EECS 203: Discrete Mathematics
Individual Homework 3 (Winter 2016)

Answer Key

Due: May 19, 2016

1. (2 points) Section 1.6 Problem 28

Solution:

1.	$\forall x(P(x) \vee Q(x))$	premise
2.	$\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$	premise
	Let c be an arbitrary value	
3.	$P(c) \vee Q(c)$	Universal instantiation, 1
4.	$(\neg P(c) \wedge Q(c)) \rightarrow R(c)$	Universal instantiation, 2
5.	$\neg(\neg P(c) \wedge Q(c)) \vee R(c)$	Definition of $\rightarrow$
6.	$P(c) \vee \neg Q(c) \vee R(c)$	De Morgans, 5
7.	$(P(c) \vee R(c)) \vee \neg Q(c)$	Commutative, Associative, 6
8.	$P(c) \vee P(c) \vee R(c)$	Resolution, 3, 7
9.	$P(c) \vee R(c)$	Idempotent, 8
10.	$\neg R(c) \rightarrow P(c)$	Definition of $\rightarrow$
11.	$\forall x(\neg R(x) \rightarrow P(x))$	Universal generalization, 10

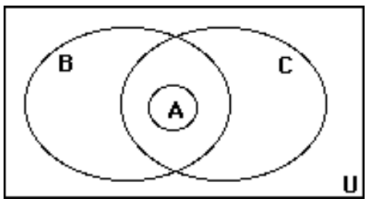
2. (2 points) Prove that there are no integer solutions for y and z to the equation $2z^2+5y^2 = 14$

Solution: Because both z^2 and y^2 are nonnegative, each term on the left must be no greater than 14. $2 \cdot 3^2 = 2 \cdot 9 = 18 > 14$, so $|z| < 3$. $5 \cdot 2^2 = 5 \cdot 4 = 20 > 14$, so $|y| < 2$. Because y and z are integers, $|z| \leq 2$ and $|y| \leq 1$. This means $2z^2+5y^2 \leq 2 \cdot 2^2+5 \cdot 1^2 = 2 \cdot 4+5 \cdot 1 = 8+5 = 13 < 14$. The biggest the expression can get is still 1 less than 14.

Alternately, because of the absolute value inequalities, we can write out a few cases. We know $|z| \in \{0,1,2\}$ and $|y| \in \{0,1\}$, so we have only 6 cases:
$2 \cdot 0^2 + 5 \cdot 0^2 = 0 \neq 14$
$2 \cdot 1^2 + 5 \cdot 0^2 = 2 \neq 14$
$2 \cdot 2^2 + 5 \cdot 0^2 = 8 \neq 14$
$2 \cdot 0^2 + 5 \cdot 1^2 = 5 \neq 14$
$2 \cdot 1^2 + 5 \cdot 1^2 = 7 \neq 14$
$2 \cdot 2^2 + 5 \cdot 1^2 = 13 \neq 14$

3. (1 point) Section 2.1 Problem 16

Solution: We allow B and C to overlap, because we are told nothing about their relationship. The set A must be a subset of each of them, and that forces it to be positioned as shown. We cannot actually show the properness of the subset relationships in the diagram, because we dont know where the elements in B and C that are not in A are locatedthere might be only one (which is in both B and C), or they might be located in portions of B and/or C outside the other. Thus the answer is as shown, but with the added condition that there must be at least one element of B not in A and one element of C not in A.



4. (2 points) Section 2.1 Problem 18

Solution: Since the empty set is a subset of every set, we just need to take a set B that contains $\emptyset$ as an element. Thus we can let $A = \emptyset$ and $B = \{\emptyset\}$ as the simplest example.

5. (2 points) Section 2.1 Problem 42 (b,d)

Solution:

b) There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is truein fact, every integer satisfies this statement.

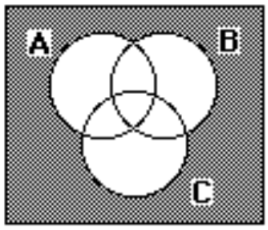
d) The square of every integer is an integer. This is true.

6. (2 points) Section 2.2 Problem 14

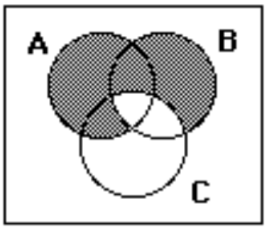
Solution: Since $A = (A - B) \cup (A \cap B)$, we conclude that $A = \{1, 5, 7, 8\} \cup \{3, 6, 9\} = \{1, 3, 5, 6, 7, 8, 9\}$. Similarly $B = (B - A) \cup (A \cap B) = \{2, 10\}\{3, 6, 9\} = \{2, 3, 6, 9, 10\}$.

7. (2 points) Section 2.2 Problem 26 (b,c)

Solution: The set is shaded in each case.



(b)



(c)

8. (2 points) Section 2.2 Problem 48 (a,b)

Solution: We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$ Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

a) $A_n = \{..., -2, -1, 0, 1, ..., n\}$

b) $A_1 = \{..., -2, -1, 0, 1\}$

9. (3 points) Give an example of a function from R to R that is

   a) one-to-one but not onto.

   b) onto but not one-to-one.

   c) both onto and one-to-one (but different from the identity function).

Solution: There are many solutions to this problem, so here is just one of them.

1. $f(x) = e^x$. Given $e^x = e^y$, we know $\ln e^x = \ln e^y$ or in other words, $x = y$. However, no matter the value of x, we never have $e^x = -1$
2. $f(x) = x^3 - x$. This is a continuous function that approaches infinity as x does and -infinity as x does, so it must be onto. However, $0^3 - 0 = 1^3 - 1$, but $0 \neq 1$
3. x^3. $f(y^{1/3}) = (y^{1/3})^3 = y^{3/3} = y$, so f is onto. Given $x^3 = y^3$, we can take the cuberoot of both sides to get $(x^3)^{1/3} = (y^3)^{1/3}$ or in other words, $x = y$

10. (3 points) Section 2.3 Problem 28

Solution: For the function to be invertible, it must be a one-to-one correspondence. This means that it has to be one-to-one, which it is, and onto, which it is not, because, its range is the set of positive real numbers, rather than the set of all real numbers. When

we restrict the codomain to be the set of positive real numbers, we get an invertible function. In fact, there is a well-known name for the inverse function in this case-the natural logarithm function (g(x) = ln x).

11. (2 points) Section 2.3 Problem 36

Solution: We have $(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$, whereas $(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$. Note that they are not equal.

12. (2 points) Section 2.4 Problem 16 (a,b)

Solution: In the iterative approach, we write a_n in terms of a_{n-1} , then write a_{n-1} in terms of a_{n-2} (using the recurrence relation with n - 1 plugged in for n), and so on. When we reach the end of this procedure, we use the given initial value of a_0. This will give us an explicit formula for the answer or it will give us a finite series, which we then sum to obtain an explicit formula for the answer.

a) $a_n = -a_{n-1} = (-1)^2 a_{n-2} = ... = (-1)^n a_{n-n} = (-1)^n a_0 = 5 \cdot (-1)^n$

b) $a_n = 3 + a_{n-1} = 3 + 3 + a_{n-2} = 2 \cdot 3 + a_{n-2} = 3 \cdot 3 + a_{n-3} = ... = n \cdot 3 + a_{n-n} = n \cdot 3 + a_0 = 3n + 1$

13. (3 points) Section 2.4 Problem 36

Solution: We use the suggestion (simple algebra shows that this is indeed an identity) and note that all the terms in the summation cancel out except for the 1/k when k = 1 and the 1/(k + 1) when k = n:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

14. (2 points) Section 2.5 Problem 2 (a,b,d,e)

Solution:

a) This set is countably infinite. The integers in the set are 11, 12, 13, 14, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 11$, $2 \leftrightarrow 12$, $3 \leftrightarrow 13$, and so on; in general $n \leftrightarrow (n + 10)$.

b) This set is countably infinite. The integers in the set are -1, -3, -5, -7, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow -1$, $2 \leftrightarrow -3$, $3 \leftrightarrow -5$, and so on; in general $n \leftrightarrow -(2n - 1)$.

d) This set is uncountable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5.

e) This set is countable. We can list its elements in the order (2, 1),(3, 1),(2, 2),(3, 2),(2, 3),(3, 3), . . ., giving us the one-to-one correspondence $1 \leftrightarrow (2, 1)$, $2 \leftrightarrow (3, 1)$, $3 \leftrightarrow (2, 2)$, $4 \leftrightarrow (3, 2)$, $5 \leftrightarrow (2, 3)$, $6 \leftrightarrow (3, 3)$, ...