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Best L_p Isotonic Regressions, $\mathbf{p} \in \{0, 1, \infty\}$

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Extended Abstract: Given a real-valued weighted function f on a finite dag G, an L_p isotonic regression of f is a nondecreasing function on G which minimizes the L_p regression error. Isotonic regression comes up in a wide array of applications, and while L_2 regression is most commonly used, there is also longstanding interest in L_1 and L_∞, and more recent interest in L_0. L_p isotonic regression is unique for all $p \in (1, \infty)$, but not when $p \in [0, 1] \cup \{\infty\}$. We are interested in determining a "best" isotonic regression for $p \in \{0, 1, \infty\}$, where by best we mean a regression satisfying stronger properties than merely having minimal norm.

One approach is to use *strict L_p regression*, which is the limit of the best L_q approximation as q approaches p. When $p = \infty$ this is known as the Polya approach, and when $p = 1$ is sometimes called the Polya-1 approach. A quite different approach is to use *lex regression*, which is based on lexical ordering of regression errors. The ordering for L_∞ uses the errors in decreasing order, while for L_0 they are in increasing order. For L_∞ the strict and lex regressions are unique and the same. For L_1, strict $q{\searrow}1$ is unique, but we show that $q{\nearrow}1$ may not be, and even when it is unique the two limits may not be the same. The strict $q{\searrow}1$ approach has also been used to determine a best median. For L_0, in general neither of the strict and lex regressions are unique, nor do they always have the same set of optimal regressions, but by expanding the objectives of L_p optimization to $p < 0$ we show $p \nearrow 0$ is the same as lex regression.

We also give algorithms for computing the best L_p isotonic regression in certain situations. One is a refinement of a previous algorithm for L_∞, based on the lex definition, and another determines L_1 for a linear order, based on the strict $q{\searrow}1$ definition. The latter uses L_1 partitioning and pool adjacent violators (PAV).

Paper

Keywords: strict isotonic regression, lex regression, monotonic, Polya approach, L_0, L_1, L_∞, Hamming distance, best median

My work on shape-constrained regression (isotonic, unimodal, step). Particularly relevant is a paper on strict L_∞ and lex ordering.