Estimation and Control with Quantized Sensors

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May 12, 2003

Outline

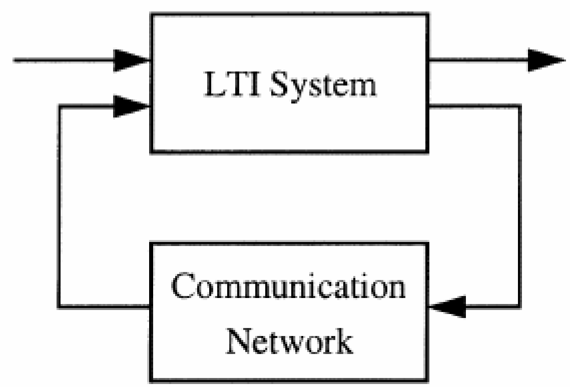
- Motivation
- Modeling
- Problem Statement
- Stochastic Linearization
- Estimation
- Estimation and Control
- Conclusions

Motivation

- Why Quantized Sensors?

1. System Constraint

   Example: Limits on information (bit) rate for systems with distributed sensors and actuators.

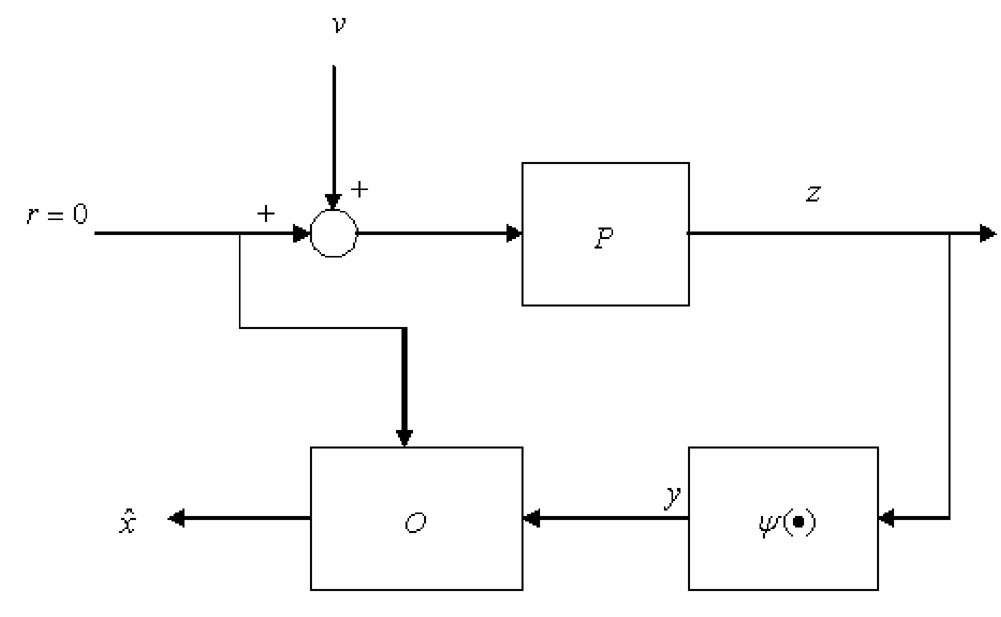


2. Budget

   Example: an inexpensive lambda Exhaust Gas Oxygen sensor might be used in air/fuel control of engines.

Modeling

- Open-Loop Environment



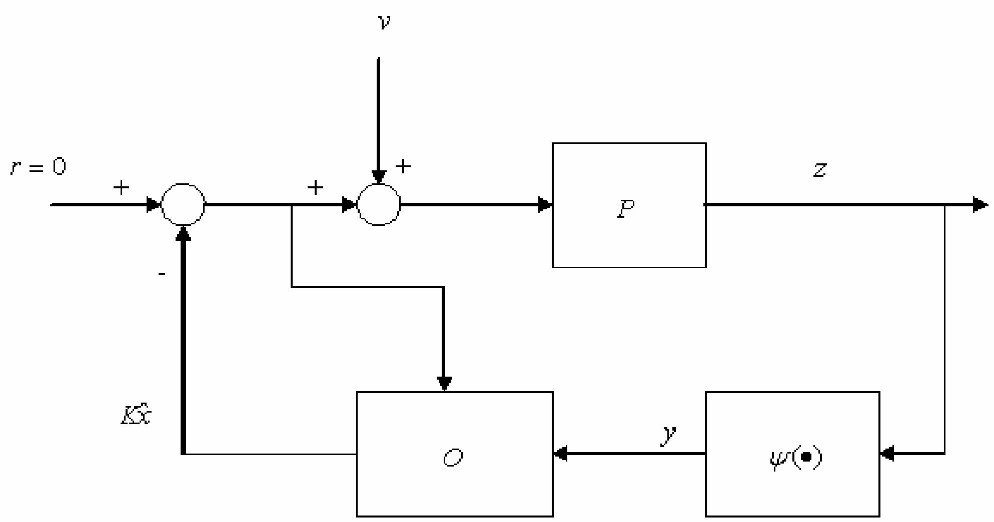
– $\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}$

– $\varphi_{vv}(\tau) = V^2 \delta(\tau)$

– In the open-loop environment, the estimation error is the figure of merit.

P : stable, SISO, LTI plant
O : asymptotic state observer
$\psi(\bullet)$: quantized sensor
r : reference input, which is assumed to be 0
v : stationary, zero mean, Gaussian white process disturbance
z : plant output, which is also the input to the quantized sensor
y : quantized sensor output
$\hat{x}$: estimated states.

- Closed-Loop Environment

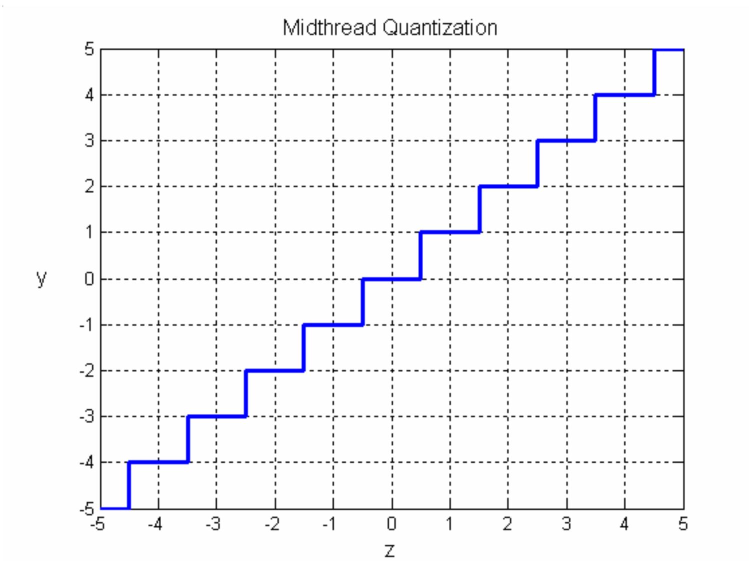


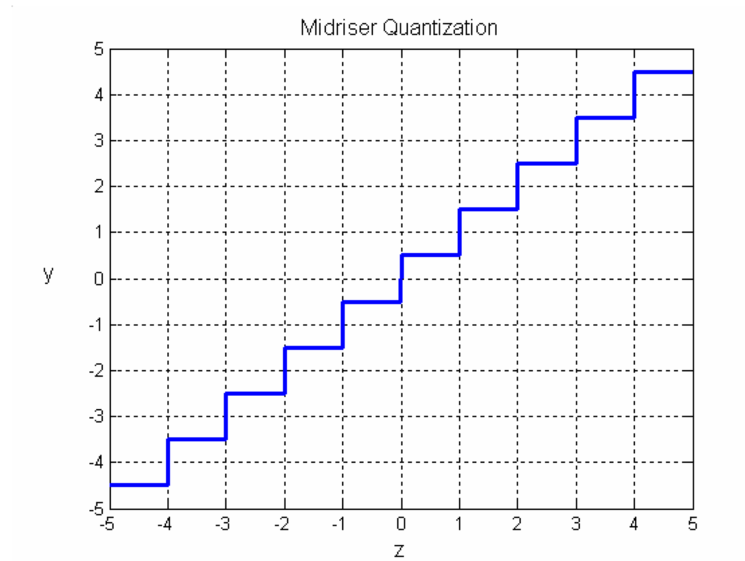
- $\begin{cases} \dot{x} = Ax + Bv - BK\hat{x} \\ z = Cx \end{cases}$
- $\varphi_{vv}(\tau) = V^2\delta(\tau)$
- In the closed-loop environment, the control objective is disturbance rejection and the figure of merit is the standard deviation of the plant output.

P : stable, SISO, LTI plant
O : asymptotic state observer
$\psi(\bullet)$: quantized sensor
r : reference input, which is assumed to be 0
v : stationary, zero mean, Gaussian white process disturbance
z : plant output, which is also the input to the quantized sensor
y : quantized sensor output
$\hat{x}$: estimated states.

• Two Types of Quantizations

- Midtread Quantization
- Midriser Quantization





$$y = \psi(z) = \frac{\Delta}{2}\sum_{k=1}^{\infty}\left[sign(2z+\Delta(2k-1)) + sign(2z-\Delta(2k-1))\right]$$

$$y = \psi(z) = \frac{\Delta}{2}\left\{sign(z) + \sum_{k=1}^{\infty}\left[sign(2z+\Delta 2k) + sign(2z-\Delta 2k)\right]\right\}$$

- In this presentation, only sensors with midriser quantization are considered.

Problem Statement

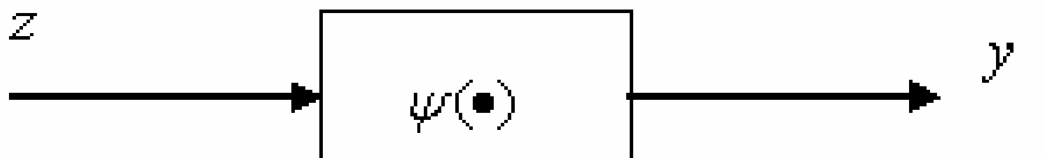
- To analyze and design the open-loop and closed-loop systems considered, need to know the estimation error and plant output.
- Due to quantized sensors, the systems are nonlinear and hard to analyze analytically.
- Use stochastic linearization to obtain a linear approximation of the nonlinear system.

The following will be addressed in this presentation:

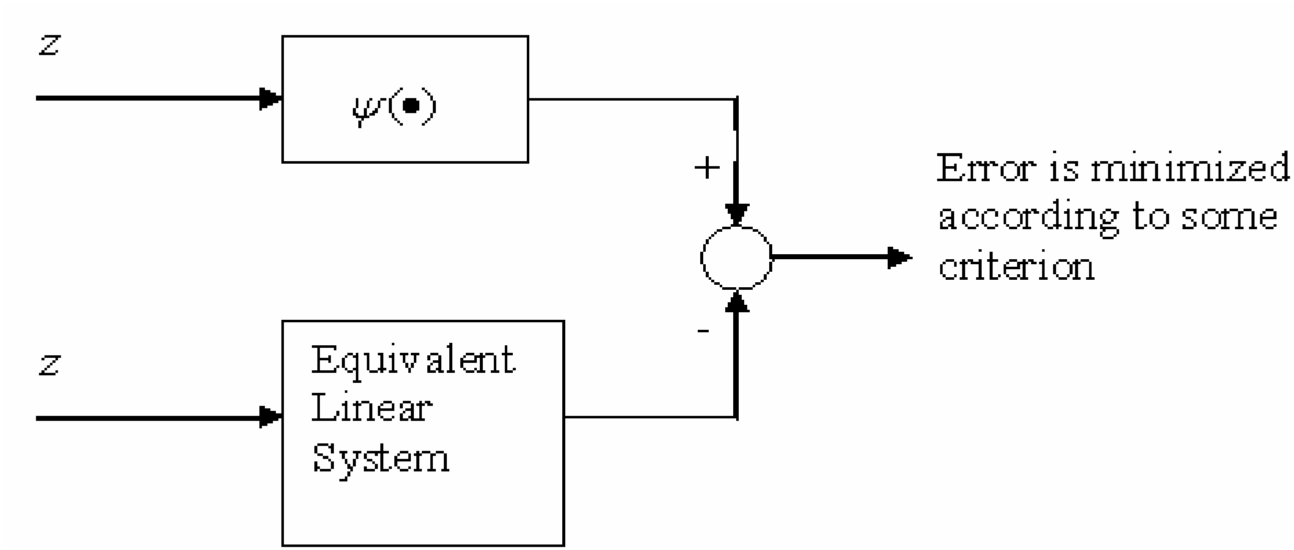
- Predict the estimation error of the observer in the open-loop environment using stochastic linearization.
- Predict the estimation error of the observer and the system output in the closed-loop environment using stochastic linearization.
- Discuss the accuracy of using stochastic linearization in both the open- and closed-loop environment by the Cramer’s expansion.

Stochastic Linearization

- The Method of Stochastic Linearization



The input z to the nonlinearity, $\psi(\bullet)$, is a random process



* In this presentation, the criterion is to minimize the mean squared error between the output of the nonlinear element and the output of the equivalent linear system

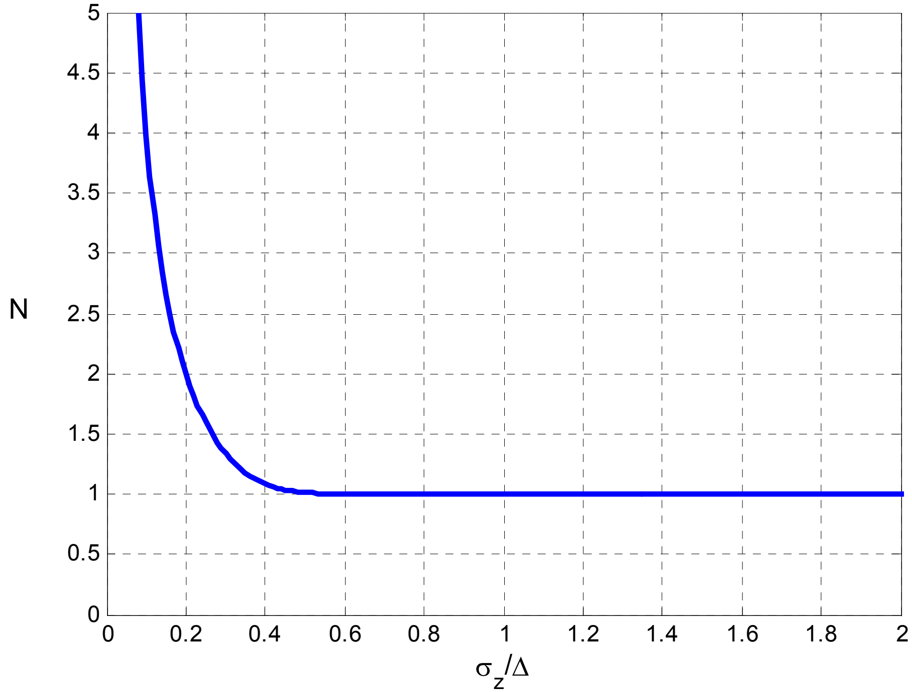
If the input is a zero-mean, stationary, Gaussian random process, and the nonlinearity is a midriser quantization, the equivalent linear system of this midriser quantization is a linear gain (equivalent gain), and can be calculated by

$$N = Q\left(\frac{\sigma_z}{\Delta}\right)$$

where

$$Q(x) = \left(\frac{1}{\sqrt{2\pi x}} + \frac{\sqrt{2}}{\sqrt{\pi x}} \sum_{k=1}^{\infty} e^{-\frac{1}{2}\left(\frac{k^2}{x^2}\right)}\right)$$

A plot of the equivalent gain of the midriser quantization as a function of $\frac{\sigma_z}{\Delta}$



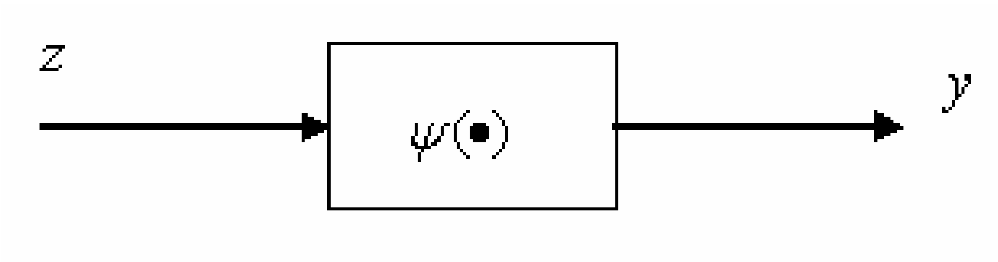
In this presentation, it is assumed that the input to the nonlinearity is

* reasonably close to a zero mean, stationary, Gaussian process so that

$N = Q(\frac{\sigma_z}{\Delta})$ can be used to calculate the equivalent gain.

- The Cramer's Expansion

  - Basically, expressing the autocorrelation function of the sensor output as a power series of the autocorrelation function of the sensor input.



$$y = \psi(z)$$

  - Assume that input is a zero-mean, stationary, Gaussian Random process.

– By definition,

$$\varphi_{yy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(z_1)\psi(z_2) \frac{1}{2\pi\sigma_z^2\sqrt{1-\rho^2}} \exp\left[-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2\sigma_z^2(1-\rho^2)}\right] dz_1 dz_2$$

where $\rho = \dfrac{\varphi_{zz}(\tau)}{\sigma_z^2}$

By introducing Hermite polynomials,

$$\varphi_{yy}(\tau) = \sum_{k=1,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2}\right)^k$$

where

$$a_k = \frac{1}{\sqrt{2\pi k!\sigma_z^2}} \int_{-\infty}^{\infty} \psi(z) \exp\left(-\frac{z^2}{2\sigma_z^2}\right) H_k\left(\frac{z}{\sigma_z}\right) dz$$

and $H_k(\bullet)$'s are the Hermite polynomials

Note that given the nonlinearity, a_k only depends on σ_z

- It can be verified that $N^2 = \frac{a_1^2}{\sigma_z^2}$ where N

  is the equivalent gain of the quantized sensor, hence,

$$\varphi_{yy}(\tau) = N^2 \varphi_{zz}(\tau) + \sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$$

This is referred to as the Cramer's Expansion

This term is what one would get by substituting the nonlinearity with an equivalent gain.

These higher order terms are what is not accounted for by substituting the nonlinearity by its equivalent gain

Typically, the Fourier Transform of the higher order terms have wider bandwidth, and hence the power spectrum of this part of the output has wider bandwidth and a larger portion of its power is contained In the tail of this spectrum.

- The power spectrum of the output of the nonlinearity is the spectrum of Nz plus the spectrum corresponding to the higher order terms which has a wide bandwidth

To give a quantitative assessment of the accuracy of stochastic linearization, consider

$$\varphi_{yy}(\tau) = N^2 \varphi_{zz}(\tau) + \sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$$

- Power of the higher order terms not accounted for $P = \sum_{k=3,odd}^{\infty} a_k^2$

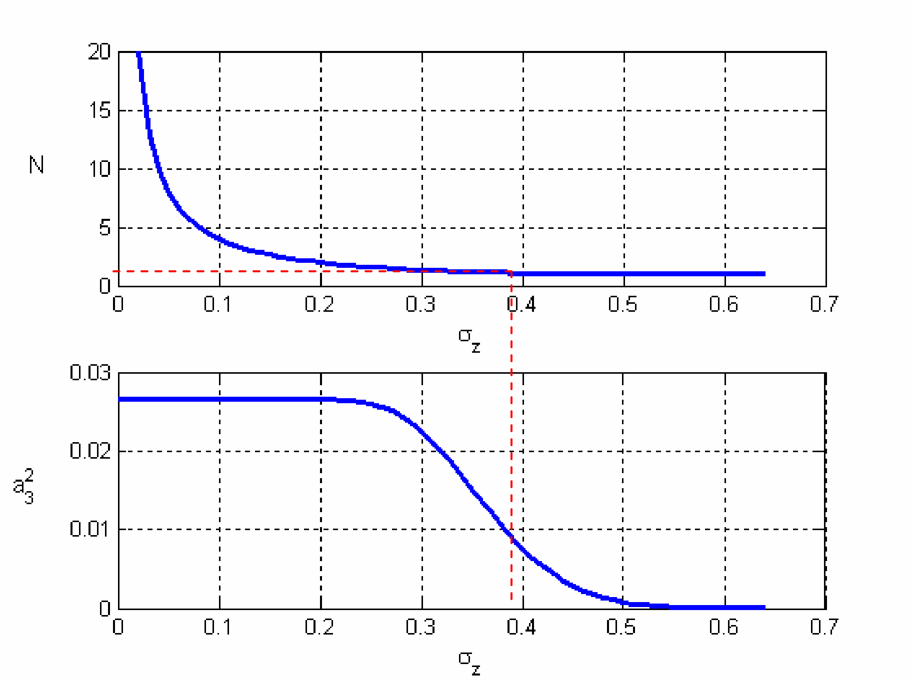
- a_3 is the largest coefficients of the higher order terms

- By calculating the value of a_3^2 a measure of how large the power of

  the terms not accounted for might be can be assessed and a rough assessment of using stochastic linearization to predict the standard deviations of signals in a system may be made.

N and a_3^2 as a Function of σ_z

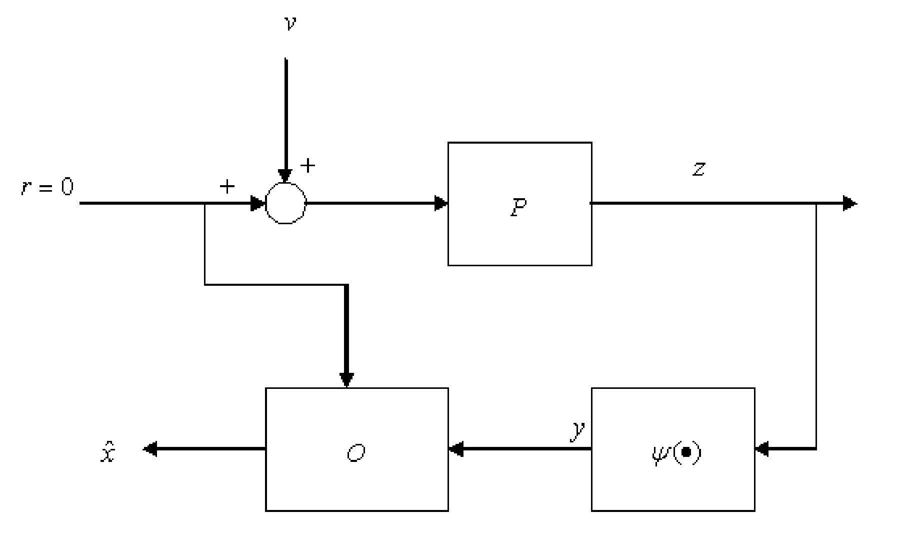
for Midriser Quantization with Quantization Step $\Delta = 1$



A way to assess the accuracy of using stochastic linearization is to find out which regime the system is operating in and look up the value of a_3^2

Estimation

- Structure of the Observer



$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

- Dynamics of the plant is

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

and the measurement is the nonlinearity $\psi(\bullet)$

- It seems reasonable to use the same nonlinearity in the observer

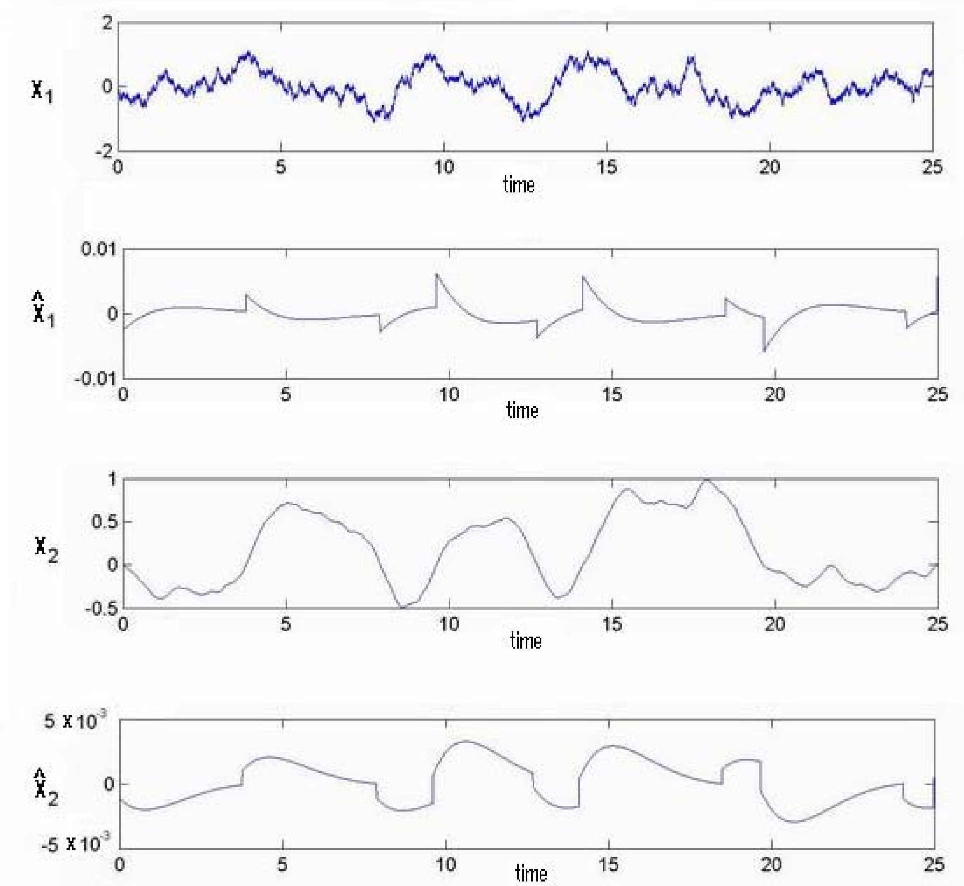
$$\begin{cases} \dot{\hat{x}} = A\hat{x} + L(\psi(z) - \psi(\hat{z})) \\ \hat{z} = C\hat{x} \end{cases}$$

- There are problems when quantization is coarse

$$A = \begin{bmatrix} -1.414 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L = \begin{bmatrix} 30.9903 \\ 12.7260 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad V = 0.5916$$

$$\sigma_z = 0.3518 \quad \Delta = 1$$



- Need a way to fix this deficiency

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

$\Rightarrow$ σ_z can be calculated by solving a Lyapunov equation

$\Rightarrow$ $N = Q(\frac{\sigma_z}{\Delta})$ can be calculated

- use this equivalent gain in observer design

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) \\ \hat{z} = C\hat{x} \end{cases}$$

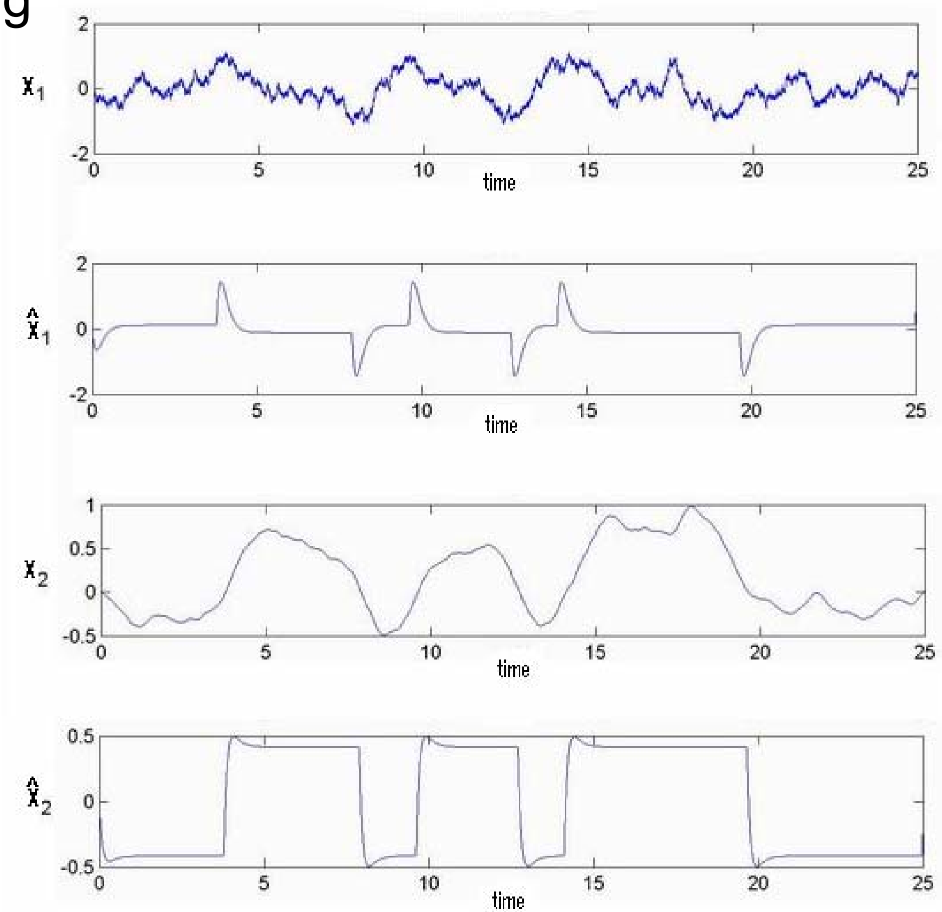
Note that we are not performing stochastic linearization yet

- The state estimation improves

$$A = \begin{bmatrix} -1.414 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L = \begin{bmatrix} 30.9903 \\ 12.7260 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad V = 0.5916$$

$$\sigma_z = 0.3518 \quad \Delta = 1$$



•Equations for Predicting the Estimation Error

- $\sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$ will be used as a measure of the estimation error of the observer, where $\tilde{x} = x - \hat{x}$

$$
\begin{cases}
\dot{x} = Ax + Bv \\
z = Cx \\
\dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) \\
\hat{z} = C\hat{x} \\
N = Q\left(\dfrac{\sigma_z}{\Delta}\right)
\end{cases}
$$

By subtraction,

$$
\begin{cases}
\dot{x} = Ax + Bv \\
z = Cx \\
\dot{\tilde{x}} = (A - LNC)\tilde{x} + LNz - L\psi(z) + Bv \\
N = Q\left(\dfrac{\sigma_z}{\Delta}\right)
\end{cases}
$$

Applying the method of stochastic linearization:

$$\begin{cases} \dot{x}_{SL} = Ax_{SL} + Bv \\ z_{SL} = Cx_{SL} \\ \dot{\tilde{x}} = (A - LNC)\tilde{x}_{SL} + Bv \\ N = \left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}$$

A prediction of $\sqrt{E\left[\tilde{x}^T\tilde{x}\right]}$ can then be calculated analytically

assuming that $A - LNC$ is stable.

$$\begin{cases} (A - LNC)R + R(A - LNC)^T + BV^2B^T = 0 \\ \sqrt{E\left[\tilde{x}_{SL}^T\tilde{x}_{SL}\right]} = \sqrt{tr(R)} \end{cases}$$

- Simulation Results

$$\begin{cases} \dot{x} = Ax + Bv \\ z = Cx \\ \dot{\tilde{x}} = (A - LNC)\tilde{x} + LNz - L\psi(z) + Bv \\ N = Q\left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}$$

Calculate (from simulation data) → $\sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

$$\begin{cases} \dot{x}_{SL} = Ax_{SL} + Bv \\ z_{SL} = Cx_{SL} \\ \dot{\tilde{x}} = (A - LNC)\tilde{x}_{SL} + Bv \\ N = \left(\dfrac{\sigma_z}{\Delta}\right) \end{cases}$$

Calculate (analytically) → $\sqrt{E[\tilde{x}_{SL}^T \tilde{x}_{SL}]}$

$$\begin{cases} (A - LNC)R + R(A - LNC)^T + BV^2 B^T = 0 \\ \sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]} = \sqrt{tr(R)} \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

- $$\begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases} \quad V = 1$$

- $$estimation \quad prediction \quad error = \frac{\sigma_{\tilde{x}_{SL}} - \sigma_{\tilde{x}}}{\sigma_{\tilde{x}}} \times 100\%$$

$$\sigma_{\tilde{x}_{SL}} = \sqrt{E\left[\tilde{x}_{SL}^T \tilde{x}_{SL}\right]}$$

$$\sigma_{\tilde{x}} = \sqrt{E\left[\tilde{x}^T \tilde{x}\right]}$$

- Try different values of L and for each L use various Δ

$L = 4$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	4.74%	7.68%	11.35%	15.89%	20.34%	24.33%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

$L = 9$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	15.08%	22.46%	29.64%	36.45%	42.16%	46.72%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

$L = 19$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
$\frac{\sigma_{\tilde{x}} - \sigma_{\tilde{x}_{SL}}}{\sigma_{\tilde{x}}} \times 100\%$	34.04%	43.45%	50.19%	56.89%	61.43%	64.84%
N	1.0001	1.0021	1.013	1.0423	1.0951	1.1679

As observer gain L gets larger, the prediction of the estimation error gets worse.

- Discussion of Simulation Results

For a given quantization step, the estimation prediction error gets worse as the observer gain gets larger

- $\dot{\tilde{x}} = (A - LCN)\tilde{x} + Bv + L(Nz - \psi(z))$ $\xrightarrow{\text{Stochastic Linearization}}$ $\dot{\tilde{x}}_{SL} = (A - LCN)\tilde{x}_{SL} + Bv$
- $L(Nz - \psi(z))$ is ignored in stochastic linearization
- Consider the characteristic of $Nz - \psi(z)$

Consider the Cramer's Expansion:

$$\varphi_{yy}(\tau) = N^2 \varphi_{zz}(\tau) + \sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$$

autocorrelation of Nz

Since $y = \psi(z) = Nz - (Nz - \psi(z))$, autocorrelation of $Nz - \psi(z)$ is part of this term

If L is small,

- the term $Nz - \psi(z)$ in $\dot{\tilde{x}} = (A - LNC)\tilde{x} + Bv + L(Nz - \psi(z))$ will be filtered out, so $\tilde{x}$ from $\dot{\tilde{x}} = (A - LCN)\tilde{x} + Bv + L(Nz - \psi(z))$ and $\tilde{x}_{SL}$ from $\dot{\tilde{x}}_{SL} = (A - LCN)\tilde{x}_{SL} + Bv$ will not differ very much.

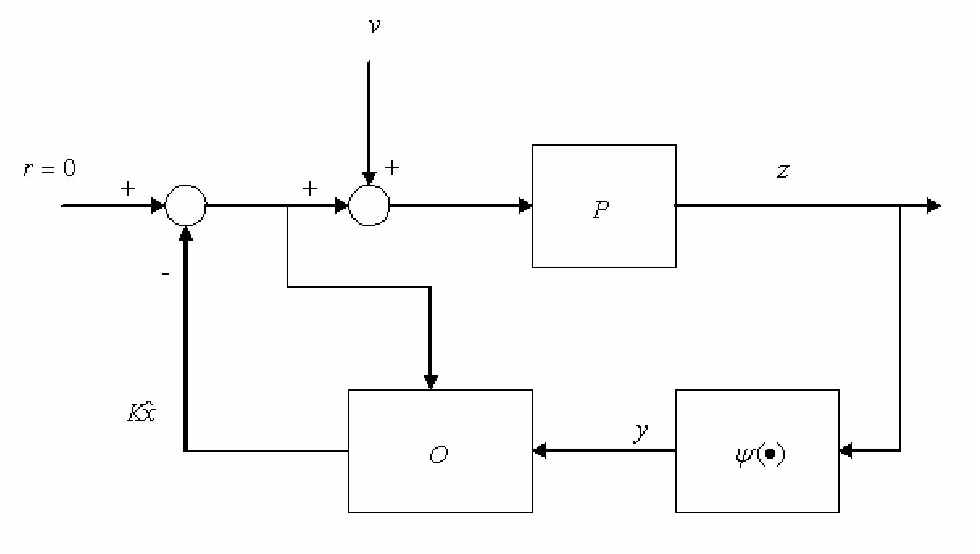
As L gets larger,

- $\dot{\tilde{x}}_{SL} = (A - LCN)\tilde{x}_{SL} + Bv$ will predict that $\tilde{x}_{SL}$ is small
- $\dot{\tilde{x}} = (A - LCN)\tilde{x} + Bv + L(Nz - \psi(z))$ will not be as small as stochastic linearization predicts since the dynamics will become less low pass and more power of $Nz - \psi(z)$ will pass through

Stochastic linearization may not give accurate prediction of the estimation error in the open-loop environment.

How about the closed-loop environment?

Estimation and Control



$$
\begin{cases}
\dot{x} = Ax + Bu + Bv \\
z = Cx \\
\dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) + Bu \\
\hat{z} = C\hat{x} \\
u = -K\hat{x}
\end{cases}
$$

$$
\varphi_{vv}(\tau) = V^2 \delta(\tau)
$$

- Equations for Predicting the Plant Output and Estimation Error

  - Plant output σ_z
  - Estimation error $\sigma_{\tilde{z}} = \sqrt{E\left[\tilde{z}^T \tilde{z}\right]} = \sqrt{E\left[\tilde{x}^T C^T C\tilde{x}\right]}$ where $\tilde{z} = C\tilde{x}$

$$\begin{cases} \dot{x} = Ax + Bu + Bv \\ z = Cx \\ \dot{\hat{x}} = A\hat{x} + L(\psi(z) - N\hat{z}) + Bu \\ \hat{z} = C\hat{x} \\ u = -K\hat{x} \end{cases} \xrightarrow{\text{subtraction}} \begin{cases} \dot{x} = (A - BK)x + BK\tilde{x} + Bv \\ z = Cx \\ \dot{\tilde{x}} = (A - LNC)\tilde{x} + Bv - L(\psi(z) - Nz) \\ \tilde{z} = C\tilde{x} \end{cases}$$

$$\xrightarrow{\text{stochastic linearization}} \begin{cases} \begin{bmatrix} \dot{x}_{SL} \\ \dot{\tilde{x}}_{SL} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LNC \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \\ z_{SL} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} \\ \tilde{z}_{SL} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} \end{cases}$$

Let, $\tilde{A} = \begin{bmatrix} A - BK & BK \\ 0 & A - LNC \end{bmatrix}$ $\quad \tilde{B} = \begin{bmatrix} B \\ B \end{bmatrix}$ $\quad \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$ $\quad \tilde{C}_2 = \begin{bmatrix} 0 & C \end{bmatrix}$

$$\longrightarrow \begin{cases} \tilde{A}\tilde{R} + \tilde{R}\tilde{A}^T + \tilde{B}V^2\tilde{B}^T = 0 \\ \sigma_{z_{SL}} = \sqrt{\tilde{C}\tilde{R}\tilde{C}^T} \\ \sigma_{\tilde{z}_{SL}} = \sqrt{\tilde{C}_2\tilde{R}\tilde{C}_2^T} \end{cases}$$

Note that we don't know N yet

$$N = Q\left(\frac{\sigma_z}{\Delta}\right) \xrightarrow{\text{Assume stochastic linearization gives a good prediction}} N = Q\left(\frac{\sigma_{z_{SL}}}{\Delta}\right)$$

$$\begin{cases} \tilde{A}\tilde{R} + \tilde{R}\tilde{A}^T + \tilde{B}V^2\tilde{B}^T = 0 \\ \sigma_{z_{SL}} = \sqrt{\tilde{C}\tilde{R}\tilde{C}^T} \\ N = Q\left(\dfrac{\sigma_{z_{SL}}}{\Delta}\right) \end{cases}$$

have to be solved simultaneously for

$\tilde{A}$, $\sigma_{z_{SL}}$ and N

- Simulation results

$$\begin{cases} \dot{x} = (A - BK)x + BK\tilde{x} + Bv \\ z = Cx \\ \dot{\tilde{x}} = (A - LNC)\tilde{x} + Bv - L(\psi(z) - Nz) \\ \tilde{z} = C\tilde{x} \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

Calculate (from simulation data) → σ_z, $\sigma_{\tilde{z}}$

$$\begin{cases} \begin{bmatrix} \dot{x}_{SL} \\ \dot{\tilde{x}}_{SL} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LNC \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \\ z_{SL} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} \\ \tilde{z}_{SL} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x_{SL} \\ \tilde{x}_{SL} \end{bmatrix} \end{cases}$$

$$\varphi_{vv}(\tau) = V^2 \delta(\tau)$$

Calculate (analytically) → $\sigma_{z_{SL}}$, $\sigma_{\tilde{z}_{SL}}$

$$\begin{cases} \tilde{A}\tilde{R} + \tilde{R}\tilde{A}^T + \tilde{B}V^2\tilde{B}^T = 0 \\ \sigma_{z_{SL}} = \sqrt{\tilde{C}\tilde{R}\tilde{C}^T} \\ \sigma_{\tilde{z}_{SL}} = \sqrt{\tilde{C}_2\tilde{R}\tilde{C}_2^T} \end{cases}$$

$$N = Q\left(\frac{\sigma_{z_{SL}}}{\Delta}\right)$$

$$A = \begin{bmatrix} -1.414 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad plant \quad output \quad prediction \quad error = \frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad V = 1 \qquad estimation \quad prediction \quad error = \frac{\sigma_{\tilde{z}} - \sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}} \times 100\%$$

Feedback gain K kept constant and tried different observer gain L
And for each L various Δ are used.

$$L = \begin{bmatrix} -1 \\ -1.414 \end{bmatrix} \qquad K = \begin{bmatrix} 0.5860 & 0 \end{bmatrix}$$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
σ_z	0.6839	0.6852	0.6844	0.6880	0.6933	0.6997
$\sigma_{z_{SL}}$	0.6762	0.6762	0.6767	0.6888	0.6818	0.6856
$\frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	1.13%	1.32%	1.12%	1.34%	1.66%	2.02%
$\frac{\sigma_{\tilde{z}} - \sigma_{\tilde{z}_{SL}}}{\sigma_{\tilde{z}}} \times 100\%$	7.37%	11.11%	15.54%	19.45%	22.81%	26.83%
N	1	1	1.01	1.05	1.11	1.19

$$L = \begin{bmatrix} 30.9903 \\ 12.7260 \end{bmatrix} \qquad K = \begin{bmatrix} 0.5860 & 0 \end{bmatrix}$$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
σ_z	0.5183	0.5789	0.5789	0.5797	0.5805	0.5812
$\sigma_{z_{SL}}$	0.5727	0.5727	0.5727	0.5727	0.5728	0.5728
$\frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	0.96%	1.07%	1.07%	1.2%	1.34%	1.44%
$\frac{\sigma_y - \sigma_{y_{SL}}}{\sigma_y} \times 100\%$	90.8%	92.4%	93.56%	94.42%	95.08%	95.61%
N	1	1.02	1.07	1.15	1.27	1.39

$$L = \begin{bmatrix} 160.9511 \\ 26.8660 \end{bmatrix} \qquad K = \begin{bmatrix} 0.5860 & 0 \end{bmatrix}$$

	$\Delta = 1$	$\Delta = 1.2$	$\Delta = 1.4$	$\Delta = 1.6$	$\Delta = 1.8$	$\Delta = 2.0$
σ_z	0.5449	0.5454	0.5458	0.5468	0.5479	0.5484
$\sigma_{z_{SL}}$	0.5391	0.5391	0.5391	0.5391	0.5390	0.5390
$\frac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	1.07%	1.16%	1.24%	1.41%	1.62%	1.71%
$\frac{\sigma_y - \sigma_{y_{SL}}}{\sigma_y} \times 100\%$	96.99%	97.52%	97.89%	98.17%	98.39%	98.56%
N	1	1.03	1.1	1.21	1.34	1.48

Stochastic linearization seems to give a good prediction of
The plant output.

$$L = \begin{bmatrix} 160.9511 \\ 26.8660 \end{bmatrix} \qquad \Delta = 1$$

	$K = [0.5860 \quad 0]$	$K = [10.586 \quad 35]$
σ_z	0.5475	0.1242
$\sigma_{z_{SL}}$	0.5391	0.0768
$\dfrac{\sigma_z - \sigma_{z_{SL}}}{\sigma_z} \times 100\%$	1.54%	38.17%
$\dfrac{\sigma_{\hat{z}} - \sigma_{\hat{z}_{SL}}}{\sigma_{\hat{z}}} \times 100\%$	96.97%	98.32%
N	1	5.19

The prediction of the plant output gets significantly worse as feedback gain K becomes larger.

- Discussion of Simulation Results

1. As observer gain L gets larger, the estimation prediction error gets larger

− The explanation of this phenomenon is basically the same as the case in the open-loop environment.

− $\dot{\tilde{x}} = (A - LCN)\tilde{x} + Bv + L(Nz - \psi(z))$ $\xrightarrow{\text{Stochastic Linearization}}$ $\dot{\tilde{x}}_{SL} = (A - LCN)\tilde{x}_{SL} + Bv$

As in the open-loop environment, we are ignoring the term $L(Nz - \psi(z))$

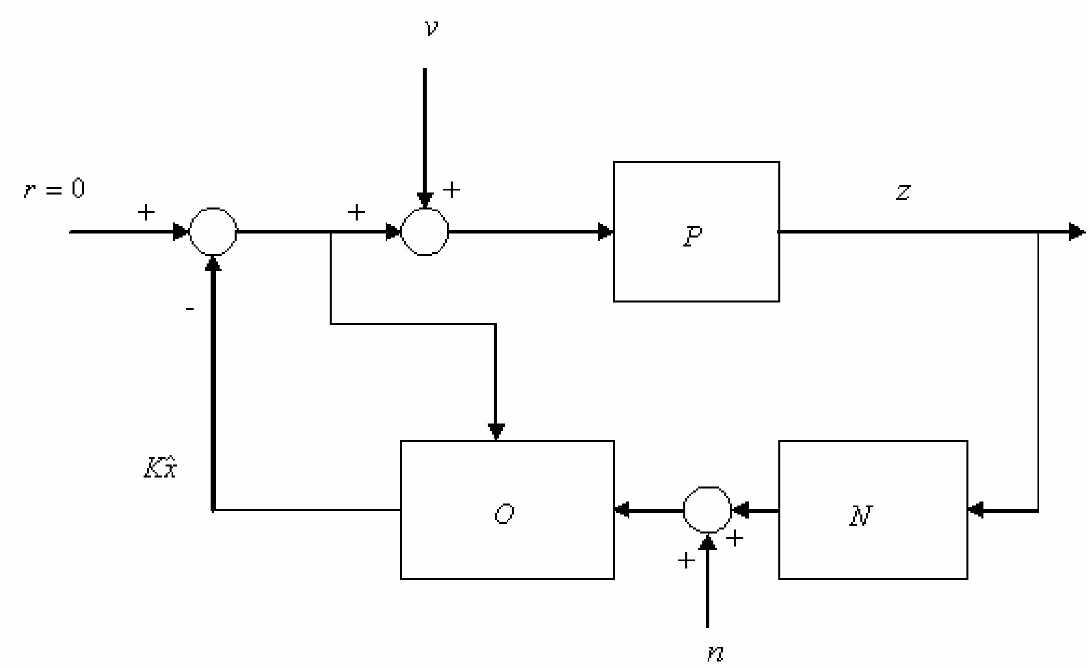
− As discussed in the open loop case, as the observer gain L gets larger, $C\tilde{x}$ and $C\tilde{x}_{SL}$ will differ very much. And as L is smaller, $C\tilde{x}$ and $C\tilde{x}_{SL}$ will not differ as much.

2. As feedback gain K gets larger and the plant output gets smaller (control is more effective), the plant output prediction error gets worse.

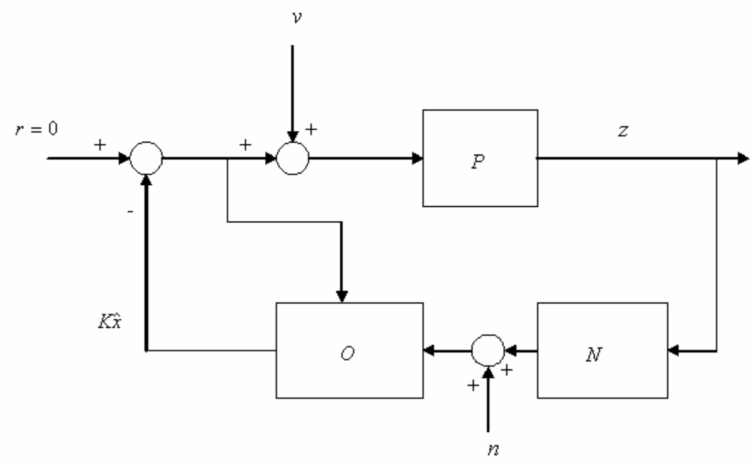
- Consider the Cramer's Expansion

$$\varphi_{yy}(\tau) = N^2 \varphi_{zz}(\tau) + \sum_{k=3,odd}^{\infty} a_k^2 \left(\frac{\varphi_{zz}(\tau)}{\sigma_z^2} \right)^k$$

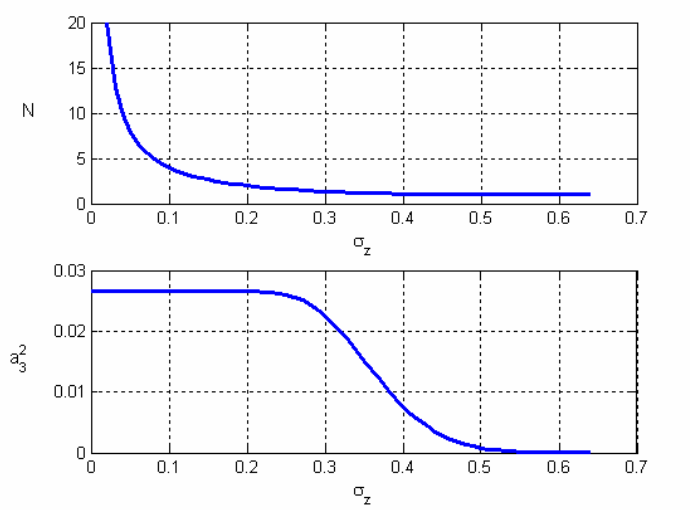
- Need to add an additional term to correctly model the variance or standard deviation at the plant output

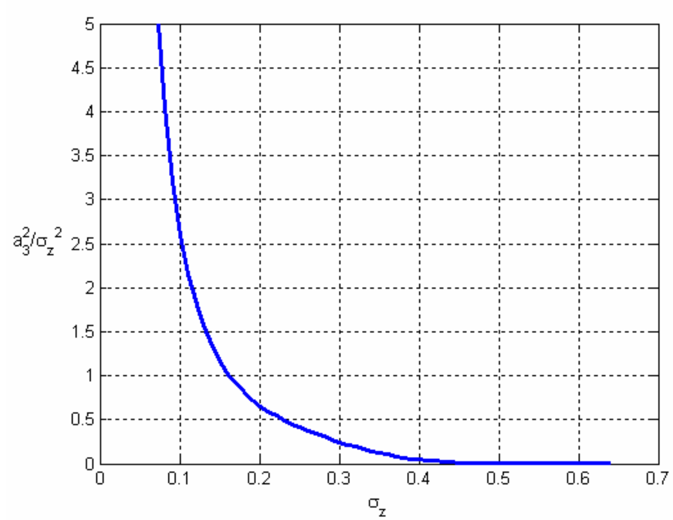


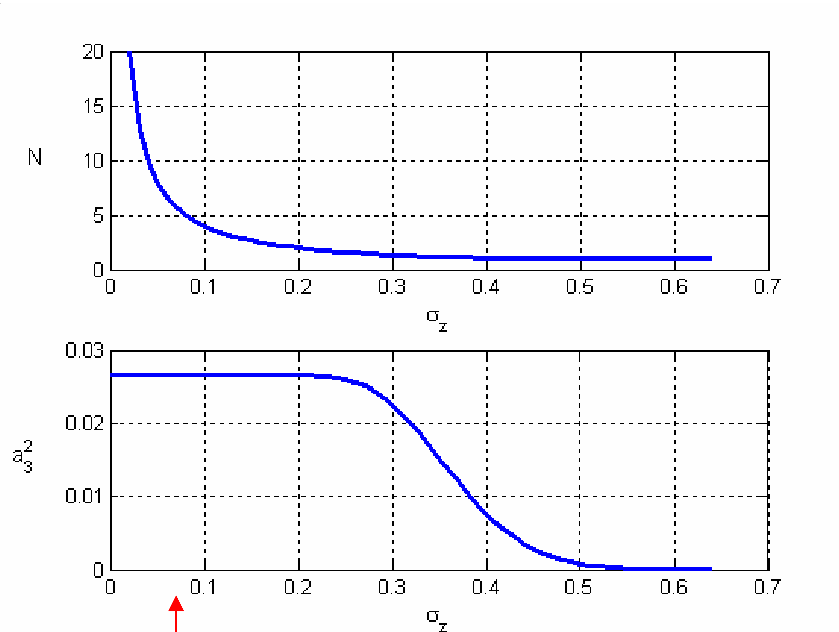
− If stochastic linearization gives a good approximation of the plant output, this additional term should have little contribution to the plant output

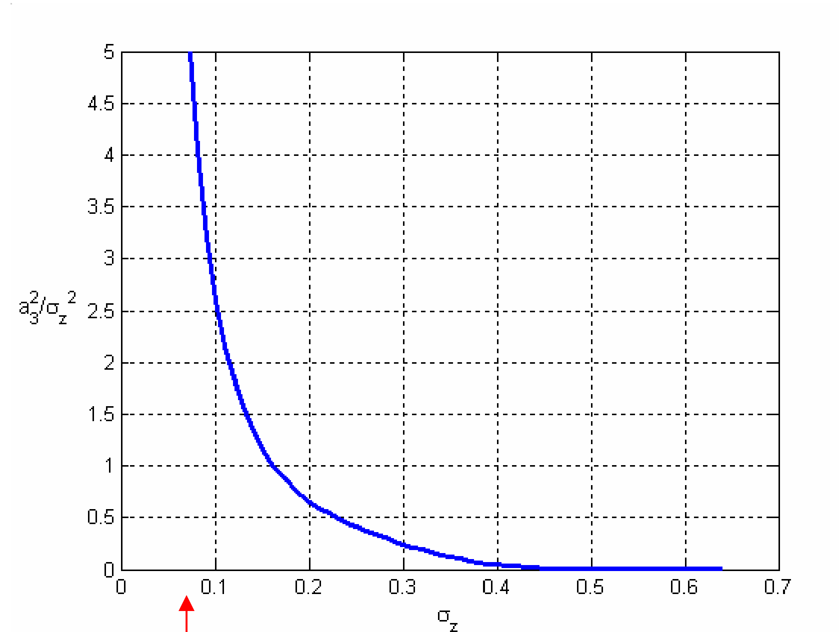


− To see how much this additional term might contribute to the plant output, consider the following plots:









The system is predicted to be operating around this regime when the plant output prediction becomes very inaccurate.

If stochastic linearization is to be used to help select parameters to achieve good disturbance rejection, one should be careful that the parameters chosen by stochastic linearization might not give as good performance as predicted.

Conclusions

- Stochastic linearization may not give accurate prediction of the estimation error in both the open- and closed-loop environments with quantized sensors.
- Stochastic linearization gives good prediction of the plant output in the closed-loop environment with quantized sensors in certain situations. A way to assess when stochastic linearization may not give good prediction of the plant output is also given.
- Cramer’s expansion is useful in explaining the inaccuracy of stochastic linearization. Further research using Cramer’s expansion to give rigorous proofs of the accuracy of stochastic linearization in certain situations might be possible.