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SCHUR MULTIPLICATION ON $\mathscr{B}(\ell_p, \ell_q)$

QUENTIN F. STOUT

1. INTRODUCTION

In 1911 Schur [17] proved that if (a_{ij}) and (b_{ij}) are bounded matrix operators on ℓ_2 then so is $(a_{ij}b_{ij})$, and $\|(a_{ij}b_{ij})\| \leqslant \|(a_{ij})\| \cdot \|(b_{ij})\|$. We call this termwise product *Schur multiplication*, although it is more often called "Hadamard multiplication". (This term apparently originated in Halmos [4] as a parallel to the Hadamard product of series.) Recently Bennett [2] extended Schur's result to show that, for $1 \leqslant p,\ q \leqslant \infty$, Schur multiplication gives a commutative Banach algebra structure to the bounded matrix operators from ℓ_p to ℓ_q. This paper studies these Banach algebras, exhibiting their maximal ideal spaces and some of their properties.

If $p = q$ the Schur-Bennett results are startling, for they have taken a highly noncommutative Banach algebra and given it a nontrivial commutative multiplication consistant with the original norm and linear structure. Varopoulos [24] is interested in such compatible Banach algebras and has asked if Schur multiplication on ℓ_2 is a Q-algebra. If $p \neq q$ Schur multiplication accomplishes even more by supplying a multiplication to a collection of operators on which there is no natural product. For these reasons alone it is an interesting subject of study, but there is more. Schur multiplication is useful in many areas of linear algebra, analysis, and statistics, and there is an increasing awareness of its role. It has been used in operator theory (Halmos and Sunder [5], Johnson and Williams [7], Shields and Wallen [20]), complex analysis (FitzGerald and Horn [3], Pommerenke [14], Shapiro and Shields [18], [19]), Banach spaces (Bennett [1], Kwapien and Pełczyński [9]), and combinatorics (Ryser [16]). Further, Styan [23] has a survey article outlining its uses in multivariate analysis, Bennett [2] uses it to unify and improve results on absolutely summing operators, and Stout [21], [22] explores connections with the essential numerical range and interpolating ideals.

We begin in Section 2 by introducing some notation and giving basic results on Schur products. In Section 3 we use these to determine the various maximal ideal spaces, allowing us to derive several properties. In Section 4 we concentrate on certain ideals and their hulls. This is of interest because the compact operators from

ℓ_p to ℓ_q are an ideal in Schur multiplication, and on Hilbert space the trace class is also a Schur ideal. In Section 5 we outline several open areas and pose some questions.

Portions of this material (for the case $p = q = 2$) appeared in the author's Ph. D. thesis written at Indiana University under the supervision of Professor John B. Conway.

2. PRELIMINARIES

If $T = (t_{ij})$ and $S = (s_{ij})$ are matrices then $T * S$ denotes their Schur product, i.e., the matrix $(t_{ij}s_{ij})$. In all matrices the indices range through $\mathbf{N} = \{1,2, \ldots\}$. All Banach spaces are complex. Throughout we assume $1 \leqslant p, q \leqslant \infty$. Using the standard bases, $\mathscr{B}(\ell_p, \ell_q)$ denotes the matrices which are bounded linear operators from ℓ_p to ℓ_q. (If $p = \infty$ this omits some of the bounded linear operators.) We use $\|\ \|_{p,q}$ to denote the operator norm in $\mathscr{B}(\ell_p, \ell_q)$.

We will use extensively the following matrix facts:

1) Let $T \in \mathscr{B}(\ell_p, \ell_q)$ with $1 < p$ and $q < \infty$. Then for any $\varepsilon > 0$ there is an n such that no row or column of T has more than n entries with absolute value greater than ε.

2) Let $T \in \mathscr{B}(\ell_1, \ell_q)$ with $q < \infty$. Then for any $\varepsilon > 0$ there is an n such that no column of T has more than n entries with absolute value greater than ε.

3) Let $T \in \mathscr{B}(\ell_p, \ell_\infty)$ with $p > 1$. Then for any $\varepsilon > 0$ there is an n such that no row of T has more than n entries with absolute value greater than ε.

4) $(t_{ij}) \in \mathscr{B}(\ell_1, \ell_\infty)$ iff $t = \sup_{i,j}|t_{ij}| < \infty$, in which case $t = \|(t_{ij})\|_{1,\infty}$.

5) Let $T \in \mathscr{B}(\ell_p, \ell_q)$ with $p > q$. Then for any $\varepsilon > 0$ there are only finitely many entries of T with absolute value greater than ε.

6) Let (t_{ij}) have no more than n nonzero entries in any row or column and suppose $t = \sup_{i,j}|t_{ij}| < \infty$. Then for $1 \leqslant p \leqslant q \leqslant \infty$, $(t_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$ and $t \leqslant \|(t_{ij})\|_{p,q} \leqslant n \cdot t$.

7) Let (t_{ij}) have no more than n nonzero entries in any row and suppose $t = \sup_{i,j}|t_{ij}| < \infty$. Then $(t_{ij}) \in \mathscr{B}(\ell_p, \ell_\infty)$ and $t \leqslant \|(t_{ij})\|_{p,\infty} \leqslant n \cdot t$.

8) Let (t_{ij}) have no more than n nonzero entries in any column and suppose $t = \sup_{i,j}|t_{ij}| < \infty$. Then $(t_{ij}) \in \mathscr{B}(\ell_1, \ell_q)$ and $t \leqslant \|(t_{ij})\|_{1,q} \leqslant n \cdot t$.

All of these facts are easy to establish, except perhaps fact 6, which needs the decomposition result stated in the following lemma. The lemma is well-known for finite sets, and the infinite version follows from the finite version by a use of Konig's lemma of infinity.

LEMMA 2.1. *Let $n \in \mathbf{N}$ and $A \subset \mathbf{N} \times \mathbf{N}$ such that A contains at most n elements in any row or column of $\mathbf{N} \times \mathbf{N}$. Then $A = A_1 \cup \ldots \cup A_n$, where each A_i has at most one element in any row or column, and $A_i \cap A_j = \emptyset$ when $i \neq j$.*

The next theorem is the foundation of this area. The case $p = q = 2$ is due to Schur and the others are Bennett's.

THEOREM 2.2. (Schur [17, Satz III], Bennett [2, Proposition 2.1]) *Let* (a_{ij}), $(b_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$. *Then* $(|a_{ij}b_{ij}|) \in \mathscr{B}(\ell_p, \ell_q)$ *and*

$$\|(|a_{ij}b_{ij}|)\|_{p,q} \leqslant \|(a_{ij})\|_{p,q} \cdot \|(b_{ij})\|_{p,q}.$$

COROLLARY 2.3. (Bennett [2, Theorem 2.2]) *Let* A, $B \in \mathscr{B}(\ell_p, \ell_q)$. *Then* $A * B \in \mathscr{B}(\ell_p, \ell_q)$ *and* $\|A * B\|_{p,q} \leqslant \|A\|_{p,q} \cdot \|B\|_{p,q}$.

Proof. For any matrix (a_{ij}), $\|(a_{ij})\|_{p,q} \leqslant \|(|a_{ij}|)\|_{p,q}$. ▨

We use $\mathscr{B}_*(\ell_p, \ell_q)$ to denote the Banach algebra of Schur multiplication on $\mathscr{B}(\ell_p, \ell_q)$.

LEMMA 2.4. *Let* A, $B \in \mathscr{B}_*(\ell_p, \ell_q)$. *Then* $\operatorname{rank}(A * B) \leqslant \operatorname{rank}(A) \operatorname{rank}(B)$.

Proof. Let $A|_n$ denote the operator on $\mathbf{C}^n$ which is the restriction of A to its upper $n \times n$ corner. Then $\operatorname{rank}(A|_n) \leqslant \operatorname{rank}(A)$ and $\lim\limits_{n \to \infty} \operatorname{rank}(A|_n) = \operatorname{rank}(A)$. Khan and Marcus [8] noted that $A|_n * B|_n$ is a principle submatrix of the tensor product $A|_n \otimes B|_n$, and hence $\operatorname{rank}(A|_n * B|_n) \leqslant \operatorname{rank}(A|_n \otimes B|_n) \leqslant \operatorname{rank}(A|_n) \cdot \operatorname{rank}(B|_n)$. ▨

PROPOSITION 2.5. a) *The finite rank operators are a subalgebra of* $\mathscr{B}_*(\ell_p, \ell_q)$, *but not an ideal.*

b) *The compact operators are an ideal in* $\mathscr{B}_*(\ell_p, \ell_q)$, *except when* $p = 1$ *and* $q = \infty$.

Proof. a) Lemma 2.4 showed that the finite ranks are a subalgebra. Let $a_{ij} = 1/(i^2j^2)$ and $b_{ij} = \delta_{ij}/i^2$. Then (a_{ij}), $(b_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$, the rank of (a_{ij}) is one, but the rank of $(a_{ij}b_{ij})$ is infinite.

b) In $\mathscr{B}(\ell_p, \ell_q)$ the compact operators are the closure of the finite rank operators. It suffices to show that if (a_{ij}), $(b_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$ and (a_{ij}) has rank one then $(a_{ij}b_{ij})$ is in the closure of the finite rank operators. Since (a_{ij}) is rank one there is a (c_i) in ℓ_q and (d_j) in ℓ_{p^*} such that $a_{ij} = c_i d_j$. Assume $q < \infty$. (If $q = \infty$ then the restriction on p insures that $p^* < \infty$, in which case the following argument will work by interchanging the roles of rows and columns.) Let $B = (b_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$ and define $A_n \in \mathscr{B}(\ell_p, \ell_q)$ to be the restriction of A to the first n rows. Then $A_n \to A$, so $A_n * B \to A * B$, and $\operatorname{rank}(A_n * B) \leqslant n$.

Finally, in $\mathscr{B}(\ell_1, \ell_\infty)$ the compact operators are not an ideal because the matrix of all ones is a rank one operators, and hence compact, but not all operators are compact. ▨

Let $\mathscr{F}(p, q)$ denote the closure, in $\mathscr{B}(\ell_p, \ell_q)$, of the matrices with only finitely many nonzero entries. $\mathscr{F}(p, q)$ is an ideal for any values of p and q. If $1 < p$ and

$q < \infty$ then $\mathscr{F}(p, q)$ is the compact operators, while if $p = 1$ or $q = \infty$ then it is strictly smaller. Notice that $\mathscr{F}(p, q) * \mathscr{F}(p, q)$ is dense in $\mathscr{F}(p, q)$.

A few more characteristics of $\mathscr{B}_*(\ell_p, \ell_q)$ can be obtained easily.

PROPOSITION 2.6. *$\mathscr{B}_*(\ell_p, \ell_q)$ is semisimple.*

Proof. Let $T = (t_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q) \setminus 0$, and let i and j be such that $t_{ij} \neq 0$. Then $\|T^n\|_{p,q} \geqslant |t_{ij}|^n$, so the spectral radius is at least $|t_{ij}|$. ▨

PROPOSITION 2.7. *Let $p \leqslant q$. Then $\mathscr{B}_*(\ell_p, \ell_q) * \mathscr{B}_*(\ell_p, \ell_q)$ is not dense in $\mathscr{B}_*(\ell_p, \ell_q)$ unless $p = 1$ and $q = \infty$.*

Proof. Assume $q < \infty$. (If $q = \infty$ then $p > 1$ and the following argument applies with the roles of rows and columns interchanged.) Let $r = -1/q$ and let $T \in \mathscr{B}(\ell_p, \ell_q)$ be defined by

$$T = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2^r & & \\ 0 & 2^r & & \\ 0 & & 3^r & \\ \cdot & & 3^r & \\ \cdot & & 3^r & \\ \cdot & & & \ddots \end{pmatrix}.$$

The ℓ_q norms of the columns of any element of $\mathscr{B}(\ell_p, \ell_q)$ are uniformly bounded and hence the $\ell_{q/2}$ norms of any element of $\mathscr{B}_*(\ell_p, \ell_q) * \mathscr{B}_*(\ell_p, \ell_q)$ are uniformly bounded. If (a_{ij}) is any matrix with columns with uniformly bounded $\ell_{q/2}$ norms then $\|T - (a_{ij})\|_{p,q} \geqslant 1$. Therefore the distance from T to the closure of $\mathscr{B}_*(\ell_p, \ell_q) * \mathscr{B}_*(\ell_p, \ell_q)$ is one. ▨

REMARKS. 1) $\mathscr{B}_*(\ell_1, \ell_\infty) * \mathscr{B}_*(\ell_1, \ell_\infty) = \mathscr{B}(\ell_1, \ell_\infty)$ since $\mathscr{B}_*(\ell_1, \ell_\infty)$ has an identity.

2) Theorem 2.2 showed that each element of $\mathscr{B}_*(\ell_p, \ell_q) * \mathscr{B}_*(\ell_p, \ell_q)$ is an absolute matrix, i.e., the entries can be replaced by their absolute values and still give a bounded operator. Since T in the proof is also an absolute matrix, if $p \leqslant q$ then $\mathscr{B}_*(\ell_p, \ell_q) * \mathscr{B}_*(\ell_p, \ell_q)$ is not even dense in the absolute matrices, except when $p = 1$ and $q = \infty$.

3) Proposition 4.4 will show that $\mathscr{B}_*(\ell_p, \ell_q)^3$ is dense in $\mathscr{B}_*(\ell_p, \ell_q)^2$.

4) A modification of the proof shows that $\mathscr{B}_*(\ell_p, \ell_q)^2$ is nowhere dense in $\mathscr{B}_*(\ell_p, \ell_q)$.

5) If $p > q$ then $\mathscr{B}_*(\ell_p, \ell_q)^2$ is dense in $\mathscr{B}_*(\ell_p, \ell_q)$ because $\mathscr{B}_*(\ell_p, \ell_q) = \mathscr{F}(p, q)$. See Pitt [13] and Littlewood [10].

An *approximate identity* of a Banach algebra $\mathscr{A}$ is a net $\{a_\lambda\}$ in $\mathscr{A}$ such that $a_\lambda x \to x$ for all x in $\mathscr{A}$.

COROLLARY 2.8. *If $p \leqslant q$ then $\mathscr{B}_*(\ell_p, \ell_q)$ has no approximate identity unless $p = 1$ and $q = \infty$, in which case it has an identity.*

Proof. If a Banach algebra $\mathscr{A}$ has an approximate identity then $\mathscr{A}^2$ is dense in $\mathscr{A}$. ▨

For $p > q$, $\mathscr{B}_*(\ell_p, \ell_q)$ has an approximate identity, but no bounded approximate identity.

3. THE MAXIMAL IDEAL SPACE OF $\mathscr{B}_*(\ell_p, \ell_q)$

The primary purpose of this section is the construction of the space of maximal ideals of $\mathscr{B}_*(\ell_p, \ell_q)$. To do this some standard terminology must be introduced (see Hoffman [6] or Loomis [11]). Let $\mathscr{A}$ denote a commutative Banach algebra without unit. An element x of $\mathscr{A}$ has an *adverse* y if $xy - x - y = 0$. The spectrum of x in $\mathscr{A}$, denoted $\sigma(x)$, is the spectrum of the image of x in the algebra obtained by adjoining a unit to $\mathscr{A}$. For every x in $\mathscr{A}$, $0 \in \sigma(x)$. For any scalar $\lambda \neq 0$, $\lambda \in \sigma(x)$ if and only if x/λ has no adverse.

The maximal ideal space of $\mathscr{A}$ will be denoted $\Delta(\mathscr{A})$, and $\Delta(p, q)$ will denote $\Delta(\mathscr{B}_*(\ell_p, \ell_q))$. Elements of $\Delta(\mathscr{A})$ will be thought of both as regular maximal ideals or multiplicative linear functions. (An ideal $\mathscr{I}$ is *regular* if there is an e in $\mathscr{A}$ such that $ex - x \in \mathscr{I}$ for all x in $\mathscr{A}$.) The former creates a *hull-kernel topology* on $\Delta(\mathscr{A})$: for any $D \subset \Delta(\mathscr{A})$, kernel$(D) = \cap \{M\colon M \in D,\ M$ being viewed as an ideal$\}$, and for any $A \subset \mathscr{A}$, hull$(A) = \{M\colon M \in \Delta(\mathscr{A}),\ A \subset M\}$. For any $D \subset \Delta(\mathscr{A})$, hull(kernel$(D)$) is the closure of D in the hull-kernel topology. The latter characterization implies that $\Delta(\mathscr{A})$ has a natural weak $*$-topology. If this topology equals the hull-kernel topology then $\mathscr{A}$ is said to be *regular*.

For any a in $\mathscr{A}$, $a^\wedge$ will denote its *Gelfand transform*, i.e., that continuous function on $\Delta(\mathscr{A})$ defined by $a^\wedge(p) = p(a)$, where here $\Delta(\mathscr{A})$ is viewed as multiplicative functionals.

If X is a locally compact Hausdorff space then $C(X)$ denotes the bounded continuous $\mathbf{C}$-valued functions on X, and $\beta(X)$ denotes the Stone-Cech compactification of X. If $A \subset \mathbf{N}\times\mathbf{N}$ and there is an n such that A has no more than n elements in any row or column, then fact 6 showed that for $p \leqslant q$, there is an isomorphism from the closed ideal $\{(t_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)\colon t_{ij} = 0 \text{ if } (i, j) \notin A\}$ onto $C(A)$, and hence $\Delta(\{(t_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)\colon t_{ij} = 0 \text{ if } (i, j) \notin A\})$ is homeomorphic to $\Delta(C(A)) \simeq \beta A \simeq A^-$ in $\beta(\mathbf{N}\times\mathbf{N})$. This result holds for any p and q if A is finite. Further, if $p = 1$ it holds if A has no more than n entries in any column, and if $q = \infty$ it holds if A has no more than n entries in any row.

The following lemma was essentially proved by Schur [17, Satz IV]. He proved a slightly restricted version for $\mathscr{B}(\ell_2, \ell_2)$, but his method of proof, combined with Theorem 2.2, yields this lemma.

LEMMA 3.1. *Let* $(a_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)$ *and let* f *be a complex-valued function defined on a set* $V \subset \mathbf{C}$ *such that*

a) $V \supset \{a_{ij}\}$.

b) *There is a neighborhood* U *of* 0 *contained in* V *such that* f *is analytic on* U.

c) $f(0) = 0$.

d) f *is uniformly bounded on* V.

Then $(f(a_{ij})) \in \mathscr{B}_*(\ell_p, \ell_q)$.

PROPOSITION 3.2. *If* $T = (t_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)$ *then* $\sigma(T) = \{t_{ij}\}^-$.

Proof. If $p = 1$ and $q = \infty$ then this is well known. Otherwise 0 is in both $\sigma(T)$ and $\{t_{ij}\}^-$. For $\lambda \neq 0$, $\lambda \in \sigma(T)$ iff T/λ has no adverse, i.e., iff there is no $S = (s_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)$ such that $T * S/\lambda - T/\lambda - S = 0$. This requires $s_{ij} = t_{ij}/(t_{ij} - \lambda)$. If $\lambda \in \{t_{ij}\}^-$ then the s_{ij} would be unbounded or undefined, so $\lambda \in \sigma(T)$. If $\lambda \notin \{t_{ij}\}^-$ then $f(x) = x/(x - \lambda)$ satisfies Lemma 3.1, so $S = f(T)$ is in $\mathscr{B}_*(\ell_p, \ell_q)$ and λ is not in $\sigma(T)$. ▨

PROPOSITION 3.3. $\mathscr{B}_*(\ell_p, \ell_q)^\wedge$ *(the Gelfand transforms of elements of* $\mathscr{B}_*(\ell_p, \ell_q)$*) is dense in* $C(\Delta(p, q))$.

Proof. From Loomis [11, p. 89], it suffices to show that $\mathscr{B}_*(\ell_p, \ell_q)$ has an involution $\overline{}$ satisfying:

i) $\overline{\overline{T}} = T$

ii) $\overline{(T + S)} = \overline{T} + \overline{S}$

iii) $\overline{(\lambda T)} = \bar{\lambda}\overline{T}$ for $\lambda \in \mathbf{C}$

iv) $\overline{(S * T)} = \overline{S} * \overline{T}$

v) $-T * \overline{T}$ has no adverse.

$\mathscr{B}_*(\ell_p, \ell_q)$ has such an involution, namely $\overline{(t_{ij})} = (\bar{t}_{ij})$, for then $-T * \overline{T}$ has as adverse $f(-T * \overline{T})$, where $f(x) = x/(x - 1)$. ▨

For a bounded function f in $C(\mathbf{N} \times \mathbf{N})$ let $f^\sim$ denote its extension to $\beta(\mathbf{N} \times \mathbf{N})$. Any T in $\mathscr{B}_*(\ell_p, \ell_q)$ can be viewed as a bounded function on $\mathbf{N} \times \mathbf{N}$ and hence has an extension $T^\sim$. For any p in $\beta(\mathbf{N} \times \mathbf{N})$ the map $T \to T^\sim(p)$ is a multiplicative linear functional (perhaps trivial).

THEOREM 3.4. $\Delta(p, q)$ *is regular, and the map* j *from* $\beta(\mathbf{N} \times \mathbf{N})$ *to (perhaps trivial) multiplicative linear functionals on* $\mathscr{B}_*(\ell_p, \ell_q)$ *given by* $j(p)(T) = T^\sim(p)$ *is a homeomorphism of* $\mathscr{D}(p, q)$ *onto* $\Delta(p, q)$, *where*

i) $\mathscr{D}(1, \infty) = \beta(\mathbf{N} \times \mathbf{N})$;

ii) $\mathscr{D}(p, q) = \mathbf{N} \times \mathbf{N}$ *if* $p > q$;

iii) $\mathscr{D}(p, q) = \cup \{D^-: D \subset \mathbf{N} \times \mathbf{N}$ *such that* D *has no more than one element in any row or column*$\}$

if $1 < p \leqslant q < \infty$;

iv) $\mathscr{D}(p,\infty) = \cup\{D^-: D \subset \mathbf{N} \times \mathbf{N}$ *such that* D *has no more than one element in any row*$\}$

if $1 < p$;

v) $\mathscr{D}(1,q) = \cup\{D^- : D \subset \mathbf{N} \times \mathbf{N}$ *such that* D *has no more than one element in any column*$\}$

if $q < \infty$.

Proof. We give only the proof for $1 < p \leqslant q < \infty$, the other cases being similar. First it will be shown that j is a homeomorphism when $\Delta(p, q)$ has the weak $*$-topology, and then it will be shown that $\Delta(p, q)$ is regular.

To show that j is an injection, let r and s be distinct elements of $\mathscr{D}(p, q)$. There is a $D \subset \mathbf{N} \times \mathbf{N}$ with $r \in D^-$, $s \notin D^-$, and such that D has no more than one element in any row or column of $\mathbf{N} \times \mathbf{N}$. If χ_D is the characteristic function of D then, interpreted as a matrix, $\chi_D \in \mathscr{B}_*(\ell_p, \ell_q)$. Then $r(\chi_D) = 1$, $s(\chi_D) = 0$, so $j(r) \neq j(s)$ and $j(r)$ is a nonzero functional.

To show that j is onto, let $b \in \Delta(p, q)$ and let $T \in \mathscr{B}_*(\ell_p, \ell_q)$ with $T^\wedge(b) = 1$. Define $S = (s_{ij})$

$$s_{ij} = \begin{cases} 1 & \text{if } |T(i, j)| > 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $S \in \mathscr{B}_*(\ell_p, \ell_q)$ by facts 1 and 6. Since $S * S = S$, for any r in $\Delta(p, q)$, $S^\wedge(r) = 0$ or $S^\wedge(r) = 1$. If $S^\wedge(b) = 0$ then $(S * T)^\wedge(b) = 0$, so $(T - S * T)^\wedge(b) = 1$. However, no entry of $T - S * T$ has absolute value greater than $1/2$, contradicting Proposition 3.2. Therefore $S^\wedge(b) = 1$. Let V be the open and closed set $S^{\wedge -1}(1)$. $j^{-1}(V \mid j(\mathscr{D}(p,q)))$ contains at most n elements in any row or column of $\mathbf{N} \times \mathbf{N}$. The comments preceeding Lemma 3.1 showed that j is a homeomorphism from $j^{-1}(V \mid j(\mathscr{D}(p, q)))$ onto V. In particular, $b \in V$, so j is onto. Further we have shown that for every b in $\Delta(p, q)$ there is an open and closed neighborhood V of b such that $j^{-1}(V)$ is open and closed in $\mathscr{D}(p, q)$ and j^{-1} is a homeomorphism of V onto its image. Therefore j is a homeomorphism of $\mathscr{D}(p,q)$ onto $\Delta(p, q)$ with its weak $*$-topology.

To show that $\Delta(p, q)$ is regular it is sufficient to show that for every weak $*$-closed set $C \subset \Delta$ and every point $p \notin C$ there is an element T of $\mathscr{B}_*(\ell_p, \ell_q)$ such that $T^\wedge(p) \neq 0$ and $T^\wedge(C) = 0$. (See [11, p. 83].) Since we have already shown that $\mathscr{B}_*(\ell_p, \ell_q)^\wedge$ is locally isomorphic to $C(\Delta(p, q))$, this has been done. ▨

COROLLARY 3.5. $\mathscr{B}_*(\ell_p, \ell_q)^\wedge \neq C(\Delta(p, q))$ *unless* $p = 1$ *and* $q = \infty$.

Proof. Let $a(i)$ be a sequence of positive numbers converging to 0 but which is not in ℓ_p for any $p < \infty$. Define f on $\Delta(p, q)$ by

$$f((i, j)) = \begin{cases} a(i + j) & \text{if } (i, j) \in \mathbf{N} \times \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in C(\Delta(p, q))$ for all p and q but $f \notin \mathscr{B}_*(\ell_p, \ell_q)$ unless $p = 1$ and $q = \infty$. ▨

COROLLARY 3.6. *$\mathscr{B}_*(\ell_p, \ell_q)$ is not isomorphic to a C^*-algebra unless $p = 1$ and $q = \infty$.*

Proof. If $\mathscr{A}$ is a commutative C^*-algebra then $\mathscr{A}^\wedge = C(\Delta(\mathscr{A}))$. ▨

This is related to Varopoulos' question [24] of whether $\mathscr{B}_*(\ell_2, \ell_2)$ is a Q-algebra; that is, a uniform algebra divided by one of its closed ideals. The question is an initial attempt to determine how the linear and norm structures of $\mathscr{B}(\ell_2, \ell_2)$ influence the Banach algebras compatible with them. Further, it is natural to ask which, if any, of the $\mathscr{B}_*(\ell_p, \ell_q)$ spaces are Q-algebras.

4. HULLS AND KERNELS

PROPOSITION 4.1. *For $1 \leqslant p \leqslant q \leqslant \infty$, $\mathscr{B}_*(\ell_p, \ell_q) \neq$ closure of elements whose Gelfand transforms have compact support in $\Delta(p, q)$, except when $p = 1$ and $q = \infty$.*

Proof. Facts 1 and 6, coupled with Theorem 3.4 show that every element with compact support has a square root in $\mathscr{B}_*(\ell_p, \ell_q)$. Proposition 2.7 showed that $\mathscr{B}_*(\ell_p, \ell_q)^2$ is not dense in $\mathscr{B}_*(\ell_p, \ell_q)$. ▨

COROLLARY 4.2. *For $1 \leqslant p \leqslant q \leqslant \infty$, except when $p = 1$ and $q = \infty$, not every proper closed ideal of $\mathscr{B}_*(\ell_p, \ell_q)$ is contained in a regular maximal ideal.*

Proof. Let $\mathscr{I}$ be the ideal of all elements with compact support. For the given values of p and q, $\mathscr{I}^-$ is not dense. For any r in $\Delta(p, q)$ the proof of Theorem 3.4 showed that there is an S in $\mathscr{I}$ with $r(S) \neq 0$. Then $\mathscr{I}^-$ is a proper closed ideal contained in no regular maximal ideal. ▨

COROLLARY 4.3. *For $1 \leqslant p \leqslant q \leqslant \infty$, except when $p = 1$ and $q = \infty$, not every closed ideal of $\mathscr{B}_*(\ell_p, \ell_q)$ is equal to the kernel of its hull.*

Proof. Again let $\mathscr{I}$ be the ideal of elements with compact support. Then $\text{kernel}(\text{hull}(\mathscr{I}^-)) = \mathscr{B}_*(\ell_p, \ell_q) \neq \mathscr{I}^-$. ▨

In the theory of commutative Banach algebras the closure of the elements with compact support is an important ideal, while in operator theory the compact operators are usually very important. If $1 < p, q < \infty$ these sets coincide in $\mathscr{B}_*(\ell_p, \ell_q)$. In particular, for $1 < p \leqslant q < \infty$, Corollary 4.3 shows that the compact operators are not equal to the kernel of their hull. This raises the question of characterizing kernel(hull(compact operators in $\mathscr{B}_*(\ell_p, \ell_q)$)). Stout [22] solved this problem for operators on a Hilbert space in the sense that he characterized the union, over all orthonormal bases, of the kernel of the hull. Since the answer there has interesting connections to operator theory, it seems useful to investigate this in other ℓ_p spaces. To do so it is necessary to know what happens when different bases are used.

QUESTION. Pick bases (e_n) of ℓ_{p^*} and (f_n) of ℓ_q with $\|e_n\| = \|f_n\| = 1$ for all n. In these bases does Schur multiplication make $\mathscr{B}(\ell_p, \ell_q)$ a Banach algebra?

Let $T \in \mathscr{B}(\ell_2, \ell_2)$ be the operator

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \oplus \cdots.$$

If S is an element of $\mathscr{B}(\ell_2, \ell_2)$ with compact support then it can be shown that $\|T^*T - S\|_{2,2} \geqslant 1$, which shows that the elements with compact support are not even dense in $\mathscr{B}_*(\ell_2, \ell_2)^2$. They are, however, dense in $\mathscr{B}_*(\ell_2, \ell_2)^3$, a fact obtained from the following result.

PROPOSITION 4.4. *Let* $\mathscr{I}$ *be a closed ideal of* $\mathscr{B}_*(\ell_p, \ell_q)$ *and let* $\mathscr{J} = \text{kernel}(\text{hull}(\mathscr{I}))$. *Then*

$$(\mathscr{I} * \mathscr{I} * \mathscr{I})^- = (\mathscr{J} * \mathscr{J} * \mathscr{J})^-.$$

Proof. Let $\mathscr{K}$ denote the ideal of elements of $\mathscr{J}$ with compact support. Then $\mathscr{K} \subset \mathscr{I}$, and also $\mathscr{K}^2 = \mathscr{K}$, so it will be enough to show that $\mathscr{K}$ is dense in $\mathscr{J}^3$. For any $A = (a_{ij})$ in $\mathscr{J}$ and any $\varepsilon > 0$ let $A_\varepsilon = (\alpha_{ij}) \in \mathscr{B}_*(\ell_p, \ell_q)$ be given by

$$\alpha_{ij} = \begin{cases} a_{ij} & \text{if } |a_{ij}| \geqslant \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

and let $|A|$ denote the matrix $|A|(i,j) = |A(i,j)|$. Notice $A_\varepsilon \in \mathscr{K}$.

Let $B, C \in \mathscr{J}$ and let $D = A * B * C$. $A_\varepsilon * (B * C) \in \mathscr{K}$, and

$$\|D - A_\varepsilon * (B * C)\| = \|(A - A_\varepsilon) * (B * C)\| \leqslant$$

$$\leqslant \varepsilon \| \, |B*C| \, \| \leqslant \varepsilon \|B\| \, \|C\|,$$

where one goes from the first to second lines by noticing that $B*C$ is an absolute operator, by Theorem 2.2, and so for any matrix M,

$$\|M * B * C\|_{p,q} \leqslant \|M\|_\infty \cdot \||B * C|\|_{p,q},$$

where $\|M\|_\infty$ denotes the supremum of the entries of M. ▧

If D is a closed subset of $\Delta(\mathscr{A})$ such that $\text{hull}(\mathscr{I}) = D = \text{hull}(\mathscr{J})$ implies $\mathscr{I}^- = \mathscr{J}^-$ for all ideals $\mathscr{I}$ and $\mathscr{J}$, then D is said to be a set of *spectral synthesis*. Corollary 4.3 showed that if $1 \leqslant p \leqslant q \leqslant \infty$, except when $p = 1$ and $q = \infty$, then $\emptyset$ is not a set of spectral synthesis. On the other hand, if $p > q$ then every subset of $\Delta(p, q)$ is a set of spectral synthesis, and every closed subset of $\Delta(1, \infty)$ is a set of spectral synthesis. Let $A \subset \mathbf{N} \times \mathbf{N}$. Then A is a *(p, q)-line* if

i) $p > q$ and A is arbitrary.

ii) $p = 1$, $q = \infty$, and A is arbitrary.

iii) $1 < p \leqslant q < \infty$ and A is contained in one row, or one column, or contains no more than one element of any row or column.

iv) $1 = p \leqslant q < \infty$ and A is contained in one column, or contains no more than one element of any column.

v) $1 < p \leqslant q = \infty$ and A is contained in one row, or contains no more than one element of any row.

THEOREM 4.5. *Let $\mathscr{I}$ be an ideal of $\mathscr{B}_*(\ell_p, \ell_q)$. The following conditions are equivalent:*

i) $\operatorname{hull}(\mathscr{I})$ *is a set of spectral synthesis.*

ii) *There are (p, q)-lines $L_1, \ldots, L_n$ of $\mathbf{N} \times \mathbf{N}$ such that* $\operatorname{hull}(\mathscr{I}) = \Delta(p, q) \setminus (\cup L_i)^-$.

iii) $\mathscr{I} * \mathscr{I} + F(\mathscr{I}) = \mathscr{I}$, *where $F(\mathscr{I})$ is all finite rank operators in $\mathscr{I}$.*

Proof. We will prove this for $1 < p \leqslant q < \infty$, the other cases being similar. Suppose that condition ii) holds, and let the L_i be chosen so that they are disjoint. Let $\mathscr{I}/L$ denote those elements T of $\mathscr{I}$ such that the support $(T^{\sim}) \cap \mathbf{N} \times \mathbf{N} \subset L$. If some L_i is finite then $\mathscr{I}/L_i$ is isomorphic to $\ell_\infty(L_i)$. If some L_i is infinite and is not contained in a single row or column then $\mathscr{B}_*(\ell_p, \ell_q)/L_i$ is $\ell_\infty(L_i)$, $\mathscr{I}/L_i$ is just an ideal of $\ell_\infty(L_i)$, and $\mathscr{I}/L_i$ is isomorphic to $C(L_i^- \setminus \operatorname{hull}(\mathscr{I}/L_i))$. If some L_i is infinite and contained in a single row (column) then $\mathscr{I}/L_i$ is just $\ell_{p^*}(L_i)$ $(\ell_q(L_i))$. In all cases, ii) implies i) and iii) since these properties remain true under finite unions.

If ii) is false then the support of $\mathscr{I}$ contains a set like

$$\begin{array}{llllll} x & & & & & \\ & \ddots & & & & \\ & & xx & & & \\ & & & \ddots & & \\ & & & & xxx & \\ & & & & & \ddots \end{array}$$

or its transpose. (This uses Lemma 2.1.) Then

$$T = \begin{pmatrix} 1 & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & \\ & & 0 & & & & & & & & & & \\ & & & 0 & 2^{1/p-1} & 2^{1/p-1} & & & & & & & \\ & & & & & & 0 & & & & & & \\ & & & & & & & 0 & & & & & \\ & & & & & & & & 0 & & & & \\ & & & & & & & & & 3^{1/p-1} & 3^{1/p-1} & 3^{1/p-1} & \\ & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & \quad 0 \\ & & & & & & & & & & & & \qquad 0 \end{pmatrix}$$

is in kernel(hull($\mathscr{I}$)). (For the transpose case use $-1/q$ instead of $1/p-1$.) However, T is not in the closure of the elements with compact support. Let $\mathscr{J}$ = those elements in $\mathscr{I}$ in the closure of the elements with compact support. Then hull($\mathscr{J}$) = = hull($\mathscr{I}$) but $\mathscr{I} \neq$ kernel(hull($\mathscr{I}$)). To show that iii) must also be false let r be such that $2(1/p - 1) < r < 1/p - 1$, and define the compact operator S by

$$S = \begin{pmatrix} 1^r & & & & & & \\ & \ddots & & & & & \\ & & 2^r & 2^r & & & \\ & & & \ddots & & & \\ & & & & 3^r\ 3^r\ 3^r & & \\ & & & & & \ddots & \end{pmatrix}.$$

S, being compact, is in $\mathscr{I}$. In an operator of the form $A*B$ the rows must have uniformly bounded $p^*/2$ norm. Using this fact it is easy to show that $A*B - S$ cannot be finite rank for any A and B in $\mathscr{B}_*(\ell_p, \ell_q)$, and therefore iii) is false. ▨

5. FURTHER REMARKS

Despite its many uses, the systematic study of Schur multiplication has only recently begun. Because of this there are many fundamental questions which are unanswered, a few of which will be outlined here. An immediate question is to determine which Banach spaces X and Y have a Schur multiplication in $\mathscr{B}(X, Y)$. To even make sense X and Y must have bases, but what additional conditions are required? Ruckle [15] has worked some with such questions.

Proposition 2.5 showed that the compact operators in $\mathscr{B}_*(\ell_p, \ell_q)$ are a Schur ideal, except when $p = 1$ and $q = \infty$. In general, if $\mathscr{B}(X, Y)$ has a Schur product when do the compact operators form a Schur ideal? In $\mathscr{B}(\ell_2, \ell_2)$ use of duality shows that the trace class is also a Schur ideal (see Johnson and Williams [7]), and interpolation shows that, for $1 \leqslant r \leqslant \infty$, the Schatten r-class ideal is also a Schur ideal. (Stout [22] characterized those ideals of $\mathscr{B}(\ell_2, \ell_2)$ which are also Schur ideals.) A trace class can be defined in $\mathscr{B}(\ell_p, \ell_q)$ by using the s-numbers of Pietsch [12]. When is the trace class in $\mathscr{B}(\ell_p, \ell_q)$ a Schur ideal? Actually, there are many different s-numbers, so there may be many different trace classes.

A *Schur multiplier* on $\mathscr{B}(\ell_p, \ell_q)$ is a matrix (m_{ij}) such that if $(a_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$ then $(m_{ij}a_{ij}) \in \mathscr{B}(\ell_p, \ell_q)$. Schur multipliers are important members of $\mathscr{B}(\mathscr{B}(\ell_p, \ell_q))$, and many deep results in analysis are expressible in terms of them. This point is forcefully demonstrated in Bennett [2], who gives precise norm bounds and lists some open questions. Results on multipliers are generally difficult and rarer than results on Schur products. For example, there is no characterization of the spectrum of a multiplier. Each element of $\beta(\mathrm{N} \times \mathrm{N})$ is a nontrivial multiplicative functional on the multipliers but it can be shown that they are not all unique for the multipliers on $\mathscr{B}(\ell_2, \ell_2)$. A characterization of the equivalence classes has not been completed, nor is it known that all maximal ideals correspond to points of $\beta(\mathbf{N} \times \mathbf{N})$. A problem whose solution may help with these questions is to give a useful characterization of the multipliers which have only 0's and 1's as entries. It is a well-known result that, on $\mathscr{B}(\ell_2, \ell_2)$, the matrix of 0's above the diagonal and 1's on and below it is not a multiplier (see Kwapien and Pełczyński [9] for a proof and its connections with several areas of analysis). To date, every 0-1 matrix which is known not to be a multiplier on $\mathscr{B}(\ell_2, \ell_2)$ contains arbitrarily large upper left corners of this matrix, perhaps rearranged. Bennett [2] has given some characterizations of multipliers, but they are difficult to apply to specific matrices.

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QUENTIN F. STOUT
Mathematical Sciences
State University of New York
Binghamton, NY 13901
U.S.A.

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