Recursive definitions of sets

• We saw last time how to define the set of propositional expressions recursively.

• The method extends to other kinds of sets: Let $S$ be the subset of $\mathbb{N}$ defined by the following rules:

  1. $3 \in S$;
  2. if $x \in S$ and $y \in S$ then $x + y \in S$;
  3. No number is in $S$ unless it can be shown to be there using (1) and (2).

• Example:

  - $3 \in S$ (rule 1)
  - $6 = 3 + 3 \in S$ by 1. and rule 2
  - $9 = 6 + 3 \in S$ by 2. and rule 2.
Using induction along with recursive definitions

**Proposition**  The set $S$ on the last slide is the set of positive multiples of 3.

Proof: Let $P$ be the set of positive multiples of 3. We show $P \subseteq S$ and $S \subseteq P$. For the first inclusion, we show that every integer of the form $3n$, for $n \geq 1$, is in $S$. We do this by induction on $n$.

**Basis.**  When $n = 1$, the number $3 \cdot 1 \in S$ by rule 1.

**Induction step.**  Assume that $3k$ is in $S$. We want $3(k + 1) \in S$. But $3(k + 1) = 3k + 3$. By inductive hypothesis $3k \in S$, and 3 is already in $S$, so $3k + 3 \in S$ by rule 2.
Proof continued: $S \subseteq P$.

To show this, we rely on rule 3, which says nothing is in $S$ unless you can show it in a finite number of uses of rules 1 and 2. Let $n$ be the number of uses of these rules. We show by induction on $n$ that the integer proved to be in $S$ is in fact a positive multiple of 3.

**Basis.** We just apply one rule, which has to be rule 1. This rule shows $3 \in S$, and 3 is a positive multiple of 3.

**Induction step (strong form).** Assume that whenever we show that $p \in S$ by using $k$ or fewer steps, then $p$ is a positive multiple of 3. Consider a “proof” using $k+1$ steps. The last rule used in this proof is rule 2, which says that if $x$ and $y$ are in $S$, so is $x + y$. Now $x$ was shown in $S$ by $\leq k$ steps, and so was $y$. By inductive hypothesis (twice), we know that $x$ and $y$ are positive multiples of 3, and therefore so is $x + y$. 

Intuitively, relations are properties that hold among things in a world.

For example:

- “loves”
- “uncle-of”
- “friend-of”
- “left-of”
- “small”

All except the last of these hold between two things. They are called binary relations. “Small” is a unary relation.

You can also have ternary relations, etc.
Relations involving more than one world

- For example, students and courses:

  John, EECS303
  Ed, EECS376
  John, EECS280
  Mary, Math606
  Mary, Math747
  Paul, Math747
  John, Math606

- You can look at each row of the table as an ordered pair, and the table as a set of ordered pairs.

- That’s the official definition of “binary relation between two sets.”
Official Relation Definitions

• Definition Let $A, B$ be sets.
  
  – A binary relation between $A$ and $B$ is a subset $R$ of $A \times B$.
  
  – A binary relation on $A$ is a subset of $A \times A$.

• Example

  $S = \{ \text{John, Ed, Mary, Paul} \}$
  $C = \{ E203, E376, E280, M606, M747 \}$

  $\text{ENROLLED} = \{ (\text{John, E203}), (\text{John, E376}), (\text{Ed, E280}),$
  $(\text{Mary, M606}), (\text{Mary, M747}), (\text{Paul, M747}) \}$

• The relation $\text{ENROLLED} \subseteq S \times C$.

• Notation: we write $a R b$ to mean $(a, b) \in R$. Thus, John $\text{ENROLLED}$ E376.
Picturing Relations

Enrollment - a relation between students and courses

"Offered-by" - a relation between courses and departments
Array (matrix) representation of relations

- The “enrolled” information can be presented:

<table>
<thead>
<tr>
<th>S/C</th>
<th>E203</th>
<th>E376</th>
<th>E280</th>
<th>M606</th>
<th>M747</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Paul</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Mary</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Ed</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- This can be seen as a function from $S \times C$ to $\{0, 1\}$:

$$F_{ENR}(s, c) = \begin{cases} 1 \text{ if } (s, c) \in ENR, \\ 0 \text{ otherwise.} \end{cases}$$
Relations as maps to a powerset

• A relation from $A$ to $B$ can be thought of as a function from $A$ to $\mathcal{P}(B)$.

- $John \mapsto \{E203, E376\}$
  $Paul \mapsto \{M606\}$
  $Mary \mapsto \{M606, M747\}$
  $Ed \mapsto \{E280\}$

• There are lots of mathematical ways to model relations.
Relations on a set

- The set can be infinite.
- Example: $A = \mathbb{N}^+$; $DIV = \{(m, n) \mid m \text{ divides } n\}$.
- Some ordered pairs: $(1, 10), (3, 6), (4, 20), \ldots$.
- We write $m \mid n$ to indicate that $(m, n) \in DIV$. Thus $1 \mid 10$, $3 \mid 6$, $4 \mid 20$, \ldots.
- The “less-than-or-equal-to” relation on $\mathbb{R}$ is another example of a binary relation on an infinite set.
Picturing a relation on a finite set $A$

Don't need two copies of the set:

Suppose $A = \{a, b, c, d, e\}$ and $R = \{(d, a), (d, c), (a, b), (b, c), (b, e), (e, e)\}$.

Just work with one copy of $A$:
Properties of Relations on a set

• **Definition** A binary relation $R$ on a set $A$ is said to be reflexive if $(\forall a \in A)(a R a)$.

• In terms of the graph picture, there is a self-loop at every node:

![Diagram of reflexive relation]

• **Example** On every set $A$, there is the identity relation

$$id_A = \{(a, a) \mid a \in A\}.$$  

• Other examples:
  
  – the “less-than-or-equal-to” relation $\leq$ on $\mathbb{R}$;
  
  – the subset relation on $\mathcal{P}(X)$, because $Y \subseteq Y$ for any set $Y$;
  
  – the “divides” relation $|$ on $\mathbb{N}^+$.  

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Symmetric Relations

• **Definition** A binary relation $R$ on a set $A$ is said to be **symmetric** if

$$(\forall a, b \in A)(a R b \rightarrow b R a).$$

• In terms of pictures:

• **Example** The “mutual friend of” relation is symmetric. The “sister” relation isn’t. The “sibling” relation is. The identity relation is. The $\leq$ relation isn’t.
Transitive Relations

• Definition  A binary relation $R$ on a set $A$ is said to be transitive if

$$(\forall a, b, c \in A)(a R b \land b R c \rightarrow a R c)$$
• The pictures:

This picture holds *everywhere*

This is ok:

This isn't:
Examples of Transitive Relations

- The $\leq$ relation on $\mathbb{N}$.
- The divisibility relation on $\mathbb{N}^+$.
- The subset relation $\subseteq$ on $\mathcal{P}(X)$.
- The relation of logical equivalence on the set of propositional expressions.
- NOT the “father-of” relation on the set of people.
- NOT the “acquainted-with’ relation on the set of people.
- How about the “sister-of” relation?