Effective-index method and coupled-mode theory for almost-periodic waveguide gratings: a comparison

Kim A. Winick

Contradirectional propagation through active, first-order, almost-periodic, corrugated waveguide gratings is analyzed by using both coupled-mode theory and a combined effective-index/impedance-matching matrix technique. For TE-mode operation, which is near the first-order Bragg wavelength, the equivalence of the two techniques is analytically demonstrated for shallow surface corrugations.
Key words: Effective-index method, coupled-mode theory, corrugated waveguide gratings.

I. Introduction

The propagation of optical modes in periodic and almost-periodic waveguide gratings is a topic of considerable interest. Applications include waveguide filter design,[1–6] distributed feedback (DFB) laser structures,[7,8] phase matching for nonlinear interactions,[9] grating couplers,[10] pulse compression,[11] and soliton generation.[12] Propagation in periodic corrugated waveguides can be analyzed exactly by using the Floquet–Bloch theory.[13,14] For almost-periodic waveguide gratings the exact Floquet–Bloch theory can no longer be applied, and approximate methods must be used. For contradirectional coupled-wave interaction near the Bragg wavelength, both coupled-mode theory and effective-index techniques have been applied. It is of interest to know when these methods will yield equivalent results.

In coupled-mode theory[15] a pair of first-order coupled differential equations is derived that approximately relates the amplitudes of the forward- and backward-propagating modes. These equations include a dephasing term δ, a coupling coefficient κ, and a material gain factor g as parameters. The δ equals the deviation of the mode propagation constant from the local Bragg condition, and κ indicates the strength of the coupling between the modes. For an almost-periodic grating, δ, κ, and g may be functions of the position that is measured along the guide. The κ can be evaluated from an overlap integral that involves the induced polarization fields in the grating region and the field profiles of the unperturbed or local guided modes. In general the pair of coupled differential equations must be solved numerically, although approximate solutions have been developed under restrictive assumptions.[3,16] The pair of first-order differential equations can also be combined into a single second-order Ricatti differential equation,[2] which too must be solved numerically. When the grating is periodic, the coupled-mode equations can be solved analytically. Closed-form solutions are obtained, and these are in agreement with the more exact Floquet–Bloch theory.

For almost-periodic gratings an alternate approach is to divide the grating into a large number of thin sections, each of which is assumed to have a constant value of δ, κ, and g. Within each of these sections analytic solutions are obtained by using coupled-mode theory for periodic gratings, and these solutions are used to generate a 2×2 transfer matrix for the section. This matrix relates the forward- and backward-propagating field amplitudes that are measured at the front of the section to those that occur at the back. The transfer matrix for the entire corrugation is generated by multiplying the individual transfer matrices together.[17–19] It should be recognized that this approach is simply a numerical method for solving the coupled-mode equations.

Wang[20] proposed an effective-index technique for solving the contradirectional mode-coupling problem. This technique is conceptually simpler than the coupled-mode formalism described above. Basu and Ballantyne[21] recast the effective-index method into a

The author is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan 48109-2122.

Received 4 April 1990.
0003-6935/92/060757-08$05.00/0.

convenient matrix form and used it to study random fluctuations in first-order grating filters. Recently Bjork and Nilsson[22] used this approach to study the properties of asymmetric phase-shifted DFB lasers. Basu and Ballantyne's approach is a direct application of the effective-index method combined with impedance matching. Each period of the waveguide grating is divided into thin sections, and the guide height and material gain are assumed to be constant within the section. The section is treated as a three-layer waveguide, and the standard b versus V dispersion relationships[15] are used to compute the propagating modes that are supported by this guide. A 2×2 transfer matrix for the section is then derived by matching the tangential **E** and **H** fields (which correspond to these modes) at the interfaces between sections. Finally, the transfer matrix for the complete structure is obtained by multiplying together the individual transfer matrices.

Verly *et al.*[23,24] derived the effective-index method in periodic corrugated gratings directly from Maxwell's wave equation and a local normal mode expansion of the field.[25] In this way they were able to examine the approximations inherent in the effective-index technique. They demonstrated that it was a relatively accurate theory for TE modes and could be modified to give correct results for the TM case. Finally, they combined the effective-index technique with coupled-wave theory for one-dimensional dielectrics. Using this combined approach, they obtained results that were identical to those obtained by use of the coupled-mode theory for periodic waveguide gratings.

We analytically demonstrate by direct computation that the coupled-mode theory technique is equivalent to the TE-mode effective-index/impedance-matching method. We do this for almost-periodic waveguide gratings, which may have gain. As opposed to Verly's method, the coupled-wave theory results for one-dimensional dielectrics are not used in our development. Contradirectional propagation through an almost-periodic, corrugated waveguide grating is assumed, as is operation near the first-order Bragg wavelength.

II. Impedance-Matching Matrix Method

Consider the planar, thin-film waveguide shown in Fig. 1. This is a three-layer dielectric guide, consisting of cover, film, and substrate layers, with refractive indices n_c, n_f, and n_s, respectively. The interface between the film and cover layers has a shallow, surface corrugation, which can be modeled as an almost-periodic square-wave grating. For simplicity we divide the grating into slabs, numbered 0 through $N + 1$ as indicated in Fig. 1. The width of the kth slab is w_k, and its height is h_k. Light of free-space wavelength λ is confined to the film region by total internal reflection at the film–cover and film–substrate interfaces ($n_c < n_s < n_f$). It is assumed that the corrugation effectively couples only two contradirectional, TE-polarized, guided waves. These TE-polarized waves have an electric field component only along the

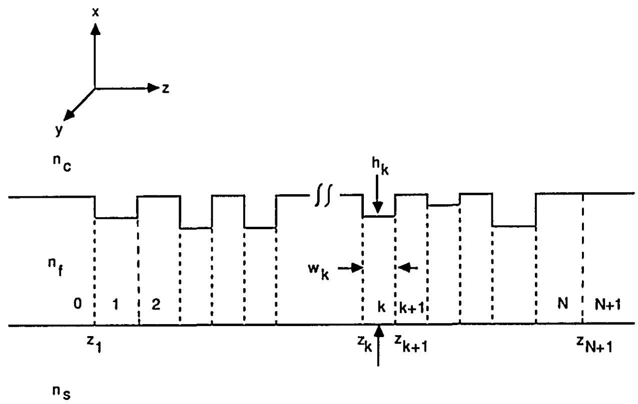

Fig. 1. Almost-periodic waveguide grating.

y direction. We write the y component of the electric field in the kth slab as

$$E_k(x, y, z) = E_{Fk}(x, y, z) + E_{Bk}(x, y, z), \tag{1}$$

where

$$E_{Fk}(x, y, z) = A_k E_{Tk}(x) \exp(-i\beta_k z), \tag{2}$$

$$E_{Bk}(x, y, z) = B_k E_{Tk}(x) \exp(i\beta_k z), \tag{3}$$

and E_{Fk} and E_{Bk} denote the forward ($+z$ direction) and backward ($-z$ direction) propagating fields. The time dependence of the fields is assumed to be $\exp(i\omega t)$. $E_{Tk}(x)$ and β_k are the transverse-mode profile and the propagation constant, respectively, for an uncorrugated guide of height h_k.

The propagation constant β_k can be computed by using the standard, three-layer guide, dispersion equation given below[15]:

$$V_k(1 - b_k)^{1/2} = \tan^{-1}[(b_k + a)/(1 - b_k)]^{1/2} + \tan^{-1}[b_k/(1 - b_k)]^{1/2} + \nu\pi, \tag{4}$$

where

$$V_k = \frac{2\pi}{\lambda} h_k (n_f^2 - n_s^2)^{1/2}, \tag{5}$$

$$a = (n_s^2 - n_c^2)/(n_f^2 - n_s^2), \tag{6}$$

$$b_k = \frac{N_k^2 - n_s^2}{n_f^2 - n_s^2}, \tag{7}$$

$$\beta_k = \frac{2\pi}{\lambda} N_k, \tag{8}$$

and ν is an integer denoting the mode number. Observe that N_k is the effective index of the kth slab.

Using Maxwell's equation, we obtain

$$\nabla \times \mathbf{E} = -i\omega u \mathbf{H}, \tag{9}$$

where u is the permeability of free space. The x component of the magnetic field in the kth slab,

$H_k(x, y, z)$, is given by

$$H_k(x, y, z) = H_{Fk}(x, y, z) + H_{Bk}(x, y, z), \tag{10}$$

where

$$H_{Fk}(x, y, z) = \frac{-\beta_k}{\omega u} E_{Fk}(x, y, z), \tag{11}$$

$$H_{Bk}(x, y, z) = \frac{\beta_k}{\omega u} E_{Bk}(x, y, z). \tag{12}$$

Let z_k^- and z_k^+ denote the values of z just to the left of and to the right of z_k, respectively, where z_k lies on the boundary between the $(k-1)$th and kth slabs. The electromagnetic boundary conditions require that E_k and H_k be continuous across the interface between the $(k-1)$th and kth slabs. Therefore, it follows from Eqs. (1), (8), and (10)–(12) that

$$\mathscr{V}_{k-1}\begin{bmatrix} E_{Fk-1}(x, y, z_k^-) \\ E_{Bk-1}(x, y, z_k^-) \end{bmatrix} = \mathscr{V}_k \begin{bmatrix} E_{Fk}(x, y, z_k^+) \\ E_{Bk}(x, y, z_k^+) \end{bmatrix}, \tag{13}$$

where the matrix $\mathscr{V}_k$ is given by

$$\mathscr{V}_k = \begin{bmatrix} 1 & 1 \\ -N_k & N_k \end{bmatrix}. \tag{14}$$

It also follows immediately from Eqs. (2) and (3) that propagation across the kth slab can be written as

$$\begin{bmatrix} E_{Fk}(x, y, z_k^+) \\ E_{Bk}(x, y, z_k^+) \end{bmatrix} = \mathscr{U}_k \begin{bmatrix} E_{Fk}(x, y, z_{k+1}^-) \\ E_{Bk}(x, y, z_{k+1}^-) \end{bmatrix}, \tag{15}$$

where the matrix $\mathscr{U}_k$ is given by

$$\mathscr{U}_k = \begin{bmatrix} \exp\left[w_k\left(i\frac{2\pi}{\lambda}N_k - g_k\right)\right] & 0 \\ 0 & \exp\left[-w_k\left(i\frac{2\pi}{\lambda}N_k - g_k\right)\right] \end{bmatrix}, \tag{16}$$

and g_k denotes the material gain, if any, in the kth slab. Combining Eqs. (13) and (15) yields

$$\begin{bmatrix} E_{Fk-1}(x, y, z_k^-) \\ E_{Bk-1}(x, y, z_k^-) \end{bmatrix} = (\mathscr{V}_{k-1}^{-1}\mathscr{V}_k\mathscr{U}_k) \begin{bmatrix} E_{Fk}(x, y, z_{k+1}^-) \\ E_{Bk}(x, y, z_{k+1}^-) \end{bmatrix}. \tag{17}$$

Defining the one period transfer matrix $\mathscr{M}_k$ as

$$\mathscr{M}_k = \mathscr{V}_{k-1}^{-1}\mathscr{V}_k\mathscr{U}_k \tag{18}$$

and using Eq. (17) repeatedly, we can write

$$\begin{bmatrix} E_{F0}(x, y, z_1^-) \\ E_{B0}(x, y, z_1^-) \end{bmatrix} = \left(\prod_{k=1}^{N/2} \mathscr{M}_{2k-1}\mathscr{M}_{2k}\right) \begin{bmatrix} E_{FN}(x, y, z_{N+1}^-) \\ E_{BN}(x, y, z_{N+1}^-) \end{bmatrix}, \tag{19}$$

where we have assumed that N is even. Thus the $\mathscr{M}_k$ matrices completely characterize the propagation through the waveguide grating. Their product relates the fields at the beginning of the grating to those at the end. For example, the reflection coefficient $r(\lambda)$, as defined by

$$r(\lambda) = \frac{E_{B0}(x, y, z_1^-)}{E_{F0}(x, y, z_1^-)}, \tag{20}$$

with $E_{BN}(x, y, z_{N+1}^-)$ set to 0, can be calculated directly from the $\mathscr{M}_k$ matrices by using Eq. (19).

III. Evaluation of $\mathscr{M}_k$

Combining Eqs. (8), (14), (16), and (18) yields

$$\mathscr{M}_k = \begin{bmatrix} \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\exp[w_k(i\beta_k - g_k)] & \frac{\Delta\beta_k}{2\beta_{k-1}}\exp[-w_k(i\beta_k - g_k)] \\ \frac{\Delta\beta_k}{2\beta_{k-1}}\exp[w_k(i\beta_k - g_k)] & \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\exp[-w_k(i\beta_k - g_k)] \end{bmatrix}, \tag{21}$$

where

$$\Delta\beta_k = \beta_{k-1} - \beta_k. \tag{22}$$

Therefore, the transfer matrix for a single period of the surface corrugation is given by

$$\mathscr{M}_k\mathscr{M}_{k+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad k = 1, 3, 5 \ldots, \tag{23}$$

where

$$a = \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\left(1 - \frac{\Delta\beta_{k+1}}{2\beta_k}\right)\exp[i(\gamma_k w_k + \gamma_{k+1}w_{k+1})] + \frac{\Delta\beta_k}{2\beta_{k-1}}\frac{\Delta\beta_{k+1}}{2\beta_k}\exp[-i(\gamma_k w_k - \gamma_{k+1}w_{k+1})], \tag{24}$$

$$b = \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\frac{\Delta\beta_{k+1}}{2\beta_k}\exp[i(\gamma_k w_k - \gamma_{k+1}w_{k+1})] + \left(1 - \frac{\Delta\beta_{k+1}}{2\beta_k}\right)\frac{\Delta\beta_k}{2\beta_{k-1}}\exp[-i(\gamma_{k+1}w_{k+1} + \gamma_k w_k)], \tag{25}$$

$$c = \left(1 - \frac{\Delta\beta_{k+1}}{2\beta_k}\right)\frac{\Delta\beta_k}{2\beta_{k-1}}\exp[i(\gamma_k w_k + \gamma_{k+1}w_{k+1})] + \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\frac{\Delta\beta_{k+1}}{2\beta_k}\exp[-i(\gamma_k w_k - \gamma_{k+1}w_{k+1})], \tag{26}$$

$$d = \left(1 - \frac{\Delta\beta_k}{2\beta_{k-1}}\right)\left(1 - \frac{\Delta\beta_{k+1}}{2\beta_k}\right)\exp[-i(\gamma_k w_k + \gamma_{k+1}w_{k+1})] + \frac{\Delta\beta_k}{2\beta_{k-1}}\frac{\Delta\beta_{k+1}}{2\beta_k}\exp[i(\gamma_k w_k - \gamma_{k+1}w_{k+1})], \tag{27}$$

$$\gamma_k = \beta_k + ig_k. \tag{28}$$

The expressions for a, b, c, and d given above can be simplified. First, we observe that for an almost-periodic grating,

$$w_k \approx w_{k+1}, \tag{29}$$

$$\Delta\beta_k \approx -\Delta\beta_{k+1}, \tag{30}$$

$$g_k \approx g_{k+1}. \tag{31}$$

Second, we assume that the depth of the surface corrugation is small compared with the average waveguide height so that

$$\frac{|\Delta\beta_k|}{\beta_{k-1}} \ll 1 \tag{32}$$

and that the gain per slab section is small so that

$$g_k w_k \ll 1. \tag{33}$$

Finally the deviation δ_k from the local first-order Bragg condition is defined by

$$2\delta_k = (\beta_k + \beta_{k+1}) - \frac{\pi}{w_k}, \qquad k = 1, 3, 5, \ldots, \tag{34}$$

and we assume that the Bragg condition is nearly satisfied, or equivalently,

$$|\delta_k w_k| \ll 1. \tag{35}$$

Combining Eqs. (24)–(35) now yields the following first-order expressions for a, b, c, and d:

$$a \approx -\exp[2w_k(i\delta_k - g_k)], \tag{36}$$

$$b \approx -\frac{\Delta\beta_k}{\beta_k}, \tag{37}$$

$$c \approx -\frac{\Delta\beta_k}{\beta_k}, \tag{38}$$

$$d \approx -\exp[-2w_k(i\delta_k - g_k)]. \tag{39}$$

In deriving relations (36)–(39) we used the fact that

$$e^q \approx 1 + q \quad \text{for } |q| \ll 1$$

and neglected terms of the form $(\Delta\beta/\beta)^{p1}(\delta w)^{p2}\,(gw)^{p3}$ when $p1 + p2 + p3 > 1$. This degree of approximation is consistent with the conditions given by inequalities (32), (33), and (35). Combining Eqs. (17), (18), and (23) and approximations (36)–(39) yields

$$\begin{bmatrix} E_{Fk-1}(x, y, z_k^-) \\ E_{Bk-1}(x, y, z_k^-) \end{bmatrix} \approx \begin{bmatrix} -\exp[2w_k(i\delta_k - g_k)] & -\dfrac{\Delta\beta_k}{\beta_k} \\ -\dfrac{\Delta\beta_k}{\beta_k} & -\exp[-2w_k(i\delta_k - g_k)] \end{bmatrix} \times \begin{bmatrix} E_{Fk+1}(x, y, z_{k+2}^-) \\ E_{Bk+1}(x, y, z_{k+2}^-) \end{bmatrix}. \tag{40}$$

We now define the functions $R(z)$ and $S(z)$ as follows:

$$E_{Fy}(x, y, z) = R(z)\exp[i\delta(z)z]\exp[-i\beta_{av}(z)z], \tag{41}$$

$$E_{By}(x, y, z) = S(z)\exp[-i\delta(z)z]\exp[i\beta_{av}(z)z], \tag{42}$$

where

$$\beta_{av}(z) = \frac{\beta_k + \beta_{k+1}}{2} \quad \text{for} \quad z_k < z \le z_{k+2}, \quad k = 1, 3, 5, \ldots, \tag{43}$$

$$\delta(z) = \frac{\beta_k + \beta_{k+1}}{2} - \frac{\pi}{2w_k} = \delta_k \quad \text{for} \quad z_k < z \le z_{k+2}, \quad k = 1, 3, 5, \ldots. \tag{44}$$

Combining relations (29), (34), and (40) and Eqs. (41)–(44) (along with the fact that $\delta_{k-1} \approx \delta_k$ and $\beta_{k-1} \approx \beta_k \approx \beta_{k+1}$ for an almost-periodic grating) yields

$$R(z_k^-) \approx \exp[2w_k(i\delta_k - g_k)\,R(z_{k+2}^-) + \frac{\Delta\beta_k}{\beta_k}\exp\left(i\frac{\pi}{w_k}z_k\right)S(z_{k+2}^-), \tag{45}$$

$$S(z_k^-) \approx \frac{\Delta\beta_k}{\beta_k}\exp\left(-i\frac{\pi}{w_k}z_k\right)R(z_{k+2}^-) + \exp[-2w_k(i\delta_k - g_k)]\,S(z_{k+2}^-). \tag{46}$$

Without loss of generality we set z_1 (the z coordinate at the beginning of the grating) to zero; then

$$z_k = \sum_{j=1}^{k-1} w_j. \tag{47}$$

The grating is assumed to be almost periodic. Therefore

$$\exp\left(\pm i\frac{\pi}{w_k}z_k\right) \approx \exp[\pm i(k-1)\pi] = \begin{cases} -1 & \text{for } k \text{ even}, \\ +1 & \text{for } k \text{ odd}. \end{cases} \tag{48}$$

Combining approximations (45), (46), and (48) yields to first order

$$\begin{bmatrix} R(z_k^-) \\ S(z_k^-) \end{bmatrix} \approx \begin{bmatrix} \exp[2w_k(i\delta_k - g_k)] & \dfrac{\Delta\beta_k}{\beta_k} \\ \dfrac{\Delta\beta_k}{\beta_k} & \exp[-2w_k(i\delta_k - g_k)] \end{bmatrix} \begin{bmatrix} R(z_{k+2}^-) \\ S(z_{k+2}^-) \end{bmatrix} \tag{49}$$

for k odd.

IV. Coupled-Mode Theory

Based on a coupled-mode theory analysis, $R(z)$ and $S(z)$, as defined by Eqs. (41) and (42), are solutions to the following pair of differential equations[15]:

$$R'(z) + [i\delta(z) - g(z)]R(z) = -\kappa(z)S(z), \tag{50}$$

$$S'(z) - [i\delta(z) - g(z)]S(z) = -\kappa(z)R(z), \tag{51}$$

where the prime denotes differentiation with respect to z and $\kappa(z)$ is the coupling coefficient. For TE-mode propagation in the guide, the coupling coefficient $\kappa(z)$ for an almost-periodic, square-wave, surface corrugation (see Fig. 1) is given by[15]

$$\kappa_k = \kappa(z) \approx \frac{2}{\lambda}\frac{h_{k+1} - h_k}{h_{\text{eff}_k}}\frac{n_f^2 - N_k^2}{N_k} \quad \text{for } z_k < z \le z_{k+2}, \quad k = 1, 3, 5, \ldots, \tag{52}$$

where h_{eff_k} is the effective height of the guide and is approximately equal to[15]

$$h_{\text{eff}_k} \approx \frac{V_k + b_k^{-1/2} + (b_k + a)^{-1/2}}{\frac{2\pi}{\lambda}(n_f^2 - n_s^2)^{1/2}}, \qquad k = 1, 3, 5, \ldots. \tag{53}$$

It is assumed in approximation (52) that the corrugation depth is shallow, and therefore the electric field of the uncorrugated waveguide mode is taken to be constant over the corrugation depth. For operation near the first-order Bragg wavelength, Eqs. (8) and (34) and inequalities (32) and (35) can be combined to yield

$$\frac{2}{\lambda} N_k 2w_k \approx 1. \tag{54}$$

Thus it follows from approximation (52) and approximation (54) that

$$\kappa_k 2w_k \approx \frac{h_{k+1} - h_k}{h_{\text{eff}_k}} \frac{n_f^2 - N_k^2}{N_k^2}, \quad k = 1, 3, 5, \ldots. \tag{55}$$

It is now shown that

$$\kappa_k 2w_k \approx \frac{\Delta\beta_k}{\beta_k}, \quad k = 1, 3, 5, \ldots. \tag{56}$$

We start by differentiating between both sides of Eq. (4) with respect to h. The result is

$$\frac{dV}{dh}[1 - b]^{1/2} - \frac{1}{2} V \frac{db}{dh} \frac{1}{(1 - b)^{1/2}} = \frac{1}{1 + \frac{b + a}{1 - b}} \frac{1}{2} \left(\frac{1 - b}{b + a}\right)^{1/2}$$
$$\times \frac{(1 - b) + (b + a)}{(1 - b)^2} \frac{db}{dh} + \frac{1}{1 + \frac{b}{1 - b}} \frac{1}{2} \left(\frac{1 - b}{b}\right)^{1/2} \frac{(1 - b) + b}{(1 - b)^2} \frac{db}{dh}$$
$$= \frac{1}{2} \frac{1}{(1 - b)^{1/2}} \left[\frac{1}{b^{1/2}} + \frac{1}{(b + a)^{1/2}}\right] \frac{db}{dh}. \tag{57}$$

It follows from Eqs. (5) and (7) that

$$\frac{dV}{dh} = \frac{2\pi}{\lambda}(n_f^2 - n_s^2)^{1/2}, \tag{58}$$

$$\frac{dN}{db} = \frac{(n_f^2 - n_s^2)}{2N}, \tag{59}$$

$$1 - b = \frac{n_f^2 - N^2}{n_f^2 - n_s^2}. \tag{60}$$

Multiplying both sides of Eq. (57) by $[1 - b]^{1/2}$ (dh) (dN/db) and then using Eqs. (58)–(60) yield

$$\frac{n_f^2 - N^2}{N} dh = \frac{[V + b^{-1/2} + (b + a)^{-1/2}]}{\frac{2\pi}{\lambda}[n_f^2 - n_s^2]^{1/2}} dN. \tag{61}$$

It now follows from Eqs. (8) and (61) and approximation (53) that

$$\frac{d\beta}{\beta} = \frac{dN}{N} = \frac{n_f^2 - N^2}{N^2} \frac{dh}{h_{\text{eff}}}. \tag{62}$$

Combining approximation (55) and Eq. (62) yields the desired result

$$\kappa_k 2w_k \approx \frac{\Delta\beta_k}{\beta_k}, \quad k = 1, 3, 5, \ldots. \tag{56}$$

Since $R(z)$ and $S(z)$ change little over the distance of one grating period, we can write

$$R(z_k^-) \approx R(z_{k+2}^-) - R'(z_{k+2}^-)2w_k, \tag{63}$$

$$S(z_k^-) \approx S(z_{k+2}^-) - S'(z_{k+2}^-)2w_k. \tag{64}$$

Combining Eqs. (50) and (51) and approximations (63) and (64) yields

$$R(z_k^-) \approx R(z_{k+2}^-) + 2w_k(i\delta_k - g_k)R(z_{k+2}^-) + \kappa_k 2w_k S(z_{k+2}^-), \tag{65}$$

$$S(z_k^-) \approx S(z_{k+2}^-) - 2w_k(i\delta_k - g_k)S(z_{k+2}^-) + \kappa_k 2w_k R(z_{k+2}^-). \tag{66}$$

Recall from inequalities (33) and (35) that $g_k w_k \ll 1$ and $|\delta_k w_k| \ll 1$. Therefore,

$$\exp[\pm 2w_k(i\delta_k - g_k)] \approx 1 \pm 2w_k(i\delta_k - g_k). \tag{67}$$

Combining approximations (56) and (65)–(67) yields

$$\begin{bmatrix} R(z_k^-) \\ S(z_k^-) \end{bmatrix} \approx \begin{bmatrix} \exp[2w_k(i\delta_k - g_k)] & \frac{\Delta\beta_k}{\beta_k} \\ \frac{\Delta\beta_k}{\beta_k} & \exp[-2w_k(i\delta_k - g_k)] \end{bmatrix} \times \begin{bmatrix} R(z_{k+2}^-) \\ S(z_{k+2}^-) \end{bmatrix}. \tag{68}$$

Finally we observe that and approximations (68) (49) are identical. Therefore an identical result for the one-period transfer matix $\mathscr{M}_k \mathscr{M}_{k+1}$ is derived by using either the effective-index/impedance-matching technique or coupled-mode theory.

Approximations (49) and (68) indicate that the effective-index method and coupled-mode theory yield nearly identical results for almost-periodic corrugated waveguides in the vicinity of the first-order Bragg wavelength. Our analysis, however, has been restricted to grating profiles that have a square shape. In coupled-mode theory an arbitrary grating profile is handled by decomposing the profile into its Fourier series components. The fundamental component has the same grating period as the original profile, and the periods of the remaining components are integer fractions of the fundamental. It is well known in coupled-mode theory that these remaining components can be neglected, since they do not achieve phase matching at the first-order Bragg wavelength. Thus a shallow grating profile of arbitrary shape can be replaced by an equivalent square profile.

The only restriction is that the fundamental Fourier components of the two profiles have the same amplitude and period. Therefore the effective-index method and coupled-mode theory will produce nearly identical results even for shallow grating profiles of arbitrary shape. The effective-index analysis, however, must be performed on the equivalent square profile as defined above.

V. Recursion Methods

A simple recursive technique is widely used in thin-film design to determine the reflection coefficient of a multilayer dielectric structure in terms of the reflection coefficients of the individual layers.[26] This recursive procedure is often referred to as the Airy summation technique or Rouard's method. In Refs. 17 and 18 an Airy-like summation procedure is used to analyze corrugated waveguides, but no mathematical justification for the technique is provided. Below we show that the recursion method is equivalent to first order to a numerical integration of the coupled-mode equations.

Using approximation (67), we can write approximations (65) and (66) in the following matrix form:

$$\begin{bmatrix} R(z_k^-) \\ S(z_k^-) \end{bmatrix} \simeq \begin{bmatrix} \exp[2w_k(i\delta_k - g_k)] & 2w_k\kappa_k \\ 2w_k\kappa_k & \exp[-2w_k(i\delta_k - g_k)] \end{bmatrix} \times \begin{bmatrix} R(z_{k+2}^-) \\ S(z_{k+2}^-) \end{bmatrix}. \quad (69)$$

If we define the reflection coefficient r_k of the combined k and $k+1$ layers as

$$r_k = \left.\frac{S(z_k^-)}{R(z_k^-)}\right|_{S(z_{k+2}^-)=0}, \quad (70)$$

it follows from Eq. (69) that

$$r_k = 2w_k\kappa_k \exp[-2w_k(i\delta_k - g_k)]. \quad (71)$$

Furthermore, if we let ρ_k denote the reflection coefficient of the grating as seen when looking in the positive z direction at position $z = z_k^-$, then

$$\rho_k = \left.\frac{S(z_k^-)}{R(z_k^-)}\right|_{S(z_{N+1}^-)=0}. \quad (72)$$

Combining Eqs. (69) and (72) yields

$$\begin{bmatrix} R(z_k^-) \\ \rho_k R(z_k^-) \end{bmatrix} \simeq \begin{bmatrix} \exp[2w_k(i\delta_k - g_k)] & 2w_k\kappa_k \\ 2w_k\kappa_k & \exp[-2w_k(i\delta_k - g_k)] \end{bmatrix} \times \begin{bmatrix} R(z_{k+2}^-) \\ \rho_{k+2} R(z_{k+2}^-) \end{bmatrix}. \quad (73)$$

It immediately follows from approximation (73) that

$$\rho_k \simeq \frac{2w_k\kappa_k + \rho_{k+2}\exp[-2w_k(i\delta_k - g_k)]}{\exp[2w_k(i\delta_k - g_k)] + \rho_{k+2}2w_k\kappa_k}. \quad (74)$$

Equation (71) and approximation (74) may now be combined to yield

$$\rho_k \simeq \frac{r_k + \rho_{k+2}\exp[-4w_k(i\delta_k - g_k)]}{1 + r_k\rho_{k+2}}. \quad (75)$$

Approximation (75) is the desired recursion relationship, and the above analysis indicates that it is equivalent to first order to a numerical solution of the coupled-mode equations.

VI. TM Case

We have shown that for TE polarization the combined effective-index/impedance-matching technique yields the same result as the coupled-mode theory does. The development shown above can be paralleled for the TM polarization, and approximation (49) is obtained unaltered. It is known, however, that the coupling coefficients are different for TE and TM polarization.[15] Therefore the effective-index method and the coupled-mode theory methods do not yield the same results for TM polarization.[24] As noted by Verly *et al.*[24] this occurs because the effective-index method does not account for the boundary conditions at the grating–cover plate interface. For TE polarization this is no problem, since the **E** and **H** fields are continuous across this interface. For TM polarization, however, there is a periodic discontinuity in E_x at the interface. This periodic discontinuity gives rise to an additional coupling term, which the effective-index method neglects. It is easy to show that the coupled-mode theory and the combined effective-index/impedance-matching technique will yield identical results provided that the $\Delta\beta_k/\beta_k$ term in approximation (49) is reduced by the following factor:

$$\frac{N_{av}^2/n_f^2 - N_{av}^2/n_c^2 + 1}{N_{av}^2/n_f^2 + N_{av}^2/n_c^2 - 1}, \quad (76)$$

where

$$N_{av} = \frac{\lambda}{4\pi}(\beta_k + \beta_{k+1}), \quad k = 1, 3, 5 \ldots. \quad (77)$$

VII. Examples

Three corrugated waveguides have been chosen to compare the effective-index technique with the coupled-mode theory. We implemented the effective-index technique, using Eqs. (4)–(7), (14), (16), and (19), as described in Section II. The coupled-mode theory results were found by numerically integrating Eqs. (50) and (51), starting at the back of the grating and moving forward. The integration step size was $\sim 0.2\%$ of the grating length, and $\kappa(z)$ and $\delta(z)$ were evaluated by using Eqs. (4)–(8) and (34) and approximations (52) and (53). The results are shown in Figs. 2–4, where the reflectivity, $|E_{B0}(z_1^-)/E_{F0}(z_1^-)|^2$, is plotted versus wavelength. As expected there is excellent agreement between the two approaches.

The waveguide parameters shown in Figs. 2–4 are specified by using the notation of Fig. 1. In all three cases the film, substrate, and cover plate refractive indices are 1.55, 1.5, and 1.0, respectively, and the

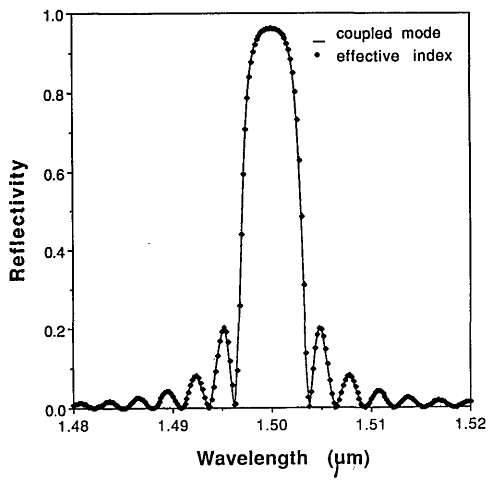

Fig. 2. Periodic grating $\{n_c = 1.0, n_f = 1.55, n_s = 1.5, w_k = 0.247$ μm, $h_k = [1.5 + 0.15(-1)^k]$μm, $k = 1, 2, \ldots, 1000$ layers$\}$.

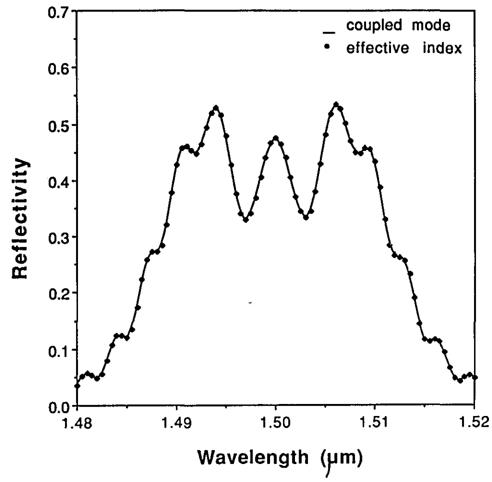

Fig. 4. Frequency-chirped grating $\{n_c = 1.0, n_f = 1.55, n_s = 1.5, w_k = [(-0.0025 + 0.0025k/500) + 0.247]$ μm, $h_k = [1.5 + 0.15(-1)^k]$ μm, $k = 1, 2, \ldots, 1000$ layers$\}$.

average film height, $(h_{k+1} + h_k)/2$, is 1.5 μm. These values correspond to silver ion-exchanged waveguides that are made in glass.[27] At a wavelength of 1.5 μm the effective index N_{eff} and the effective guide height h_{eff} are found to be 1.51822 and 2.724 μm, respectively. Waveguide 1 (see Fig. 2) has a constant corrugation period equal to 0.494 μm and a constant corrugation depth, i.e., $|h_{k+1} - h_k|$, of 0.30 μm. The device is 247 μm long and consists of 500 corrugation periods. The second waveguide (see Fig. 3) is also 247 μm long and consists of 500 corrugation periods. The corrugation depth, however, has a raised cosine taper, which yields a maximum depth of 0.30 μm at the center of the guide and no depth at either end. As expected, the amplitude taper reduces the sidelobe levels of the waveguide filter. The third waveguide (Fig. 4) has a constant corrugation depth of 0.30 μm. The period, however, varies linearly along the length of the guide, ranging from 0.2445 μm at the beginning to 0.2495 μm at the end. This variation corresponds to a 2% linear chirp over a length of 500 corrugation periods. As indicated in Fig. 4, the chirp broadens the spectral response of the waveguide filter.

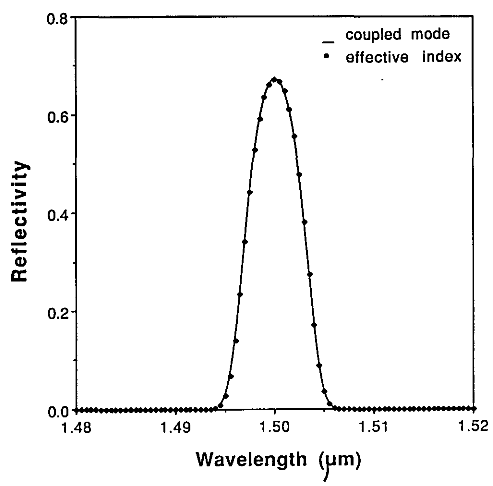

Fig. 3. Amplitude-tapered grating $(n_c = 1.0, n_f = 1.55, n_s = 1.5, w_k = 0.247$ μm, $h_k = [1.5 + 0.15(-1)^k]\ \{0.5 + 0.5\cos[\pi(k - 500)/500]\}$ μm, $k = 1, 2, \ldots, 1000$ layers).

VIII. Conclusions

Periodic and almost-periodic waveguide gratings have found applications as filters, DFB laser structures, phase-matching elements, grating couplers, pulse compressors, and soliton generators. Both coupled-mode theory and a combined effective-index/impedance-matching technique have been used to analyze propagation through these devices. The coupled-mode theory methods are older and have been applied more widely. The effective-index/impedance-matching technique, however, is conceptually simpler. In the matrix-based effective-index technique, the grating period region is divided into thin-slab sections. The mode propagation constants in each of the slabs are evaluated by using the standard b versus V dispersion relationships for a three-layer waveguide. A two-by-two transfer matrix for each section is then derived by matching the tangential **E** and **H** field components at the interface between slabs. We have shown by direct computation that the matrix-based effective-index/impedance-matching technique is equivalent to coupled-mode methods for TE polarization.

I thank the anonymous reviewers for their helpful comments. This work was supported in part by National Science Foundation grant ECS-8909802.

References

1. D. C. Flanders, H. Kogelnik, R. V. Schmidt, and C. V. Shank, "Grating filters for thin-film optical waveguides," Appl. Phys. Lett. **24**, 194–196 (1974).
2. H. Kogelnik, "Filter response of nonuniform almost-periodic structures," Bell Syst. Tech. J. **55**, 109–126 (1976).
3. M. Matsuhara, K. O. Hill, and A. Watanabe, "Optical-waveguide filters: synthesis," J. Opt. Soc. Am. **65**, 804–808 (1975).

4. G-H. Song and S-Y. Shin, "Design of corrugated waveguide filters by the Gel'Fand Levitan-Marchenko inverse-scattering method," J. Opt. Soc. Am. **A2,** 1905–1915 (1985).
5. K. A. Winick and J. E. Roman, "Design of corrugated waveguide filters By Fourier transform techniques," IEEE J. Quantum Electron. **26,** 1918–1929 (1990).
6. T. Schrans, M. Mittelstein, and A. Yariv, "Tunable chirped-corrugation waveguide filters," Appl. Phys. Lett. **55,** 212–214 (1989).
7. H. Kogelnik and C. V. Shank, "Coupled wave theory of distributed feedback lasers," J. Appl. Phys. **43,** 2327–2335 (1972).
8. H. A. Haus and C. V. Shank, "Antisymmetric taper of distributed feedback lasers," IEEE J. Quantum Electron. **QE-12,** 532–539 (1976).
9. B. U. Chen, C. C. Ghizoni, and C. L. Tang, "Phase-matched second harmonic generation in solid thin films using modulation of the nonlinear susceptibilities," Appl. Phys. Lett. **28,** 651–653 (1976).
10. M. L. Dakss, L. Kuhn, P. F. Heidrich, and B. A. Scott, "Grating coupler for efficient excitation of optical guided waves in thin films," Appl. Phys. Lett. **16,** 523–525 (1970).
11. H. G. Winful, "Pulse compression in optical fiber filters," Appl. Phys. Lett. **46,** 527–529 (1985).
12. C. Martijn de Sterke and J. E. Sipe, "Possibilities for the observation of gap solitons in waveguide geometries," J. Opt. Soc. Am. B **6,** 1722–1725 (1989).
13. S. T. Peng, H. L. Bertoni, and T. Tamir, "Analysis of periodic thin-film structures with rectangular profiles," Opt. Commun. **10,** 91–94 (1974).
14. S. T. Peng and T. Tamir, "Theory of periodic dielectric waveguides," IEEE Trans. Microwave Theory Tech. **MTT-23,** 123–133 (1975).
15. T. Tamir, ed., *Integrated Optics* (Springer-Verlag, New York, 1979), Chap. 2.
16. J. B. Shellan, C. S. Hong, and A. Yariv, "Theory of chirped gratings in broad band filters," Opt. Commun. **23,** 398–400 (1977).
17. L. A. Weller-Brophy and D. G. Hall, "Analysis of waveguide gratings: applications of Rouard's method," J. Opt. Soc. Am. **A2,** 863–871 (1985).
18. L. A. Weller-Brophy and D. G. Hall, "Analysis of waveguide gratings: a comparison of the results of Rouard's method and coupled-mode theory," J. Opt. Soc. Am. **A4,** 60–65 (1987).
19. M. Yamada and K. Sakuda, "Analysis of almost-periodic distributed feedback slab waveguides via a fundamental matrix approach," Appl. Opt. **26,** 3474–3478 (1987).
20. S. Wang, "Proposal of periodic layered waveguide structures for distributed lasers," J. Appl. Phys. **44,** 767–780 (1973).
21. A. Basu and J. M. Ballantyne, "Random fluctuations in first order waveguide grating filters," Appl. Opt. **18,** 2575–2579 (1979).
22. G. Bjork and O. Nilsson, "A new exact and efficient numerical matrix theory of complicated laser structures: properties of asymmetric phase-shifted DFB lasers," IEEE J. Lightwave Technol. **LT-5,** 140–146 (1987).
23. P. Verly, R. Tremblay, and J. W. Y. Lit, "Application of the effective index method to the study of distributed feedback in corrugated waveguides. TE polarization," J. Opt. Soc. Am. **70,** 964–968 (1980).
24. P. Verly, R. Tremblay, and J. W. Y. Lit, "Application of the effective index method to the study of distributed feedback in corrugated waveguides. TM polarization," J. Opt. Soc. Am. **70,** 1218–1221 (1980).
25. D. Marcuse, *Theory of Dielectric Waveguides* (Academic, New York, 1974).
26. M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1975), Sec. 1.6.
27. H. Nishihara, M. Haruna, and T. Suhara, *Optical Integrated Circuits* (McGraw-Hill, New York, 1988), Chap. 6.