approximation of σ_z and to solve for N. Recall that it is assumed that the equivalent gain, N, of the nonlinear sensor is given, so the stochastic linearized approximation of (5.2) written in vector-matrix form is

$$\begin{cases} \begin{bmatrix} \dot{x}_{SL} \\ \dot{\tilde{x}}_{SL} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LNC \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \\ z_{SL} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} \end{cases} . \tag{5.3}$$

To simplify notations, let

$$\tilde{A} = \begin{bmatrix} A - BK & BK \\ 0 & A - LNC \end{bmatrix}, \tag{5.4}$$

$$\tilde{B} = \begin{bmatrix} B \\ B \end{bmatrix},$$

and

$$\tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{5.5}$$

One must keep in mind that N in the above equations must satisfy

$$N = Q\left(\frac{\sigma_z}{\Delta}\right), \tag{5.6}$$

where σ_z is the standard deviation of the plant output, which is also the sensor input. An assumption is made here that (5.3) will give a reasonable approximation to the standard deviation of the plant output so that (5.6) can be re-written as

$$N = Q(\frac{\sigma_{z_{SL}}}{\Delta}), \tag{5.7}$$

where $\sigma_{z_{SL}}$ satisfies

$$\begin{cases} \tilde{A}\tilde{R} + \tilde{R}\tilde{A}^T + \tilde{B}\tilde{V}^2\tilde{B}^T = 0 \\ \sigma_{z_{SL}} = \sqrt{\tilde{C}\tilde{R}\tilde{C}^T} \end{cases}, \tag{5.8}$$

assuming that $\tilde{A}$ is stable. From (5.4), $\tilde{A}$ depends on N, thus $\tilde{R}$ and $\sigma_{z_{SL}}$ also depend on N. Hence, equations (5.7) and (5.8) must be solved simultaneously to obtain the value for N.

To sum up, equation (5.3) with the parameter N satisfying (5.7) and (5.8) simultaneously is the stochastic linearized approximation of the true nonlinear dynamics, equation (5.2). The predicted standard deviation at the plant output by stochastic linearization is given by $\sigma_{z_{SL}}$. Note that the value of N in (5.2) is obtained by solving (5.7) and (5.8) simultaneously.

In the closed-loop environment, the emphasis is on the standard deviation at the plant output, but there is still some interest in seeing how the estimation error in the state observer O behaves for various feedback gains and observer gains. In this section,

$$\sigma_{\tilde{z}} = \sqrt{E\left[\tilde{z}^T\tilde{z}\right]}, \tag{5.9}$$

is used as a measure of the estimation error in the observer O, where

$$\tilde{z} = z - \hat{z} = Cx - C\hat{x}. \tag{5.10}$$

Let

$$\tilde{C}_2 = \begin{bmatrix} 0 & C \end{bmatrix}, \tag{5.11}$$

and assuming that $\tilde{A}$ is stable, the equations for using stochastic linearization to predict this estimation error is then

$$\begin{cases} \tilde{A}\tilde{R} + \tilde{R}\tilde{A}^T + \tilde{B}\tilde{V}^2\tilde{B}^T = 0 \\ \sigma_{\tilde{z}_{SL}} = \sqrt{\tilde{C}_2\tilde{R}\tilde{C}_2^T} \end{cases}. \tag{5.12}$$

The next section provides simulation results to see whether the stochastic linearized approximation of the nonlinear system can give good predictions of the estimation error and the plant output.

Note that there might be no solutions to N or multiple solutions to N that satisfy (5.7) and (5.8) simultaneously. However, in the simulations given in this paper, the system assumed was simple and problems regarding the existence or uniqueness of N did not occur.

5.2 Simulation Results

To see whether the stochastic linearized approximation of (5.2) can give accurate predictions of the plant output and estimation error, the value of $\sigma_{z_{SL}}$ obtained by calculating (5.8) analytically is compared with the value of σ_z obtained from pure simulation of (5.2), and the value of $\sigma_{\tilde{z}_{SL}}$ obtained by calculating (5.12) analytically is compared with the value of $\sigma_{\tilde{z}}$ obtained from pure simulation of (5.2).

The SISO LTI plant P is assumed to be a second order dynamic system and the matrices in equation (5.2) are

$$A = \begin{bmatrix} -1.414 & -1 \\ 1 & 0 \end{bmatrix}, \tag{5.13}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{5.14}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \tag{5.15}$$

and the autocorrelation function of v is

$$\varphi_{vv}(\tau) = V^2 \delta(\tau), \tag{5.16}$$

where $V = 1$.

First, three different observer gains L are used in the comparison, and for each L, various quantization step sizes are also used. The feedback gain K is kept constant. Two kinds of prediction errors are used to measure the accuracy of stochastic linearization. One is the plant output prediction error and the other is the estimation prediction error. They are defined as follows.

$$plant \quad output \quad prediction \quad error = \frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\% \tag{5.17}$$

and

$$estimation \quad prediction \quad error = \frac{\sigma_{\tilde{z}} - \sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}} \times 100\%\,. \tag{5.18}$$

These two values are recorded along with the equivalent gain, N, in the comparisons. The values of σ_z from (5.2) and $\sigma_{z_{SL}}$ from (5.3) are also recorded. These values are presented in Tables 5.1, 5.2, 5.3 for the three different values observer gains.

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
σ_z	0.6839	0.6852	0.6844	0.6880	0.6933	0.6997
$\sigma_{z_{SL}}$	0.6762	0.6762	0.6767	0.6888	0.6818	0.6856
$\frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	1.13%	1.32%	1.12%	1.34%	1.66%	2.02%
$\frac{\sigma_{\tilde{z}} - \sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}} \times 100\%$	7.37%	11.11%	15.54%	19.45%	22.81%	26.83%
N	1	1	1.01	1.05	1.11	1.19

Table 5.1: Simulation Results when $L = \begin{bmatrix} -1 \\ -1.414 \end{bmatrix}$ and $K = [0.5860 \quad 0]$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
σ_z	0.5183	0.5789	0.5789	0.5797	0.5805	0.5812
$\sigma_{z_{SL}}$	0.5727	0.5727	0.5727	0.5727	0.5728	0.5728
$\frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	0.96%	1.07%	1.07%	1.2%	1.34%	1.44%
$\frac{\sigma_{\tilde{z}} - \sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}} \times 100\%$	90.8%	92.4%	93.56%	94.42%	95.08%	95.61%
N	1	1.02	1.07	1.15	1.27	1.39

Table 5.2: Simulation Results when $L = \begin{bmatrix} 30.9903 \\ 12.7260 \end{bmatrix}$ and $K = [0.5860 \quad 0]$

	$\Delta=1$	$\Delta=1.2$	$\Delta=1.4$	$\Delta=1.6$	$\Delta=1.8$	$\Delta=2.0$
σ_z	0.5449	0.5454	0.5458	0.5468	0.5479	0.5484
$\sigma_{z_{SL}}$	0.5391	0.5391	0.5391	0.5391	0.5390	0.5390
$\frac{\sigma_z-\sigma_{z_{SL}}}{\sigma_z}\times 100\%$	1.07%	1.16%	1.24%	1.41%	1.62%	1.71%
$\frac{\sigma_{\tilde{z}}-\sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}}\times 100\%$	96.99%	97.52%	97.89%	98.17%	98.39%	98.56%
N	1	1.03	1.1	1.21	1.34	1.48

Table 5.3: Simulation Results when $L=\begin{bmatrix}160.9511\\26.8660\end{bmatrix}$ and $K=\begin{bmatrix}0.5860 & 0\end{bmatrix}$

One can see that the estimation prediction error is very large, but the plant output prediction error is very small. It seems that stochastic linearization can give a good prediction of the plant output despite inaccurate prediction of the estimation error. Since how the feedback gain K affects the plant output prediction error is also of interest, the next step is to see if the plant output prediction error is also small for different values of K.

Hence, for a constant L, the feedback gain K is increased. The results of the comparison are recorded in Table 5.4 in the case where the quantization step is 1.

	$K=\begin{bmatrix}0.5860 & 0\end{bmatrix}$	$K=\begin{bmatrix}10.586 & 35\end{bmatrix}$
σ_z	0.5475	0.1242
$\sigma_{z_{SL}}$	0.5391	0.0768
$\frac{\sigma_z-\sigma_{z_{SL}}}{\sigma_z}\times 100\%$	1.54%	38.17%
$\frac{\sigma_{\tilde{z}}-\sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}}\times 100\%$	96.97%	98.32%
N	1	5.19

Table 5.4: Simulation Results when $L=\begin{bmatrix}160.9511\\26.8660\end{bmatrix}$ and $\Delta=1$ but K Changes

From Table 5.4, the plant output prediction error increases significantly when K is increased. An observation is that $\sigma_{z_{SL}}$ is small in this case. Since the goal is to use the predictions from stochastic linearization to select L and K to improve disturbance rejection properties, care must be taken when the predicted output is small. This will be discussed in detail in the next subsection.

5.3 Discussion of Simulation Results

There are two observations from the data recorded in Tables 5.1, 5.2, 5.3 and 5.4 that need to be discussed. First, for a given quantization step size and feedback gain, K, as L becomes larger, the estimation prediction error becomes larger. Second, for a given quantization step size, and observer gain, L, as K becomes larger and $\sigma_{z_{SL}}$ becomes small, the plant output prediction error becomes larger. In the arguments below, it is assumed that the plant output is reasonable close to a Gaussian process, so that (3.9) in section 3.2 can be used.

The reasoning behind the explanation for the first observation is much the same as in section 4.4. Consider the dynamics of $\tilde{x}$ in (5.2), re-written as

$$\dot{\tilde{x}} = (A - LNC)\tilde{x} + Bv + L(Nz - \psi(z)), \quad (5.19)$$

and the dynamics of $\tilde{x}_{SL}$ from the stochastic linearized approximation (5.3)

$$\dot{\tilde{x}}_{SL} = (A - LNC)\tilde{x}_{SL} + Bv. \quad (5.20)$$

As in the open-loop environment, the $L(Nz - \psi(z))$ term is ignored in the approximation of (5.19), hence by the same reasoning in section 4.4, when L becomes larger, $\sigma_{\tilde{z}_{SL}} = \sqrt{E\left[\tilde{x}_{SL}^T C^T C \tilde{x}_{SL}\right]}$ will become smaller, but $\sigma_{\tilde{z}} = \sqrt{E\left[\tilde{x}^T C^T C \tilde{x}\right]}$ would not become as small due to the term, $L(Nz - \psi(z))$, ignored by stochastic linearization. And when L is small, more of the components in the term, $L(Nz - \psi(z))$, is filtered out by the dynamics and the approximation of $\sigma_{\tilde{z}}$ by $\sigma_{\tilde{z}_{SL}}$ will become better.

The second observation can be explained by the following argument. Recall that the method of stochastic linearization is to replace the nonlinear sensor in Fig. 2 by an equivalent gain, N. Consider the autocorrelation function of the sensor output given in (3.9). One can see that only the first term, $N^2\varphi_{zz}(\tau)$, in the autocorrelation function of the sensor output is accounted for, since $N^2\varphi_{zz}(\tau)$ is the autocorrelation function of Nz. The higher order terms starting from $k=3$ in (3.9) are ignored. Hence, a term, n, having a wide power spectrum and power of $\sum_{k=3,odd}^{\infty} a_k^2$ must be added to more accurately predict the standard deviation of the plant output, which is the square root of the power of the plant output. This term is uncorrelated with Nz. This is depicted in Fig. 12.

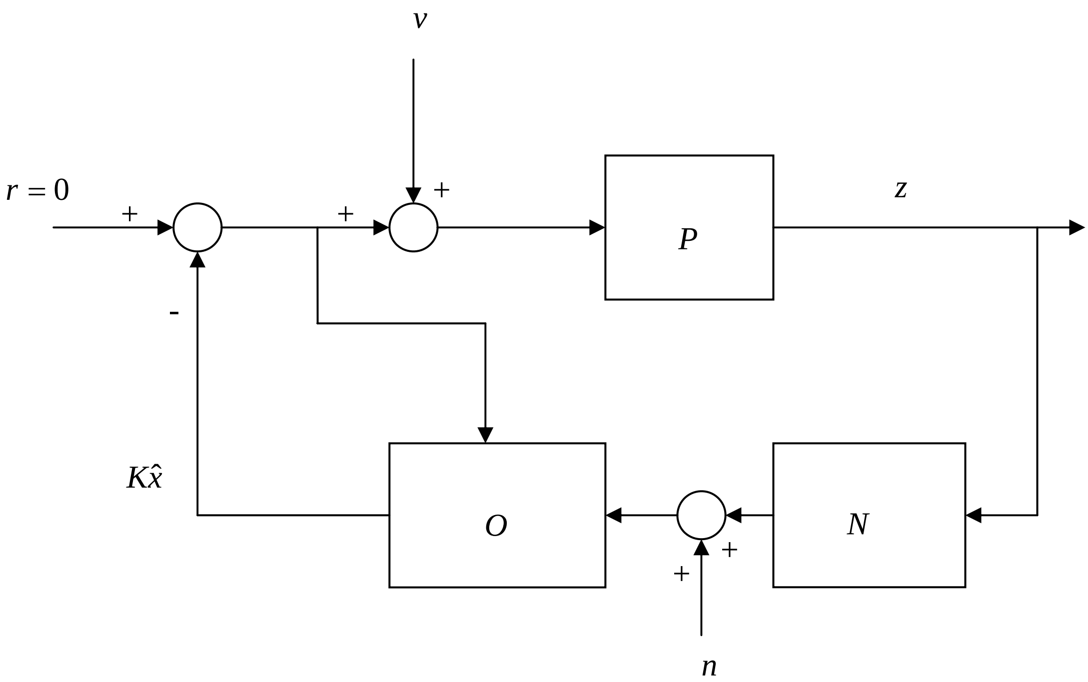

Figure 12: Adding Additional Term to Correctly Predict the Standard Deviation of the Plant Output

If stochastic linearization is to give an accurate prediction of the standard deviation of the plant output, this additional term should be filtered and have small contribution to the power of the plant output, which is the square of the standard deviation of the plant output, when passed through the closed-loop of the system. Fig. 9 in section 3.2 gives a measure of the power, $\sum_{k=3,odd}^{\infty} a_k^2$. To appreciate how large this power is relative to the power of the plant output, a plot of $\frac{a_3^2}{\sigma_z^2}$ as a function of σ_z, assuming the quantization step is 1, is plotted in Fig. 13.

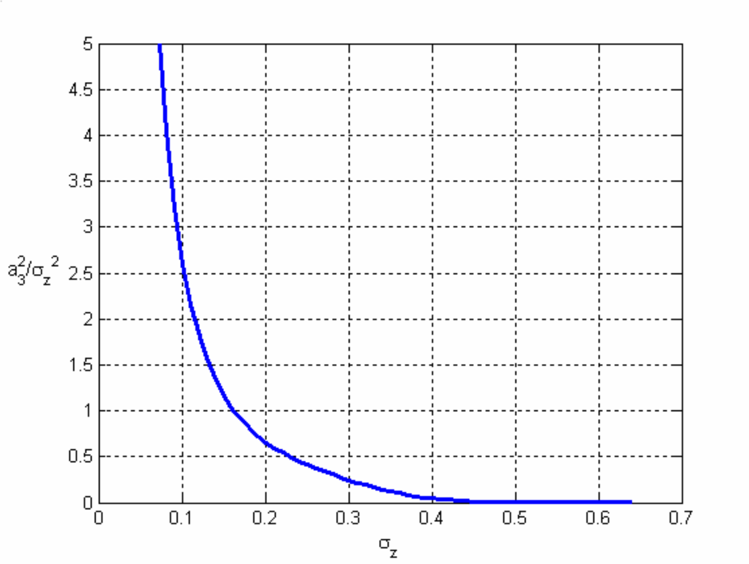

Figure 13: $\frac{a_3^2}{\sigma_z^2}$ as a Function of σ_z for Midriser Quantization with Quantization Step $\Delta = 1$

Since a_3^2 is used as a measure of $\sum_{k=3,odd}^{\infty} a_k^2$, it can be inferred from Fig. 13 that as σ_z becomes smaller, the power of the higher order terms, $\sum_{k=3,odd}^{\infty} a_k^2$, tend to become very large relative to the power of the plant output. Consider the case when feedback gain K gets larger, and the prediction $\sigma_{z_{SL}}$ becomes very small, which is the case in Table 5.4. If σ_z is indeed as small as what the prediction, $\sigma_{z_{SL}}$, is, the power passed through the closed loop of the system should be filtered out. However, from Fig. 13, there is high probability that the higher order terms will have a significant contribution to the power of the plant output after passing through the closed loop of the system, since the power of the higher order terms are very large relative to the power of the plant output. Thus, the plant output prediction might be inaccurate.

To sum up, a way to assess the accuracy of using stochastic linearization to predict the plant output is to check the value of $\frac{a_3^2}{\sigma_z^2}$ from Fig. 13 given the predicted standard deviation of the plant output, $\sigma_{z_{SL}}$. If $\frac{a_3^2}{\sigma_z^2}$ is large, then there is a high probability that this prediction of the standard deviation of the output of the plant is inaccurate. And if $\frac{a_3^2}{\sigma_z^2}$ is small, then there is a high probability that this prediction of the standard deviation of the output of the plant is accurate.

If stochastic linearization is used to predict the standard deviation of the plant output in order to select parameters K and L, it is important to keep in mind that when the parameters K and L give small plant output for the stochastic approximated system (5.3), the actual output of the system (5.2) might be significantly larger and the parameters selected based upon stochastic linearization would not be as good as expected.

6. Conclusion

Stochastic linearization was used to predict the behavior of systems with quantized sensors in both the open- and closed-loop environments. Simulation results show that the prediction is inaccurate in certain situations. The reasons for the inaccuracy in these situations were discussed using Cramer's expansion, equation (3.9), and a way to assess the accuracy in using stochastic linearization was provided for the closed-loop environment considered.

Stochastic linearization has been accurate in predicting the behavior of feedback systems with nonlinear actuators [1], [24]. However, there is no rigorous theorem to prove this. Equation (3.9) was very helpful in explaining the inaccuracy of using stochastic linearization in predicting the system behaviors in this paper. Equation (3.9) might be helpful in developing a theory for proving the accuracy of stochastic linearization in [1] and [24]. This might be a possible direction for further research.

7. Reference

[1] F. Zeitz, *Effects of Actuator Quantization on Disturbance Rejection*, Qual II paper, The University of Michigan, 2002.

[2] J.C. Mare, "Self-tuning Generalized Predictive Control of an Electrohydraulic Actuator," *Int. Journ. of Robotics and Automation*, vol. 11, no. 1, pp. 41-47, 1996.

[3] K.S. Lee, A.H. Haddad, "Approximate Models of Stochastic Linear Quantized Control Systems," *9th Mediterranean Electrotechnical*

Conference. Proceedings, May 1998.

[4] L. Xiao, M. Johansson, H. Hindi, S. Boyd, and A. Goldsmith, "Joint Optimization of Communication Rates and Linear Systems," *Proc. of 40th Conf. on Dec. and Control*, Dec. 2001.

[5] H. Ishii, T. Basar, "Remote control of LTI systems over networks with state quantization," *Proc. 41st IEEE Conference on Decision and Control*, pp. 830-835, 2002.

[6] J.H. Visser, R.E. Soltis, "Automotive Exhaust Gas Sensing Systems," *IEEE Trans. on Instr. and Measurement*, vol. 50, no. 6, Dec 2001.

[7] C.F. Chang, N.P. Fekete, A. Amstutz, J.D. Powell, "Air-Fuel Ratio Control in Spark-Ignition Engines Using Estimation Theory," *IEEE Trans. on Contr. Sys. Tech.* vol. 3, no. 1, pp. 22-31, 1995.

[8] S. Merhav, *Aerospace Sensor Systems and Applications*, NY: Springer, 1996

[9] Benjamin C. Kuo, *Automatic Control Systems*, NJ: Prentice-Hall, 1995.

[10] P.R. Belanger, P. Dobrovolny, A. Helmy, X. Zhang, "Estimation of Angular Velocity and Acceleration from Shaft Encoder Measurements," *Int. J. of Robotics Research*, vol. 17, no. 11, pp. 1225-33, 1998.

[11] J.G. Proakis, D.G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications, NJ: Prentice Hall*, 1996.

[12] P. Carbone, D. Petri, "Marginal Probability Density Function of Granular Quantization Error in Uniform Quantizers," *IEEE Trans. on Info. Theory*, vol 47, no. 3, pp. 1237-42, 2001.

[13] G..F. Franklin, J.D. Powell, A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Mass: Addison-Wesley, 2002.

[14] R.E. Curry, *Estimation and Control with Quantized Measurements*, MIT Press, 1970.

[15] D.F. Delchamps, "Extracting State Information From a Quantized Output Record," *Sys. & Contr. Letters*, vol. 13, no. 5, pp. 365-72, 1989.

[16] R.W. Brockett, D. Liberzon, "Quantized Feedback Stabilization of Linear Systems," *IEEE Trans. on Auto. Control*, vol. 45, no. 7, pp. 1279-1289, 2000.

[17] H. Ritchter, E.A. Misawa, "Stability Analysis of Discrete Linear Systems with Quantized Input and State Measurements," *Proceedings of the American Control Conference*, May 2002.

[18] B.E. Hitz, B.D.O. Anderson, "Discrete Positive-Real Functions and Their Applications to System Stability," *IEE Proceedings*, 116(1):153-155, 1969.

[19] J. Raisch, "Qualitative Control with Quantitative Models," *Intelligent Systems Engineering*, 5-9 September 1994, Conference Publication No. 395.

[20] P. Carbone, D. Petri, "Mean Value and Variance of Noisy Quantized Data,"

Measurement, vol. 23, no. 3, pp. 131-44, 1998.
[21] P.R. Kumar, P. Viraiya, “Stochastic Systems”, Prentice-Hall, 1986.
[22] T.T. Soong, *Random Differential Equations in Science and Engineering*. NY: Academic Press, 1973
[23] A. Gelb and W. Vander Velde, *Multi-Input Describing Functions and Nonlinear System Design*, NY: McGraw-Hill, 1968
[24] Cevat Gokcek, *Disturbance Rejection and Reference Tracking in Control Systems with Saturating Actuators*, Ph.D. thesis, The University of Michigan.
[25] J.B. Roberts and P. Spanos, *Random Vibration and Statistical Linearization*, NY: John Wiley and Sons, 1990.