Stellar scintillation technique for the measurement of tilt anisoplanatism

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The significance of tilt anisoplanatism is established, and a measurement theory, based on aperture-averaging intensity scintillation, is developed. The theory is a direct extention of the technique currently used to determine the isoplanatic angle as defined by Fried [J. Opt. Soc. am. 72, 52 (1982)]. By using this theory, a physically realizable binary aperture-weighting function is derived for a particular case of interest. It is noted that direct quantitative measurements of tilt anisoplanatism can also be made, under specific circumstances, by tracking the relative centroid motion of a binary star pair. Thus independent verification of the remote-sensing theory for tilt anisoplanatism, based on aperture-averaging scintillation measurements, should be possible.

1. INTRODUCTION

The performance of all optical systems depends on the field of view over which they are constrained to operate. In general, the term isoplanatic angle is used to note explicitly the field angle region over which the performance does not degrade by more than some specified amount. For conventional optical systems it is often loosely defined as the field of view over which spatial invariance, and hence a well-defined optical transfer function, can be assumed to exist.[1] Outside this limited region, the system is said to be anisoplanatic. Beyond what is implied by the loose definition given above, the terms isoplanatism and anisoplanatism are rarely quantitatively applied to conventional systems.

Isoplanatism takes on a fundamental role in the field of atmospheric optics, however, where its effects are apparent and of considerable importance.[1-9] For instance, even with a perfect ground-based adaptive optics telescope it is impossible to form a diffraction-limited image of an angularly extended astronomical body.[1-3] Optical rays from different regions of the body will pass through distinct sections of the atmosphere, each of which has its own optical transfer function. Fundamentally, the telescope cannot correct for these multiple transfer functions simultaneously, and a distorted image will result. This effect may become noticeable for different rays subtending angles as small as several arcseconds.

A similar problem exists when a ground-based adaptive optics transmitter is pointed toward some distant object. In most cases the adaptive compensation for the transmitter would be determined by observing the atmospherically induced image distortion present in light received form a nearby optical source, such as a star.[4] Unfortunately, if the angle between the star and the object exceeds the isoplanatic angle, then the receive and transmit paths will experience different atmospheric distortions. Therefore the required transmitter correction cannot be precisely inferred from this observation, and the transmitted power reaching the object will not be maximum. Under these conditions and assuming a large transmit–receive aperture, Fried[4] has defined the isoplanatic angle ϑ_0 as

> the angle between the received and transmitted paths for which the average far-field transmit power will be reduced from its maximum value by a factor of $1/e$.

In space-to-Earth optical communications systems, anisoplanatism also plays a significant role.[5-7] Because both the satellite and the ground station are in relative motion in inertial space, the uplink and downlink paths will not be collinear. In fact, the uplink beam will have to be pointed ahead of the received downlink beam in order to reach the moving satellite. This situation is analogous to that of a marksman pointing ahead to strike a moving target. Thus a ground-based adaptive optics communications transmitter cannot perfectly correct the uplink beam by observing the downlink from the satellite.

In many situations, the dominant effect of atmospheric turbulence is to introduce random angle-of-arrival fluctuations into the transmitted or received beam.[1,10,11] Some adaptive optical systems are designed to compensate only for these tilt fluctuations. This is particularly true when the diameter of the receive–transmit aperture is not much larger than transverse coherence length of the atmosphere, because tilt is the dominant degrading effect in this case. Therefore a useful quantity is the anisoplanatic tilt error θ_0, defined as the root-mean-squared difference in the angle-of-arrival fluctuations observed along two distinct propagation paths. This quantity will be a function of the receive–transmit aperture diameter D, the zenith angle α, and the angular separation $\delta\alpha$ between the two paths as well as of the properties of the propagation medium. In principle, θ_0 can be measured directly by tracking the relative centroid motion of a binary star pair, having angular separation $\delta\alpha$, when viewed through a single telescope.[12] Of course, such a binary star pair of sufficient brightness may not exist, and thus an alternative measurement technique is required.

The importance of isoplantatic effects in the field of atmospheric optics has been recognized for some time, but it was only recently that quantitative measurement techniques were proposed. Two problems have hindered this effort. First, some quantities, such as the isoplanatic angle ϑ_0, are virtually impossible to measure directly because spaceborne equipment is required. Second, in order to evaluate the theoretical expressions corresponding to these quantities, the refractive structure constant profile of the atmosphere must be known, and in general it is not. Therefore techniques for quantifying isoplanatic effects such as ϑ_0 must be indirect.

It has been noted by many authors (see, e.g., Refs. 13–21) that the spatial and temporal properties of stellar scintillations, as observed on Earth, can be used to infer information about the atmospheric refractive structure profile. In particular, Loos and Hogge,[19] using aperture-averaging results derived by Fried,[22] have developed an elegant approach for determining ϑ_0. Walters[23] and Stevens,[24] along with Eaton *et al.*,[25] have refined the method and constructed equipment based on the theory. The basic idea is to relate the isoplanatic angle to the variance of stellar scintillations measured through a telescope whose aperture has been weighted by a spatially varying intensity mask.

The purpose of this paper is to show that the above aperture-averaging scintillation approach can be extended to measure the anisoplanatic tilt error θ_0. As noted above, θ_0 is a fundamental quantity for adaptive optic systems that compensate only for tilt. In addition, for point-ahead angles $\delta\alpha$ corresponding to binary star pair separations, the anisoplanatic tilt error can be measured directly by tracking the relative centroid motion of these two stars. Thus independent verification of the remote-sensing theory for anisoplanatism, based on aperture-averaging scintillation measurements, should be possible. This same opportunity for verification does not present itself when only isoplanatic-angle (ϑ_0) data are available.

The remainder of this paper is divided into four sections. In Section 2 an analytic expression for the anisoplanatic tilt error θ_0 is derived. A similar expression was published previously,[1] but the result contains a typographical error. Section 3 derives the variance of the stellar scintillations in terms of the telescope's spatial aperture intensity mask. Section 4 combines the results of the two previous sections to produce an integral equation representation of the mask required to measure the anisoplanatic tilt error associated with a 17.4-μrad point-ahead angle. This point-ahead angle corresponds to an equatorial ground station (located at the subsatellite point) communicating with a satellite in geosynchronous orbit. A solution for this integral equation is also given. Finally, Section 5 summarizes the results of the paper.

2. ANISOPLANATIC TILT ERROR

The atmospherically induced, random tilt imparted to a wave front propagating at a zenith angle α_i can be modeled as

$$\frac{2\pi}{\lambda}(a_{xi}x + a_{yi}y), \tag{1}$$

where λ is the wavelength, x and y are the wave-front spatial coordinates, and a_{xi} and a_{yi} are the x and y atmospherically induced tilt components, respectively. For relatively small apertures, removal of the tilt component given by expression (1) closely corresponds to tracking the centroid of the degraded image.[26–28] Consider now a transmit–receive aperture whose size is defined by the pupil function

$$P_0(\mathbf{r}) = \begin{cases} 1, & |\mathbf{r}| \le 0.5D \\ 0, & |\mathbf{r}| > 0.5D \end{cases}, \tag{2}$$

where D is the diameter of the aperture and $\mathbf{r} = (x, y)$. Fried[10] has shown that over the aperture the x and y tilts are given by

$$a_{xi} = \frac{\lambda}{2\pi}\left(\frac{\pi D^4}{64}\right)^{-1} \int xP_0(\mathbf{r})\phi_i(\mathbf{r})\mathrm{d}\mathbf{r}, \tag{3a}$$

$$a_{yi} = \frac{\lambda}{2\pi}\left(\frac{\pi D^4}{64}\right)^{-1} \int yP_0(\mathbf{r})\phi_i(\mathbf{r})\mathrm{d}\mathbf{r}, \tag{3b}$$

where $\phi_i(\mathbf{r})$ is the phase disturbance introduced into the ith wave front by atmospheric turbulence.

The anisoplanatic tilt error θ_0 for two wave fronts having angular separation $\delta\alpha = \alpha_1 - \alpha_2$ is defined as

$$\theta_0^2 = \langle (a_{x1} - a_{x2})^2 + (a_{y1} - a_{y2})^2 \rangle, \tag{4}$$

where $\langle\ \rangle$ denotes the expectation operator. Fried[1] has evaluated Eq. (4), producing the following result:

$$\theta_0^2 = 2.91\left(\frac{64}{\pi}\right)^2 D^{-1/3} \int_0^\infty C_n^2(z) \iint W(\mathbf{v} + 0.5\mathbf{u}) \times W(\mathbf{v} - 0.5\mathbf{u})(v^2 - 0.25u^2)\left[0.5\left|\mathbf{u} + \frac{z\boldsymbol{\delta\alpha}}{D}\right|^{5/3} + 0.5\left|\mathbf{u} - \frac{z\boldsymbol{\delta\alpha}}{D}\right|^{5/3} - u^{5/3} - \left(\frac{z\delta\alpha}{D}\right)^{5/3}\right]\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{u}\mathrm{d}z, \tag{5}$$

where

$$u = |\mathbf{u}|, \qquad v = |\mathbf{v}|, \qquad \delta\alpha = |\boldsymbol{\delta\alpha}|,$$
$$W(\mathbf{r}) = \begin{cases} 1, & |\mathbf{r}| \le 0.5 \\ 0, & |\mathbf{r}| > 0.5 \end{cases}, \tag{6}$$

$C_n^2(z)$ is the atmospheric refractive-structure constant given as a function of slant range z measured along the propagation path from the Earth, and $\boldsymbol{\delta\alpha}$ is a vector of magnitude $\delta\alpha$ equal to the angular separation between the paths and with direction defined by the intersection of the plane containing the propagation paths for wave fronts 1 and 2 and the transmit–receive aperture plane. In Ref. 1 the $\mathbf{v}$ integration in Eq. (5) above is performed without derivation, and the published result contains a typographical error. The correct expression, which is derived in Appendix A, should read as follows:

$$\int W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})(v^2 - 0.25u^2)\mathrm{d}\mathbf{v} = \tfrac{1}{16}[\cos^{-1}(u) - (3u - 2u^3)(1 - u^2)^{1/2}]. \tag{7}$$

When Eqs. (5) and (7) are combined, the anisoplanatic tilt error becomes

$$\theta_0^2 = 2.91\left(\frac{16}{\pi}\right)^2 D^{-1/3} \int_0^\infty C_n^2(z) f_\alpha(z)\mathrm{d}z, \tag{8}$$

where

$$f_\alpha(z) = \int_0^{2\pi}\int_0^1 u[\cos^{-1}(u) - (3u - 2u^3)(1 - u^2)^{1/2}] \times \{0.5[u^2 + 2us\cos(\omega) + s^2]^{5/6} + 0.5[u^2 - 2us\cos(\omega) + s^2]^{5/6} - u^{5/3} - s^{5/3}\}\mathrm{d}u\mathrm{d}\omega, \tag{9}$$

$$s = z\delta\alpha/D. \tag{10}$$

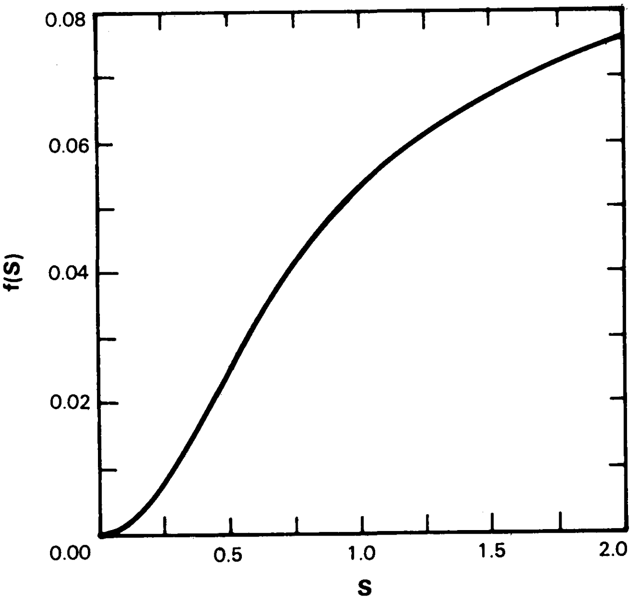

Fig. 1. Anisoplanatic tilt weighting function.

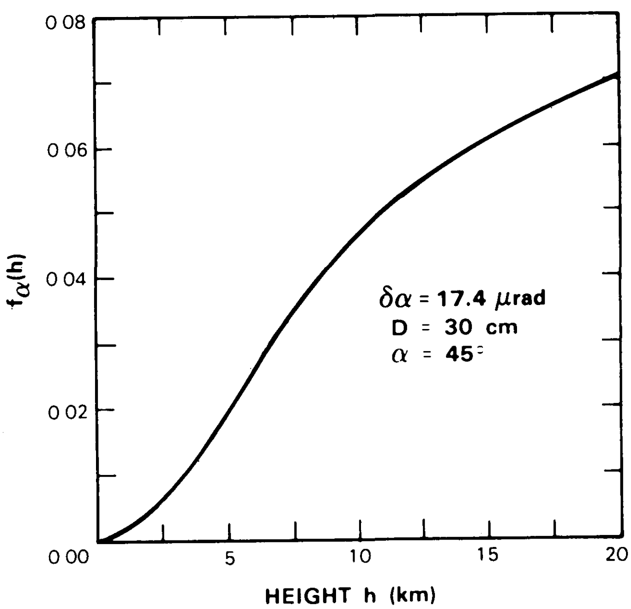

Fig. 2. Anisoplanatic tilt weighting function versus altitude.

It is observed from Eq. (8) that the anisoplanatic tilt error is independent of wavelength.

Since $\alpha_1 \cong \alpha_2$, the nominal zenith propagation angle α is given by

$$\alpha = 0.5(\alpha_1 + \alpha_2). \tag{11}$$

Therefore, if h denotes the height above the Earth, the zenith-angle dependence for the anisoplanatic tilt error can be written as

$$\theta_0{}^2 = 2.91\left(\frac{16}{\pi}\right)^2 D^{-1/3}\sec(\alpha)\int_0^\infty C_n{}^2(h)f_\alpha(h)\mathrm{d}h, \tag{12}$$

where

$$h = z\cos(\alpha). \tag{13}$$

Although we do not know how to simplify Eq. (9), we can evaluate it numerically as a function of s. This evaluation has been performed for $0 \le s \le 2$, and the results are plotted in Fig. 1. This range of s is adequate for a number of practical cases. As an example, the point-ahead angle ($\delta\alpha$) for a ground station-to-geosynchronous-satellite path lies between 17 and 21 μrad. The exact value depends on the relative position between the ground station and the satellite. Considering a transmit–receive aperture diameter (D) of 30 cm and a slant-path zenith angle of 45°, it follows that s is less than 2 for all altitudes at which there is any significant atmosphere (i.e., $h < 20{,}000$ m). The parameters described above correspond to an optical heterodyne communications system designed by the MIT Lincoln Laboratory. Using the parameter values given above, $f_\alpha(h)$ is plotted in Fig. 2.

3. APERTURE AVERAGED INTENSITY SCINTILLATION

Equation (12) indicates how the anisoplanatic tilt error θ_0 can be computed. Unfortunately, this computation requires knowledge of the $C_n{}^2(h)$ profile, and such information is usually not available. Using results derived by Fried,[22] however, Loos and Hogge[19] noted that a related quantity, the isoplanatic angle ϑ_0, can be approximated by measuring the received power fluctuations (i.e., intensity scintillations) of the stars observed through a specially modified telescope. We show below that the applicability of this technique can be extended to the anisoplanatic tilt error measurements.

At any arbitrary point in the aperture of a telescope, the amplitude and the phase of a received beam from a distant object, such as a star, are statistical quantitites. Their statistical nature is due to the effects of atmospheric turbulence. Unlike the phase, which remains correlated over points widely separated in the aperture, the amplitude rapidly decorrelates in distances of the order of millimeters or centimeters. Thus the total power received by a large aperture fluctuates little because many small independent cells within the aperture are averaged. It is for this reason that the eye, which has a small aperture, can easily detect the twinkling of stars that go unnoticed by a large telescope.

At a time t and position $\mathbf{x}$ in the telescope aperture, let $I(\mathbf{x}, t)$ denote the intensity of the received light from a star viewed at a zenith angle β. In addition, let $S(t)$ denote the total instantaneous power collected by a telescope, which has an attenuating mask placed over its aperture. At each point within the aperture, this mask can transmit between 0 and 100% of the incident light. Consequently, the mask will be defined mathematically by the function $P(\mathbf{x})$ whose values lie between zero and one. It immediately follows that

$$S(t) = \int P(\mathbf{x})I(\mathbf{x}, t)\mathrm{d}\mathbf{x}. \tag{14}$$

If we let $\langle\ \rangle$ denote the statistical expectation operator over time and σ_S^2 is the normalized variance defined by

$$\sigma_S^2 = \frac{\langle[S(t) - \langle S(t)\rangle]^2\rangle}{[\langle S(t)\rangle]^2}, \tag{15}$$

then, assuming a Kolmogorov turbulence spectrum, it follows that (see Appendix B)

$$\sigma_S^2 = 4(2\pi)^4\, 0.033\left(\frac{2\pi}{\lambda}\right)^2 A^{-2}\sec(\beta)\int_0^\infty C_n{}^2(h)G_\beta(h)\mathrm{d}h, \tag{16}$$

where

$$A = \int P(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{17}$$

$$N(L) = \left| \int_0^\infty \rho J_0(L\rho)P(\rho)\mathrm{d}\rho \right|^2, \tag{18}$$

and

$$G_\beta(h) = \int_0^\infty N(L)L^{-8/3} \sin^2\left[\frac{L^2 h \sec(\beta)\lambda}{4\pi}\right]\mathrm{d}L. \tag{19}$$

It has been assumed in Eq. (18) that $P(\mathbf{x})$ is a circularly symmetric function [i.e., $P(\mathbf{x}) = P(|\mathbf{x}|)$].

The similarity between Eqs. (12) and (16) is apparent. If $G_\beta(h)$ could be made proportional to the $f_\alpha(h)$ term in Eq. (12), then the normalized variance of the scintillations (i.e., σ_S^2) would give a measure of the anisoplanatic tilt error. Note that $G_\beta(h)$ is not a function of the $C_n^2(h)$ profile. $G_\beta(h)$ depends only on λ and on $N(L)$, which is the modulus squared of the Fourier–Bessel transform of the aperture attenuation mask. The idea is to choose an aperture mask [i.e., pick $P(\rho)$] such that

$$G_\beta(h) \approx c f_\alpha(h), \tag{20a}$$

where c is a constant of proportionality chosen such that the average difference,

$$\text{average difference} = \int_0^{20\ \mathrm{km}} [G_\beta(h) - c f_\alpha(h)]\mathrm{d}h, \tag{20b}$$

is zero. If an accurate approximation to $cf_\beta(h)$ can be realized, then the measurement of the received power fluctuations will give a measure of the anisoplanatic tilt error. Specifically, it follows from Eqs. (12) and (16)–(20) that

$$\theta_0 = 0.0964 c^{-1/2} A D^{-1/6} \lambda \left(\frac{\sec\alpha}{\sec\beta}\right)^{1/2} \sigma_S. \tag{21}$$

It can be shown from Eqs. (18) and (19) that $G_{\beta'}(h)$ is proportional to $G_\beta(h)$, provided that $P(\rho)$ is replaced by

$$P[\rho \cos^{1/2}(\beta)/\cos^{1/2}(\beta')]. \tag{22}$$

Therefore, if a suitable intensity aperture mask can be found for a zenith viewing angle β, then some linearly scaled version of this mask can be used at any other viewing angle β'.

4. APERTURE MASK SOLUTION

The required telescope aperture intensity mask, $P(\rho)$, is the solution of an integral equation. This integral equation follows immediately from Eqs. (18)–(20) and is given by

$$c f_\alpha(h) = G_\beta(h) = \int_0^\infty \left| \int_0^\infty \rho J_0(L\rho)P(\rho)\mathrm{d}\rho \right|^2 \times L^{-8/3} \sin^2\left[\frac{L^2 h \sec(\beta)\lambda}{4\pi}\right]\mathrm{d}L. \tag{23}$$

Although we are not able to solve Eq. (23) analytically for an arbitrary $f_\alpha(h)$, an approximate solution has been found for the practical case given in Fig. 2. The solution was derived by performing an exhaustive search of circular aperture masks [defined by Eq. (24) below] of different radii R_0. This form of mask was chosen for exhaustive search because it had been previously used successfully under similar circumstances.[19] Recall that the function shown in Fig. 2 corresponds to an optical communications link between a ground station and a geosynchronous satellite. Assuming that the scintillation measurements are made at a wavelength (λ) of 0.5 μm and a zenith angle (β) of 0 deg, then an approximate aperture mask solution is given by the following 4.7-cm-diameter circle:

$$P(\rho) = \begin{cases} 1, & |\rho| \le R_0 = 2.35\ cm \\ 0, & |\rho| > R_0 = 2.35\ cm \end{cases}. \tag{24}$$

With this solution Eq. (23) becomes

$$f_\alpha(h) \approx G_\beta(h) = \int_0^\infty R_0^4 \frac{J_1^2(LR_0)}{(LR_0)^2} L^{-8/3} \sin^2\left[\frac{L^2 h \sec(\beta)\lambda}{4\pi}\right]\mathrm{d}L. \tag{25}$$

The functions $f_\alpha(h)$ and $c^{-1}G_\beta(h)$ are plotted for comparison in Fig. 3 for h between 1 and 20 km.

Radiosonde balloon measurements of the $C_n^2(h)$ profile indicate that 1 to 20 km is usually the principal range of interest for isoplanatic effects,[29,30] with the major contributions coming from atmospheric layers at altitudes between 7 and 11 km. This is illustrated by Table 1, which shows $C_n^2(h)$ profile data collected by Barletti and Ceppatelli.[30] By using these data in Eq. (12), along with the $f_\alpha(h)$ weighting shown in Fig. 2, the contribution to the square of the anisoplanatic tilt error (i.e., θ_0^2) is plotted in Fig. 4. It is clear from Fig. 4 that contributions below 1 km and above 20 km are not large. Although Table 1 does not give data above 22.5 km, this altitude region is not significant, since $C_n^2(h)$ is proportional to the square of the atmospheric pressure at altitude h,[30] and atmospheric pressure decreases exponentially with height.

Local ground effects (not shown in Table 1) can cause atmospheric turbulence to be quite high within the first few meters of ground level.[31] This ground-level turbulence

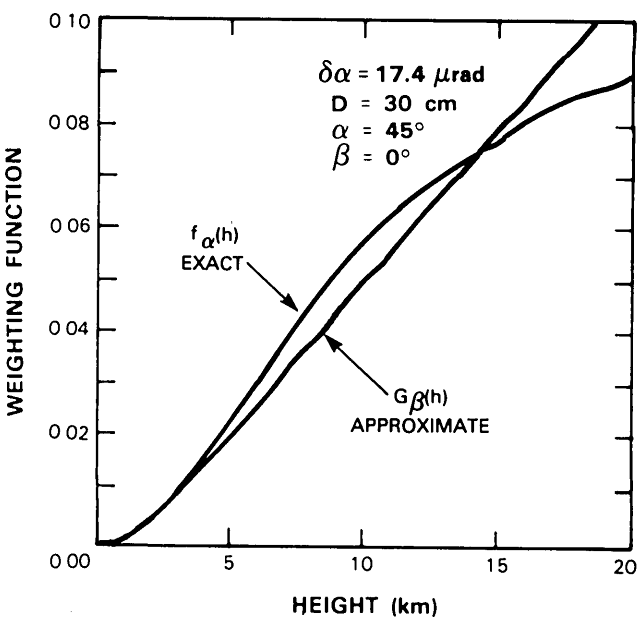

Fig. 3. Approximate anisoplanatic tilt weighting function.

Table 1. Measured Atmospheric $C_n^2(h)$ Profile[a]

Altitude Range (km)	$C_n^2(h)$ m$^{-2/3}$
0–1	1.1×10^{-15}
1–2	1.3×10^{-16}
2–3	6.1×10^{-17}
3–4	2.3×10^{-17}
4–5	1.5×10^{-17}
5–7.5	1.8×10^{-17}
7.5–10	1.4×10^{-17}
10–12.5	1.8×10^{-17}
12.5–15	2.1×10^{-17}
15–17.5	1.3×10^{-17}
17.5–20	8.0×10^{-18}
20–22.5	4.0×10^{-18}

[a] Source: Ref. 30.

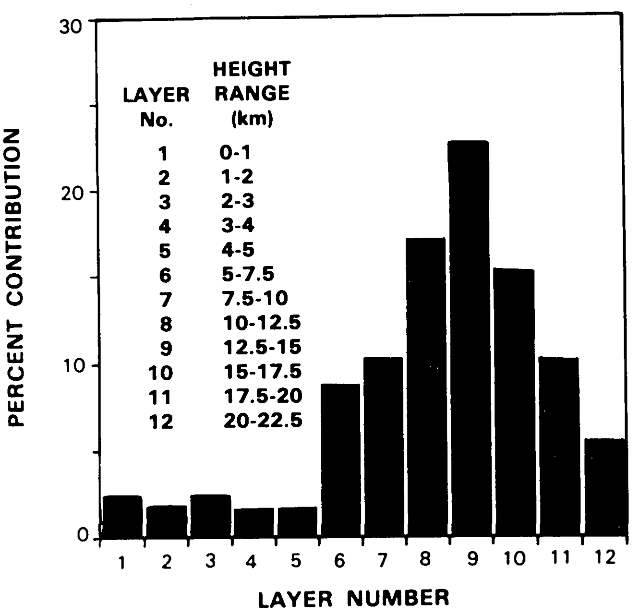

Fig. 4. Anisoplanatic tilt angle contribution versus altitude.

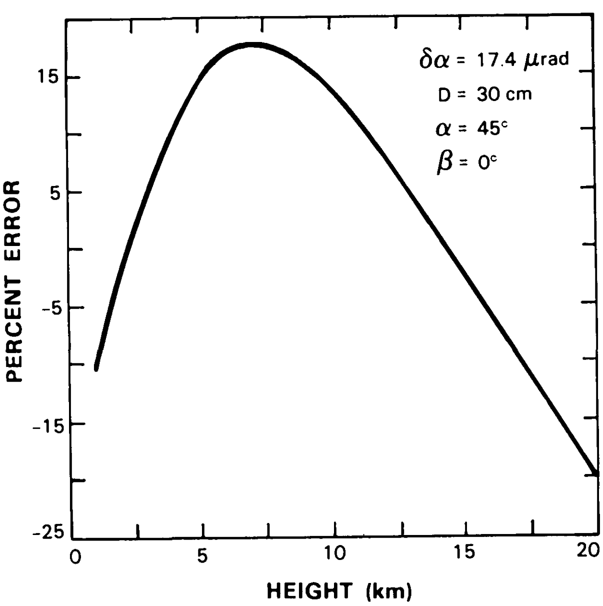

Fig. 5. Error in tilt weighting function approximation.

should not contribute appreciably to the anisoplanatic tilt, because the $f_\alpha(h)$ weighting function is small in this region. It is possible that atmospheric conditions may exist for which high- and low-altitude contributions cannot be neglected. Under these circumstances, an aperture mask having a closer fit to the desired weighting function should be used.

The percent error, defined as

$$\text{percent error} = \frac{f_\alpha(h) - c^{-1}G_\beta(h)}{f_\alpha(h)}, \tag{26}$$

is plotted in Fig. 5. It can be seen that the percent error between $\theta_0{}^2$ and $\sigma_S{}^2$ is less than ~20% over the 1- to 20-km range. Similar results are obtained (by using slightly different aperture mask diameters) for other point-ahead angles lying between 17 and 21 μrad.

Finally, it is noted that the analysis in this paper assumes a classical Kolmogorov turbulence spectrum, and the use of this spectrum is customary. The Kolmogorov assumption is predicated on the fact that an inertial subrange exists for which the turbulence is locally isotropic and homogeneous. The Kolmogorov model is supported by a large amount of evidence when the turbulence is well developed. The model's range of applicability throughout the entire atmosphere, however, remains an area for research.

5. SUMMARY

The significance of isoplanatism for optical systems operating within the atmosphere is discussed. The anisoplanatic tilt error θ_0 is shown to be an important parameter for adaptive optical systems that correct only for angle-of-arrival fluctuations. This error is defined as the rms difference in the angle-of-arrival fluctuations observed along two distinct propagation paths that form an angle $\delta\alpha$ at the transmit–receive aperture.

It is demonstrated that the previously published analytic expression for θ_0 contains a typographical error, and the correct equation is derived. An indirect method for determining θ_0, based on aperture-averaged stellar intensity scintillation measurements, is proposed. The method is a direct extension of the scintillation technique used to determine the isoplanatic angle as defined by Fried. In contrast to the isoplanatic technique, a separate spatial aperture mask is required for each zenith propagation angle at which the intensity scintillations are measured. Each mask, however, is shown to be identical up to a linear scaling factor, as indicated by expression (22).

Although direct techniques for determining the required aperture-intensity mask for arbitrary $\delta\alpha$ are not presented, a mask is found for a problem of practical interest. The problem is one of adaptive tilt correction in the presence of a 17.4-μrad point-ahead angle $\delta\alpha$. This point-ahead angle corresponds to a transmitter on the ground and a satellite in geosynchronous orbit. The aperture solution for a 30-cm-diameter transmit antenna and a 17.4-μrad point-ahead angle is found to be a 4.7-cm-diameter circle.

Unlike the isoplanatic angle, the anisoplanatic tilt error derived from scintillation data can be compared directly with binary star measurements. This provides an opportunity to validate further the theoretical framework on which several remote-sensing techniques are based.

APPENDIX A

In this appendix Eq. (7) of the main text is derived:

$$\int W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})(v^2 - 0.25u^2)\mathrm{d}\mathbf{v} = g(\mathbf{u}) + s(\mathbf{u}), \tag{A1}$$

where

$$g(\mathbf{u}) = \iint W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})v^2\mathrm{d}\mathbf{v}, \tag{A2}$$

$$s(\mathbf{u}) = \iint W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})(-0.25u^2)\mathrm{d}\mathbf{v}. \tag{A3}$$

Note that $g(\mathbf{u})$ is invariant under a coordinate rotation in $\mathbf{v}$ space. Therefore we can assume that $\mathbf{u}$ lies along the x-coordinate axis in $\mathbf{v}$ space and write

$$g(u) = 4\int_0^{(1-u)/2} \mathrm{d}x \int_0^{[0.25-(0.5u+x)^2]^{1/2}} (x^2 + y^2)\mathrm{d}y \tag{A4}$$

$$= 4\int_0^{(1-u)/2} \{x^2[0.25 - (0.5u + x)^2]^{1/2} + \tfrac{1}{3}[0.25 - (0.5u + x)^2]^{1.5}\}\mathrm{d}x. \tag{A5}$$

On making the following coordinate substitution in Eq. (A5):

$$x = 0.5(s - u), \tag{A6}$$

it follows that

$$g(u) = 0.25\int_u^1 (s^2 - 2us + u^2)(1 - s^2)^{1/2}\mathrm{d}s + \frac{1}{12}\int_u^1 (1 - s^2)^{3/2}\mathrm{d}s. \tag{A7}$$

Equation (A7) can be evaluated by using the following easily derivable definite integrals:

$$\int_u^1 (1 - s^2)^{1/2}\mathrm{d}s = -0.5u(1 - u^2)^{1/2} + 0.5\cos^{-1} u, \tag{A8}$$

$$\int_u^1 s(1 - s^2)^{1/2}\mathrm{d}s = \tfrac{1}{3}(1 - u^2)^{3/2}, \tag{A9}$$

$$\int_u^1 s^2(1 - s^2)^{1/2}\mathrm{d}s = 0.25u(1 - u^2)^{3/2} + 0.125\cos^{-1} u - 0.125u(1 - u^2)^{1/2}, \tag{A10}$$

$$\int_u^1 (1 - s^2)^{3/2}\mathrm{d}s = \tfrac{3}{8}\cos^{-1} u - \tfrac{5}{8}u(1 - u^2)^{1/2} + \tfrac{1}{4}u^3(1 - u^2)^{1/2}. \tag{A11}$$

From Eq. (A3) it follows that

$$s(\mathbf{u}) = -0.25u^2 \iint W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})\mathrm{d}\mathbf{v}. \tag{A12}$$

The integral on the right-hand side of Eq. (A12) is recognized as the area of overlap of two circles (diameters = 1) separated by a distance u. On using simple geometry, it then follows that

$$s(u) = -0.25u^2[0.5\cos^{-1} u - 0.5u(1 - u^2)^{1/2}]. \tag{A13}$$

Combining Eqs. (A1), (A7)–(A11), and (A13) produces the desired result:

$$\iint W(\mathbf{v} + 0.5\mathbf{u})W(\mathbf{v} - 0.5\mathbf{u})(v^2 - 0.25u^2)\mathrm{d}\mathbf{v} = \tfrac{1}{16}[\cos^{-1} u - (3u - 2u^3)(1 - u^2)^{1/2}]. \tag{A14}$$

APPENDIX B

In this appendix Eqs. (16)–(19) of the main text are derived. The analysis closely follows that given by Loos and Hogge in Ref. 19. The inner and outer scales of the turbulence are neglected in the derivation given below, and weak-scintillation conditions are assumed. For a more thorough discussion of these issues see Ref. 24.

Let $I(\mathbf{x})$ denote the intensity at a point in the telescope aperture (before the attenuation mask), and let I_0 denote the average value of $I(\mathbf{x})$. Then the log amplitude $\chi(\mathbf{x})$ at $\mathbf{x}$ is defined by

$$I(\mathbf{x}) = I_0 \exp[2\chi(\mathbf{x})]. \tag{B1}$$

The covariance $C_\chi(\rho)$ of the log amplitude is given by

$$C_\chi(\rho) = \langle[\chi(\mathbf{x}) - \langle\chi\rangle][\chi(\mathbf{x}') - \langle\chi\rangle]\rangle, \tag{B2}$$

where

$$\rho = |\boldsymbol{\rho}| = |\mathbf{x} - \mathbf{x}'|. \tag{B3}$$

Similarly, the covariance $C_I(\rho)$ of the intensity is defined by

$$C_I(\rho) = \langle[I(\mathbf{x}) - I_0][I(\mathbf{x}') - I_0]\rangle. \tag{B4}$$

It can be shown [see Eq. (1.11) of Ref. 22)] that the log amplitude and the intensity covariance functions are related by

$$C_I(\rho) = I_0^2\{\exp[4C_\chi(\rho)] - 1\}. \tag{B5}$$

It will be assumed that $C_\chi(\rho) \ll 1$ (i.e., weak turbulence); then Eq. (B5) becomes

$$C_I(\rho) \approx I_0^2 4C_\chi(\rho). \tag{B6}$$

From Table II, Eq. (T21) of Ref. 32, the log-amplitude covariance function is given by

$$C_\chi(\rho) = 4\pi^2k^2 \int_0^\infty C_n^2(z) \int_0^\infty LJ_0(L\rho)\Phi_n^0(L)\sin^2\left(\frac{L^2z}{2k}\right)\mathrm{d}L\mathrm{d}z, \tag{B7}$$

where $\Phi_n^0(L)$ is the normalized power spectral density of the turbulence refractive index. Assuming Kolmogorov turbulence (and neglecting inner and outer scale effects), the normalized power spectral density is[33]

$$\Phi_n^0(L) = 0.033L^{-11/3}. \tag{B8}$$

Let $P(\mathbf{x})$ denote the transmittance (in percent/100) of the attenuation mask in front of the telescope. Then the total signal power S collected by the telescope is given by

$$S = \int P(\mathbf{x})I(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{B9}$$

and the normalized power variance σ_S^2 is defined by

$$\sigma_S^2 = \langle (S - \langle S \rangle)^2 \rangle / \langle S \rangle^2. \tag{B10}$$

Combining Eqs. (B9) and (B10) yields

$$\sigma_S^2 = \left\langle \left\{ \int P(\mathbf{x})[I(\mathbf{x}) - I_0]\mathrm{d}\mathbf{x} \right\} \left\{ \int P(\mathbf{x}')[I(\mathbf{x}') - I_0]\mathrm{d}\mathbf{x}' \right\} \right\rangle \Big/ \langle S \rangle^2. \tag{B11}$$

With Eqs. (B4) and (B5) and relation (B6), Eq. (B11) becomes

$$\sigma_S^2 \approx 4A^{-2} \iint P(\mathbf{x})P(\mathbf{x}')C_\chi(\rho)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}', \tag{B12}$$

where

$$A = \int P(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{B13}$$

Make the change of variables

$$\boldsymbol{\rho} = \mathbf{x} - \mathbf{x}' \tag{B14}$$

and

$$\boldsymbol{\rho} = 0.5(\mathbf{x} + \mathbf{x}') \tag{B15}$$

in relation (B12). Then

$$\sigma_S^2 = 4A^{-2} \iint P(\boldsymbol{\rho}' + 0.5\boldsymbol{\rho})P(\boldsymbol{\rho}' - 0.5\boldsymbol{\rho})\mathrm{d}\boldsymbol{\rho}' C_\chi(\rho)\mathrm{d}\boldsymbol{\rho}. \tag{B16}$$

If we assume that $P(\mathbf{x})$ is circularly symmetric, $M(\boldsymbol{\rho})$ can be defined as

$$M(\boldsymbol{\rho}) = M(\rho) = \int P(\boldsymbol{\rho}' + 0.5\boldsymbol{\rho})P(\boldsymbol{\rho}' - 0.5\boldsymbol{\rho})\mathrm{d}\boldsymbol{\rho}'. \tag{B17}$$

Combining Eqs. (B16) and (B17) yields

$$\begin{aligned} \sigma_S^2 &= 4A^{-2} \int M(\rho)C_\chi(\rho)\mathrm{d}\boldsymbol{\rho} \\ &= 4A^{-2} \int_0^\infty 2\pi\rho M(\rho)C_\chi(\rho)\mathrm{d}\rho. \end{aligned} \tag{B18}$$

It now follows from Eqs. (B7), (B8), and (B18) that

$$\begin{aligned} \sigma_S^2 = {} & 4(2\pi)^2\, 0.033A^{-2}\, k^2 \int_0^\infty C_n{}^2(z) \\ & \times \int_0^\infty \left[\int_0^\infty 2\pi\rho J_0(L_\rho M(\rho)\mathrm{d}\rho \right] L^{-8/3} \sin^2\!\left(\frac{L^2 z}{2k}\right)\mathrm{d}L\mathrm{d}z. \end{aligned} \tag{B19}$$

Define $N(L)$ by

$$4\pi^2 N(L) = 2\pi \int_0^\infty \rho J_0(L\rho)M(\rho)\mathrm{d}\rho. \tag{B20}$$

The right-hand side of Eq. (B20) is the Fourier transform of the circularly symmetric function $M(\boldsymbol{\rho})$. Since $M(\boldsymbol{\rho})$ is the autocorrelation of $P(\boldsymbol{\rho})$, it immediately follows that

$$N(L) = \left| \int_0^\infty \rho J_0(L\rho)P(\rho)\mathrm{d}\rho \right|^2. \tag{B21}$$

Combining Eqs. (B19) and (B20) yields

$$\begin{aligned} \sigma_S^2 = {} & 4(2\pi)^4\, 0.033k^2 A^{-2} \int_0^\infty C_n{}^2(z) \int_0^\infty N(L)L^{-8/3} \\ & \times \sin^2\!\left(\frac{L^2 z}{2k}\right)\mathrm{d}L\mathrm{d}z. \end{aligned} \tag{B22}$$

Making the substitution

$$z = h \sec\beta \tag{B23}$$

in Eq. (B22) produces the desired result:

$$\begin{aligned} \sigma_S^2 = {} & 4(2\pi)^4\, 0.033\left(\frac{2\pi}{\lambda}\right)^2 A^{-2} \sec(\beta) \int_0^\infty C_n{}^2(h) \int_0^\infty N(L)L^{-8/3} \\ & \times \sin^2\!\left[\frac{L^2 h \sec(\beta)\lambda}{4\pi}\right]\mathrm{d}L\mathrm{d}h. \end{aligned} \tag{B24}$$

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