Part 2: Sinusoidal Signals

Outline

- Introduction to three representations:
  - formula $x(t) = A\cos(2\pi f_0 t + \phi)$
  - amplitude, frequency (or period), phase
  - graph/plot
- Converting between these three representations
- Signal characteristics for sinusoids
- Operations on sinusoids: adding / multiplying
- Simplify sums of sinusoids *of same frequency*
  - trigonometry
  - phasors
- Complex arithmetic
  - cartesian / polar / complex exponential form
  - Euler's identities
  - addition/subtraction
  - multiplication / division
  - polynomial roots
- Complex exponential signals
- Beat frequencies

Reading: Ch. 2 of textbook, Appendix A.

Overview of sinusoids

Why?

- Occur in nature
  - tuning fork
  - flute
  - spring-mass system
  - solution to many differential equations
- Engineering systems
  - power generation (rotating equipment)
  - laser
  - resonator circuit (capacitor and inductor)
  - oscillator (modulators for comm)
- Linear time-invariant (LTI) systems, aka filters
  - sinusoidal signal in $\rightarrow$ $\boxed{\text{LTI system}}$ $\rightarrow$ sinusoidal signal out
    This property is unique to sinusoidal signals!
  - motivates considering other signals as sums of sinusoids

Example. Audio recording of tuning fork from across a room in presence of multitude of reflections. Still sinusoidal!

Sinusoidal signals

For a while now we will focus on **continuous-time** sinusoidal signals, described by the following general formula:

$$x(t) = A\cos(\underbrace{2\pi f_0}_{\omega_0} t + \phi) \tag{2-1}$$

This signal, which is a function of the continuous-time variable t, is described by three parameters.

- A is the **amplitude** (signal units, *e.g.*, volts, Amperes, etc.).
- f_0 is the **frequency** (Hz=cycle/second, kilohertz: kHz=10^3Hz, megahertz: MHz=10^6Hz)
- ϕ is the **phase** (in radians)

Certain properties of the cosine function determine the sensible ranges for the three parameters.

- We always choose the amplitude $A \geq 0$ and usually $A > 0$.
  Why? Because of this property: $-\cos(\theta) = \cos(\theta + \pi) = \cos(\theta - \pi)$. So:
  $$-A\cos(2\pi f_0 t + \phi') = A\cos(2\pi f_0 t + \underbrace{\phi' + \pi}_{\text{new } \phi})$$
  So a negative sign can be absorbed into the phase term.
- For sinusoidal signals, we always choose the frequency $f_0 \geq 0$.
  Why? Because of the property $\cos(-\theta) = \cos(\theta)$. So:
  $$\cos(2\pi(-f_0)t + \phi') = \cos(-[2\pi f_0 t - \phi']) = \cos(2\pi f_0 t \underbrace{- \phi'}_{\text{new } \phi}).$$
  So a sinusoid with a negative frequency is indistinguishable from a sinusoid with a positive frequency but with the opposite phase. So we always just use the positive frequency.
  Later in the chapter we will consider complex exponential signals that can have positive or negative frequencies. But not for sinusoidal signals!
- We usually focus on values of the phase ϕ that are in the range $(-\pi, \pi]$.
  Why? Because of this particularly important property of the cosine function: $\cos(\theta + n2\pi) = \cos(\theta)$ for $n \in \mathbb{Z}$. In other words, the cosine function is periodic with fundamental period 2π. In other words, we can add or subtract multiples of 2π from the phase without changing the sinusoidal signal at all.
  So we may as well add or subtract multiples of 2π from the phase until the phase satisfies $-\pi < \phi \leq \pi$. This phase is called the **principal value**.

Example:

$$\cos(2\pi t + 31\pi/3) = \cos(2\pi t + 31\pi/3 - 5 \cdot 2\pi) = \cos(2\pi t + \pi/3).$$

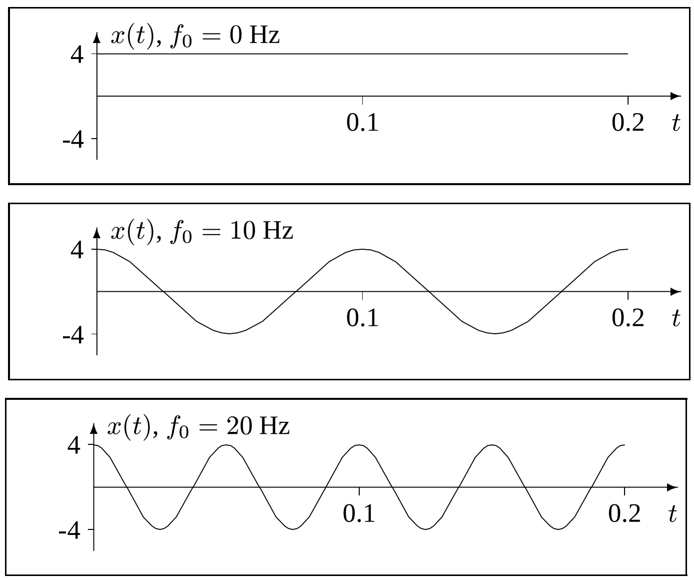
- Why do we use only cosine rather than either cosine or sin?
  Because of this property: $\sin(\theta) = \cos(\theta - \pi/2)$.
  So we can take any signal expressed in terms of the $\sin$ function and rewrite it in terms of the $\cos$ form given above as follows:

$$A \sin(2\pi f_0 t + \phi') = A \cos\Big(2\pi f_0 t + \underbrace{[\phi' - \pi/2]}_{\text{new } \phi}\Big).$$

When $A > 0$ and $f_0 \geq 0$ and $-\pi < \phi \leq \pi$, we say that (2-1) is in **standard form**.

Of these three parameters, the **frequency** is particularly important. The frequency determines the rate of oscillation of the sinusoid, the number of cycles per second. A larger frequency value (we say: "a higher frequency") corresponds to more oscillations per unit time.

The following figures show $x(t) = 4\cos(2\pi f_0 t)$ for various frequencies f_0.



Here are a few other comments about sinusoidal signals.

- $\omega_0 = 2\pi f_0$ is the **radian frequency** in units of radians/second.
  Conventional engineering units for frequency are Hz, not radians per second.
  There is little reason to use the notation ω_0 over $2\pi f_0$ except perhaps laziness...
- We say "sinusoid" even though we usually write $\cos$.
  The reasons for choosing $\cos$ rather than $\sin$ will be clear when we discuss complex signals later in this chapter.
- All continuous-time sinusoidal signals are **periodic**, so f_0 is in fact the **fundamental frequency**, but we usually just say **frequency** when discussing sinusoids.
- The (fundamental) **period** of a sinusoid is $T_0 = 1/f_0$. Why? Because:

$$x(t + T_0) = A\cos(2\pi f_0 (t + T_0) + \phi) = A\cos(2\pi f_0 t + \phi + 2\pi \underbrace{f_0 T_0}_{1}) = A\cos(2\pi f_0 t + \phi) = x(t)\,.$$

  So an alternate general form for a sinusoidal signal would be:

$$x(t) = A\cos\left(2\pi \frac{1}{T_0} t + \phi\right).$$

  As long as you put the argument in the form "$2\pi \cdot$ something $\cdot\, t + \phi$" then the "something" will be the frequency and its reciprocal will be the period.

The three representations

At this point we have three representations of a sinusoidal signal:

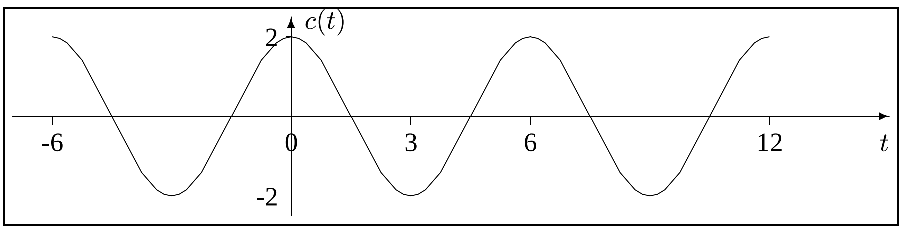
- formula $x(t) = A\cos(2\pi f_0 t + \phi)$
- list of 3 parameters: amplitude, frequency (or period), phase
- graph/plot
- (Later we will have a very important fourth representation: its **spectrum**.)

One must be able to convert between these representations.
Converting between the formula and the list of three parameters is obvious by inspection.

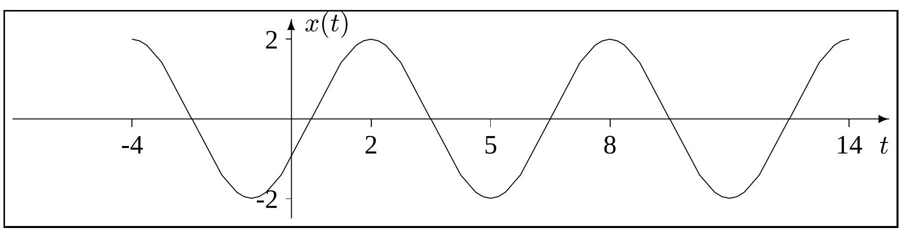
For manual graphing, (given the formula or the parameters) the following procedure can be helpful.

- First plot the sinusoid *without* the phase shift ϕ, *i.e.*, plot the signal $c(t) = A\cos(2\pi f_0 t)$.
  This is easy since the period is $T_0 = 1/f_0$.
- Then notice that $x(t) = A\cos(2\pi f_0 t + \phi)$, so we simply need to **phase shift** the signal $c(t)$ by the amount ϕ, keeping in mind that a 2π phase shift would be a complete cycle of the sinusoid.

Example. Sketch $x(t) = 2\cos\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right)$.
First draw $c(t) = 2\cos\left(2\pi\frac{1}{6}t\right)$, which has period $T_0 = 6$.



Now we shift this signal by a phase of $2\pi/3$ which is 1/3th of a period, or 2 time units in this case.



Formula from graph

To complete the story, we must also be able to examine a graph of a sinusoidal signal and determine its parameters. The amplitude A and the period T_0 are easily determined by inspection.

To determine the phase, first find the time location of the peak that is nearest to $t = 0$, call it, say t_p. Now a maximum of a cosine occurs when its argument is 0, *i.e.*, when $2\pi f_0 t_p + \phi = 0$. Thus the phase is: $\boxed{\phi = -2\pi f_0 t_p = -2\pi \frac{t_p}{T_0}.}$ Since the location of the peak nearest to $t = 0$ will be within $\pm T_0/2$ of $t = 0$, the phase computed according to the above formula will always be between $-\pi$ and π, as desired.

Example. Consider the signal $x(t)$ pictured above, but suppose we only had the graph and not the formula. From the graph we see that $A = 2$ and $T_0 = 6$. The nearest peak is at $t_p = 2$, so the phase is

$$\phi = -2\pi\frac{t_p}{T_0} = -2\pi\frac{2}{6} = -2\pi/3$$

so we have $x(t) = 2\cos(2\pi\frac{1}{6}t - 2\pi/3)$ which indeed agrees with the original formula.

Signal characteristics of sinusoids

In Part1 we defined about a dozen signal characteristics. Some of them are obvious for sinusoidal signals: the support is the reals, the duration is infinite, the minimum is $-A$ and the maximum is A, the energy is infinite (unless $A = 0$), and the period is $T_0 = 1/f_0$.

Here are some of the more interesting ones:
- $\mathrm{M}(x) = 0$. Sinusoids are symmetric about the horizontal axis so the average value is zero.
- $\mathrm{MS}(x) = A^2/2$.
  The derivation of this very important average power relationship is left as an exercise.
- From the two preceding characteristics, one can work out the RMS value, the variance, and the standard deviation.
- The signal value distribution of a sinusoidal signal was shown in a figure in Part1.
- The natural definition of the envelope would be simply a constant signal with value A.

Example. For AC power line, we know that the frequency is about $f_0 = 60$Hz. What about the amplitude A?
Is the conventional number "115V" the amplitude? No! Actually, 115V is the RMS voltage!
For a sinusoid, the RMS value is $\sqrt{\mathrm{MS}(x)} = \sqrt{A^2/2}$, so for AC power lines: $115V = A/\sqrt{2}$ so $A = 115V\sqrt{2} \approx 162.6V$.
Why is it expressed in RMS rather than in amplitude?
Because the power dissipated in a resistor with a sinusoidal voltage across it of RMS value equal to 115V is the same power that would be dissipated by that resistor with a 115V constant (DC) voltage across it. So the "effective" power is the RMS power.

Effect of simple signal operations on sinusoids

Suppose we start with a sinusoid $x(t) = A\cos(2\pi f_0 t + \phi)$ and then apply a simple signal operation to it. What happens?

- Amplitude scaling: $y(t) = cx(t)$
  $$y(t) = \begin{cases} cA\cos(2\pi f_0 t + \phi), & c \geq 0 \\ |c|A\cos(2\pi f_0 t + \phi - \pi), & c < 0. \end{cases}$$
  So amplitude scaling, scales the amplitude, naturally enough. (There is some rhyme and reason to the terminology...)
  (If c is negative, then both the amplitude and phase will change when we write the signal in standard form.)
- Time scaling: $y(t) = x(at)$
  $$y(t) = A\cos(2\pi f_0(at) + \phi) = A\cos(2\pi \underbrace{af_0}_{f_0'\text{: new frequency}} t + \phi)$$
  So the effect of time scaling is to scale the **frequency** of the sinusoidal signal.
  What happens if a is negative?
- Time shift: $y(t) = x(t - t_0)$
  $$y(t) = A\cos(2\pi f_0(t - t_0) + \phi) = A\cos(2\pi f_0 t + \underbrace{\phi - 2\pi f_0 t_0}_{\phi'\text{: new phase}})$$
  So the effect of time shift is to cause a corresponding **phase shift** of the sinusoidal signal.
  Note that the units of the expression $2\pi f_0 t_0$ is radians, as required.
- Squaring: $y(t) = x^2(t)$
  $$y(t) = A^2\cos^2(2\pi f_0 t + \phi) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\pi(2f_0)t + 2\phi)$$
  since $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

In each case, amplitude scaling, time scaling, and time shift, a sinusoidal signal remains a sinusoidal signal but one of its three parameters is changed by the operation.

Operations with two (or more) sinusoids

If we have two sinusoidal signals, $x_1(t)$ and $x_2(t)$, the two most interesting ways to "combine" them would be to add them or to multiply them.

First, consider **multiplication**. (Why should we care? AM radio is one example.)

Suppose $x_1(t) = A_1 \cos(2\pi f_1 t + \phi_1)$ and $x_2(t) = A_2 \cos(2\pi f_2 t + \phi_2)$. What happens when we multiply?

To analyze the product of these signals, recall the following identity:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta). \tag{2-2}$$

Thus

$$\begin{aligned} m(t) &= x_1(t)\, x_2(t) = [A_1 \cos(2\pi f_1 t + \phi_1)]\, [A_2 \cos(2\pi f_2 t + \phi_2)] \\ &= \frac{A_1 A_2}{2} \cos\Big(2\pi(f_1 - f_2)t + \phi_1 - \phi_2\Big) + \frac{A_1 A_2}{2} \cos\Big(2\pi(f_1 + f_2)t + \phi_1 + \phi_2\Big). \end{aligned}$$

So the result of multiplying two sinusoidal signals corresponds to a *sum* of two sinusoidal signals.

For this and other reasons, we focus almost entirely on sums of sinusoidal signals for the rest of the course!

Before we consider sums, we again ask, why should we care?
One example would be audio recording of a tuning fork from across a room in the presence of a reflection:

$$y(t) = \alpha_0 x(t - t_0) + \alpha_1 x(t - t_1)\,.$$

(Picture)

The question to be answered is: will the recorded signal be a sinusoid, or will it be some other shape?

Sums of sinusoidal signals of the *same* frequency

Case 1. Same frequency, *same phase*, different amplitudes.

$$A_1 \cos(2\pi f_0 t + \phi) + A_2 \cos(2\pi f_0 t + \phi) = (A_1 + A_2) \cos(2\pi f_0 t + \phi).$$

This case only requires arithmetic.

Case 2. Same frequency, *different* phases, *same* amplitudes.
To solve this case, we need to use tricks from trigonometry, using the identity (2-2) above.

$$\begin{aligned}
& A\cos(2\pi f_0 t + \phi_1) + A\cos(2\pi f_0 t + \phi_2) \\
&= 2A\left[\frac{1}{2}\cos\Big(\underbrace{2\pi f_0 t + \frac{\phi_1+\phi_2}{2}}_{\alpha} - \underbrace{\frac{\phi_2-\phi_1}{2}}_{\beta}\Big) + \frac{1}{2}\cos\Big(\underbrace{2\pi f_0 t + \frac{\phi_1+\phi_2}{2}}_{\alpha} + \underbrace{\frac{\phi_2-\phi_1}{2}}_{\beta}\Big)\right] \\
&= \underbrace{2A\cos\left(\frac{\phi_1-\phi_2}{2}\right)}_{\text{new amplitude } A'} \cos\Big(2\pi f_0 t + \underbrace{\frac{\phi_1+\phi_2}{2}}_{\text{new phase}}\Big).
\end{aligned}$$

So if we have same phase but different amplitudes or same amplitude but different phases, as long as the frequency is the same we end up with a new sinusoid of some different amplitude and phase (but same frequency).

Special cases
- $\phi_1 = \phi_2 \Rightarrow A' = 2A$ which is called **constructive interference**
- $\phi_1 = \phi_2 \pm \pi \Rightarrow A' = 0$ which is called **destructive interference**

Example.

What if the phases *and* the amplitudes are different? Remarkably, we still end up with a new sinusoid of some different amplitude and phase (but same frequency).

Case 3. Same frequency, *different* phases, *different* amplitudes.

Amazing fact:

$$A_1 \cos(2\pi f_0 t + \phi_1) + A_2 \cos(2\pi f_0 t + \phi_2) = A\cos(2\pi f_0 t + \phi)$$

for some amplitude A and some phase ϕ.

In words: adding together two (or more!) in sinusoidal signals *of the same frequency* yields a sinusoidal signal of that frequency with some amplitude and phase.

How do we find A and ϕ?

Hard way: trial and error trigonometry. It can be much messier than what we did in Case 2!
Systematic way: using complex phasors.

Example. Simplify the following sum of sinusoidal signals:

$$2\cos(2\pi f_0 t + \pi/4) + 2\sqrt{2}\cos(2\pi f_0 t - \pi/2).$$

Solving this by trigonometry would be painful!
Solution using phasors:

$$2\mathrm{e}^{\jmath\pi/4} + 2\sqrt{2}\mathrm{e}^{-\jmath\pi/2} = \sqrt{2} + \jmath\sqrt{2} - 2\sqrt{2}\jmath = \sqrt{2} - \jmath\sqrt{2} = 2\mathrm{e}^{-\jmath\pi/4} = A\mathrm{e}^{\jmath\phi}$$

So we conclude $A = 2$ and $\phi = -\pi/4$. Thus

$$2\cos(2\pi f_0 t + \pi/4) + 2\sqrt{2}\cos(2\pi f_0 t - \pi/2) = 2\cos(2\pi f_0 t - \pi/4).$$

To solve this example problem we use **complex numbers**. This problem illustrates one of several uses we will have for complex numbers in this course, so at this point we temporarily digress from signals to review complex numbers. After the review we will return to the study of sums of sinusoidal signals of the same frequency.

(The next chapter discusses sums of sinusoidal signals with different frequencies.)

Complex numbers

The first question one might ask is "why were complex numbers invented?"
One answer would be: to resolve a problem in algebra: finding the roots of a polynomial. If we limited ourselves to real numbers, then different polynomials of the same order would have different numbers of roots. By allowing consideration of complex numbers, all polynomials of degree M have M roots, some of which may be complex. This fact is so important that it is called the **fundamental theorem of algebra**.

Example. Consider the innocent looking polynomial: $z^2 + 1$. What are its roots? To find the roots, we equate the polynomial to zero and solve: $z^2 + 1 = 0$, so $z^2 = -1$. No real number satisfies this equality, but if we define the following **imaginary number**:

$$\jmath = \sqrt{-1},$$

then there are two roots: $z = \pm\jmath$, which is consistent with the fact that this is a second-degree polynomial.

One might say that $\jmath$ was "invented" so that the fundamental theorem of algebra works: Mth degree polynomial has M roots.

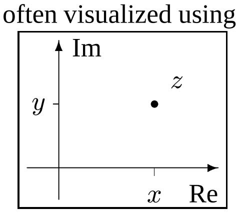
(We use $\jmath$ rather than i for complex numbers since i traditionally denotes electrical current in EE texts.)

Arithmetic

Cartesian form:

$$z = x + \jmath y = \mathrm{Re}\{z\} + \jmath\,\mathrm{Im}\{z\}\,.$$

The set of all **complex numbers** is denoted $\mathbb{C}$, and is often visualized using the **complex plane**.



Fundamental operations (for $z_1 = x_1 + \jmath y_1$ and $z_2 = x_2 + \jmath y_2$):

- Equality:

$$z_1 = z_2 \text{ iff } x_1 = x_2 \textit{ and } y_1 = y_2$$

- Addition ***(Picture)***

$$z_1 + z_2 = (x_1 + \jmath y_1) + (x_2 + \jmath y_2) = (x_1 + x_2) + \jmath\,(y_1 + y_2)$$

- Scaling by a real number c.

$$cz = c\,(x + \jmath y) = cx + \jmath\,cy \qquad \textbf{\textit{(Picture)}}$$

- Multiplication

$$z_1 \cdot z_2 = (x_1 + \jmath y_1) \cdot (x_2 + \jmath y_2) = x_1x_2 + \jmath x_1y_2 + \jmath y_1x_2 \underbrace{- y_1y_2}_{\text{since } \jmath^2 = -1} = (x_1x_2 - y_1y_2) + \jmath\,(x_1y_2 + x_2y_1)$$

- The usual properties of arithmetic apply: commutative, associative, distributive.

Complex **conjugate** ***(Picture)***

$$z^\star = x - \jmath y$$

The **real part** and **imaginary part**:

$$\mathrm{Re}\{z\} = \frac{z + z^\star}{2}, \qquad \mathrm{Im}\{z\} = \frac{z - z^\star}{2\jmath}$$

The **magnitude**:

$$|z| = \sqrt{x^2 + y^2}$$

The **squared magnitude**:

$$|z|^2 = zz^\star = x^2 + y^2$$

Division (complex conjugate of denominator simplifies this to a multiplication problem)

$$\frac{z_1}{z_2} = \frac{z_1 z_2^\star}{z_2 z_2^\star} = \frac{z_1 z_2^\star}{|z_2|^2} = \frac{x_1 + \jmath\, y_1}{x_2 + \jmath\, y_2} \cdot \frac{x_2 - \jmath\, y_2}{x_2 - \jmath\, y_2} = \frac{(x_1 x_2 + y_1 y_2) + \jmath\,(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + \jmath \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right).$$

<u>Example</u>. Find z_1/z_2 when $z_1 = -8 - \jmath\, 8\sqrt{3}$ and $z_2 = 4 - \jmath\, 4\sqrt{3}$.

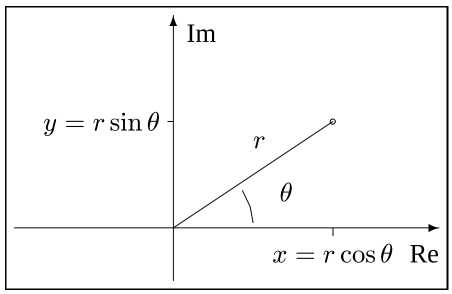
$$\frac{z_1}{z_2} = \frac{-8 - \jmath\, 8\sqrt{3}}{4 - \jmath\, 4\sqrt{3}} = \frac{(-8 - \jmath\, 8\sqrt{3})(4 + \jmath\, 4\sqrt{3})}{4^2 + (4\sqrt{3})^2} = \frac{(-32 + 32 \cdot 3) + \jmath\,(-32 - 32)\sqrt{3})}{64} = 1 - \jmath\, \sqrt{3}.$$

Polar form

The multiplication and division operations look pretty messy. Is there an easier way? Yes! Use **polar form**:

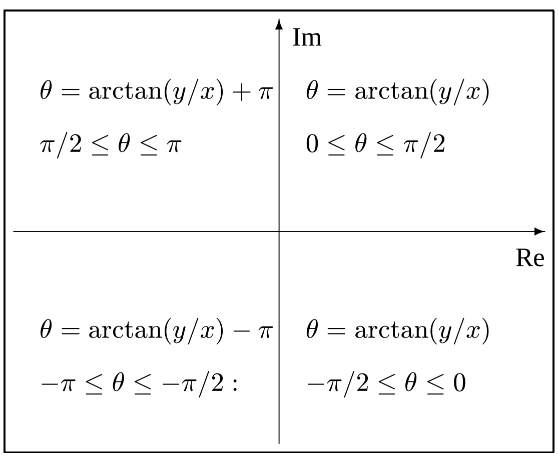
$$z = x + \jmath\, y = r\angle\theta, \qquad r = |z| = \sqrt{x^2 + y^2} \geq 0, \qquad \theta = \arctan(y/x).$$

Obviously r is the **magnitude**. And we call θ the **angle**.



The expression $\arctan(y/x)$ must be interpreted carefully! A better expression would be $\arctan(y, x)$ since the result depends not just on the ratio, but also on the *signs* of the real and imaginary parts of z. In fact, MATLAB has a command `atan2` that has two arguments precisely for this reason.

When we write "arctan" we mean "arctan but possibly with π added or subtracted" so that the result is a value in the range $-\pi \leq \theta \leq \pi$, depending on the signs of x and y. The following diagram illustrates.



$$\theta = \begin{cases} \arctan(y/x), & x > 0 \\ \arctan(y/x) + \pi, & x < 0,\ y > 0 \\ \arctan(y/x) - \pi, & x < 0,\ y < 0 \\ \pi/2, & x = 0,\ y > 0 \\ -\pi/2, & x = 0,\ y < 0 \\ 0 \text{ (irrelevant)}, & x = 0,\ y = 0 \end{cases}$$

Be careful when using your calculator's arctan function!

<u>Example</u>. Convert $z = -2 - \jmath\, 2$ to polar form.
$r = |z| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ and $\theta = \arctan(-2/-2) = \pi/4 - \pi = -3\pi/4$, because $\arctan(1) = \pi/4$.
So $z = 2\sqrt{2}\mathrm{e}^{-\jmath\, 3\pi/4}$. ***(Picture)*** .

Operations in polar form (where $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$).

- Multiplication $z_1 z_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$
- Division $z_1/z_2 = (r_1/r_2)\angle(\theta_1 - \theta_2)$
- Reciprocal $1/z = \frac{1}{r}\angle{-\theta}$

Exponential form

Even more convenient: use **exponential form**: $z = r e^{\jmath\theta}$.
So now we need to interpret the exponential function for complex arguments!
Important properties of the exponential function that we should maintain:

- $e^0 = 1$
- $e^{a+b} = e^a e^b$
- $(e^z)^n = e^{nz}$ for $n \in \mathbb{Z}$
- $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$

The following law gives the unique definition that satisfies these properties:

$$\text{Euler's law: } e^{\jmath\theta} = \cos\theta + \jmath \sin\theta.$$

Example. $e^{\jmath\pi} = \cos\pi + \jmath\sin\pi = -1$. This equality is very important! Also: $e^{-\jmath\pi} = -1$.

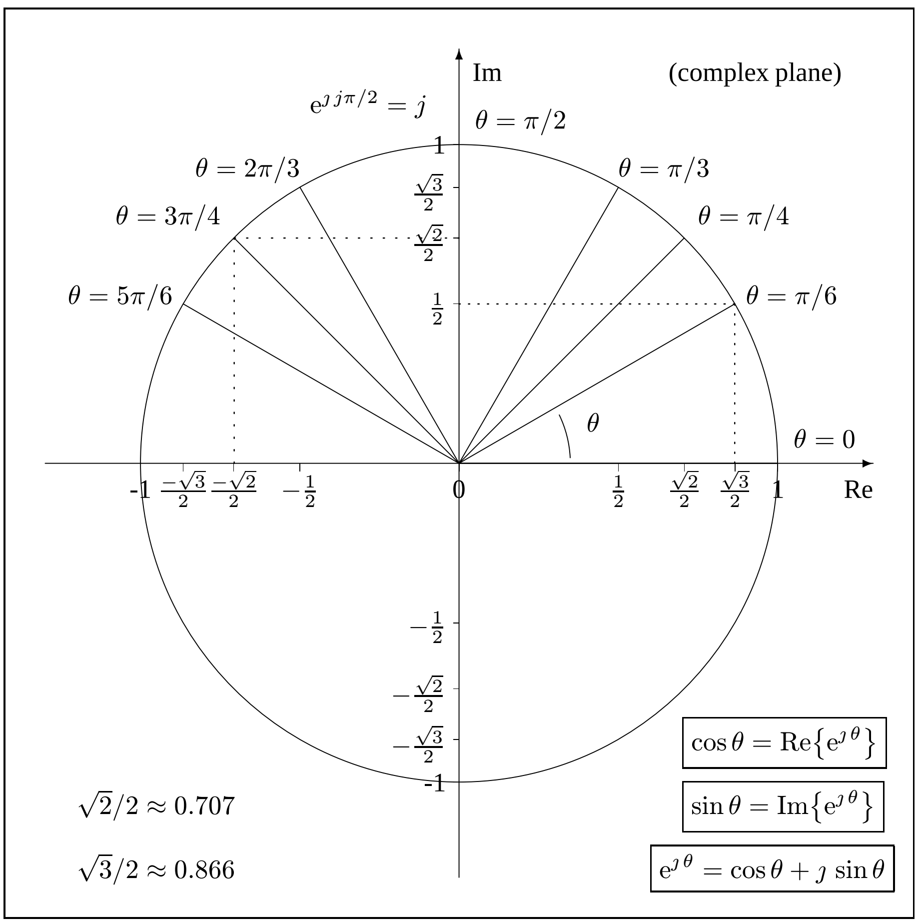
More generally: $e^{\alpha+\jmath\theta} = e^{\alpha} e^{\jmath\theta} = e^{\alpha}(\cos\theta + \jmath\sin\theta)$ for $\alpha, \theta \in \mathbb{R}$

Relationships between the three forms:

$$z = x + \jmath y = r\angle\theta = (r\cos\theta) + \jmath(r\sin\theta) = r(\cos\theta + \jmath\sin\theta) = r e^{\jmath\theta}$$

The following figure summarizes the "most important angles" around the unit circle and their sin and cos values.

Visualizing $e^{\jmath\theta}$ using the unit circle



Inverse Euler identities

$$\cos\theta = \frac{e^{\jmath\theta} + e^{-\jmath\theta}}{2}, \qquad \sin\theta = \frac{e^{\jmath\theta} - e^{-\jmath\theta}}{2\jmath}$$

Operations in complex exponential form (where $z_1 = r_1 e^{\jmath \theta_1}$ and $z_2 = r_2 e^{\jmath \theta_2}$):
- Multiplication $z_1 z_2 = (r_1 r_2) e^{\jmath (\theta_1 + \theta_2)}$ ***(Picture?)***
- Division
$$\frac{z_1}{z_2} = \frac{r_1 e^{\jmath \theta_1}}{r_2 e^{\jmath \theta_2}} = \frac{r_1}{r_2} e^{\jmath (\theta_1 - \theta_2)}$$
- Reciprocal $1/z = \frac{1}{r} e^{-\jmath \theta}$
- Power (to an integer $n \in \mathbb{Z}$): $z^n = r^n e^{\jmath n\theta}$
- Conjugate $z^\star = r e^{-\jmath \theta}$
- Magnitude $|z| = |r e^{\jmath \theta}| = |r| |e^{\jmath \theta}| = r |\cos\theta + \jmath \sin\theta| = r(\cos^2\theta + \sin^2\theta) = r$
  In particular $|e^{\jmath \theta}| = 1$ for any $\theta \in \mathbb{R}$.

Example. Simplify $2\sqrt{3} e^{\jmath \pi/6} + 2e^{-\jmath \pi/3}$.
Since we need addition, we must convert to rectangular coordinates:

$$\begin{aligned} 2\sqrt{3} e^{\jmath \pi/6} + 2e^{-\jmath \pi/3} &= 2\sqrt{3}[\cos(\pi/6) + \jmath \sin(\pi/6)] + 2[\cos(\pi/3) + \jmath \sin(-\pi/3)] \\ &= 2\sqrt{3}[\sqrt{3}/2 + \jmath 1/2] + 2[1/2 + \jmath \frac{-\sqrt{3}}{2}] = [3 + \jmath \sqrt{3}] + [1 - \jmath \sqrt{3}] = 4. \end{aligned}$$

This is exactly the type of manipulations we need for adding sinusoids of the same frequency.

Example. Find z_1/z_2 when $z_1 = 16e^{-\jmath 2\pi/3}$ and $z_2 = 4 - \jmath 4\sqrt{3}$.
- Cartesian solution desired
  $z_1 = 16e^{-\jmath 2\pi/3} = 16(-1/2 - \jmath \sqrt{3}/2) = -8 - \jmath 8\sqrt{3}$
$$\frac{z_1}{z_2} = \frac{-8 - \jmath 8\sqrt{3}}{4 - \jmath 4\sqrt{3}} = \frac{(-8 - \jmath 8\sqrt{3})(4 + \jmath 4\sqrt{3})}{4^2 + (4\sqrt{3})^2} = \frac{(-32 + 32 \cdot 3) + \jmath (-32 - 32)\sqrt{3})}{64} = 1 - \jmath \sqrt{3}$$
- Polar solution desired
  $z_2 = 4 - \jmath 4\sqrt{3} = \sqrt{4^2 + 4^2 \cdot 3}\, e^{-\jmath \pi/3} = 8e^{-\jmath \pi/3}$
$$\frac{z_1}{z_2} = \frac{16e^{-\jmath 2\pi/3}}{8e^{-\jmath \pi/3}} = 2e^{-\jmath \pi/3}$$

Sanity check: $2e^{-\jmath \pi/3} = 2[\cos(-\pi/3) + \jmath \sin(-\pi/3)] = 2[1/2 - \jmath \sqrt{3}/2] = 1 - \jmath \sqrt{3}$, so the two answers indeed agree.

Complex roots

The roots of the polynomial $z^2 + 1 = 0$ are $z = \pm \jmath = \pm\sqrt{-1}$. This is a second-degree polynomial so it has two roots.
One might say that $\jmath$ was "invented" so that the **fundamental theorem of algebra** works: an nth degree polynomial has n roots.

What are the roots of the polynomial $z^3 + 8 = 0$? It is a third-degree polynomial so it has three roots.
An equivalent question would be: determine $(-8)^{1/3}$.

Do we need to invent a $\sqrt[3]{-1}$ to solve this problem? Fortunately, no!

Strategy: solve $z^3 = -8$ by using polar form, $z = re^{\jmath \theta}$.
So $r^3 e^{\jmath 3\theta} = -8 = 8e^{\jmath \pi}$.
Equating the *magnitudes*, we see that $r^3 = 8$ and since r is real, we have $r = 2$. That's the easy part.
Fact: if $e^{\jmath \phi} = e^{\jmath \gamma}$, then $\phi = \gamma + k2\pi$ for some integer k.
Equating the *phases*, we see $3\theta = \pi + k2\pi$ so $\theta = \frac{\pi}{3} + k\frac{2\pi}{3}$. ***Picture***
Choosing three consecutive integers $k = -1, 0, 1$, we have $\theta \in \{\pm\pi/3, \pi\}$
So the roots are $z = 2e^{\jmath \pi} = -2$ and $z = 2\exp(\pm\jmath \pi/3) = 1 \pm \jmath \sqrt{3}$.

Caution: MATLAB's `(-8)^(1/3)` only gives one of the three possible values!

Caution: $\boxed{(e^{\jmath \theta})^n = e^{\jmath n\theta} \text{ when } n \in \mathbb{Z}}$ (integers). But $\boxed{(e^{\jmath \theta})^{1/n} = e^{\jmath (\theta/n + k2\pi/n)} \text{ for } k \in \mathbb{Z} \text{ and } n \in \mathbb{N}.}$

More practice

Use the `zdrill` mfile in DSP First toolbox for practice!

Sums of sinusoidal signals of *same frequency* (This is a primary motivation for complex numbers!)

Example. Find the **amplitude** A and the **phase** ϕ of the following sum-of-sinusoids signal:

$$y(t) = 2\cos(5t + \pi/4) + 2\sqrt{2}\sin(5t) \stackrel{?}{=} A\cos(5t + \phi).$$

Note that the **frequency** remains unchanged!

Most important formula: $\boxed{\cos(\theta) = \mathrm{Re}\left\{e^{\jmath\theta}\right\}.}$ So $A\cos(2\pi f_0 t + \phi) = \mathrm{Re}\left\{Ae^{\jmath(2\pi f_0 t + \phi)}\right\}$. Also recall that $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$.

$$\begin{aligned}
y(t) &= 2\cos(5t + \pi/4) + 2\sqrt{2}\cos(5t - \pi/2) \\
&= \mathrm{Re}\left\{2e^{\jmath(5t+\pi/4)} + 2\sqrt{2}e^{\jmath(5t-\pi/2)}\right\} \\
&= \mathrm{Re}\left\{\left[\underbrace{2e^{\jmath\pi/4}}_{\text{phasor1}} + \underbrace{2\sqrt{2}e^{-\jmath\pi/2}}_{\text{phasor2}}\right]e^{\jmath 5t}\right\} \qquad \text{note how the frequency term } e^{\jmath 5t} \text{ factors out!} \\
&= \mathrm{Re}\left\{\left[2(\sqrt{2}/2 + \jmath\sqrt{2}/2) + 2\sqrt{2}(-\jmath)\right]e^{\jmath 5t}\right\} \\
&= \mathrm{Re}\left\{\left[\sqrt{2} - \jmath\sqrt{2}\right]e^{\jmath 5t}\right\} \\
&= \mathrm{Re}\left\{\underbrace{2e^{-\jmath\pi/4}}_{\text{phasor}}e^{\jmath 5t}\right\} = \mathrm{Re}\left\{2e^{\jmath(5t-\pi/4)}\right\} = 2\cos(5t - \pi/4)
\end{aligned}$$

The complex values first appear in polar form, yet we must add them so cartesian form is more convenient. Then the final form requires polar form again.

This example was "cooked" for chalkboard use without a calculator.
In practice, these problems are solved easily using any scientific calculator that handles complex numbers in polar form.
You need such a calculator for the exams!

General rule for summing sinusoidal signals of the same frequency:

$$y(t) = \sum_k A_k\cos(2\pi f_0 t + \phi_k) = A\cos(2\pi f_0 t + \phi), \quad \text{where} \quad \boxed{Ae^{\jmath\phi} = \sum_k A_k e^{\jmath\phi_k}.}$$

Note that all that really enters into the calculation is the sum of the terms of the form $A_k e^{\jmath\phi_k}$. These terms are called **phasors**, particularly in the context of electrical circuits. This representation simplifies calculations with resistors, capacitors, and inductors (RLC circuits) since one can solve many problems (for sinusoidal signals) using the phasors and the (complex) impedance of each circuit element.

Summary: the key step in this approach was writing

$$x(t) = A\cos(2\pi f_0 t + \phi) = \mathrm{Re}\left\{Ae^{\jmath(2\pi f_0 t + \phi)}\right\}.$$

A (complex) signal of the form $\bar{x}(t) = Ae^{\jmath(2\pi f_0 t + \phi)}$ is called a **complex exponential signal**.
Another name for it is a **rotating phasor**.

What about a signal of the form $x(t) = \exp(-2t)$? This is an ordinary **exponential signal**; it is not "complex."

Representing **sinusoidal signals** as the real part of **complex exponential signals** allows us to **add** such signals "easily" using **complex arithmetic** rather than **trigonometry**.

Relationship between sinusoidal signals and complex exponential signals

- Viewpoint 1:

$$x(t) = A\cos(2\pi f_0 t + \phi) = \mathrm{Re}\Big\{ A\mathrm{e}^{\jmath(2\pi f_0 t + \phi)} \Big\} = \mathrm{Re}\big\{ \big(A\mathrm{e}^{\jmath\phi}\big)\, \mathrm{e}^{\jmath 2\pi f_0 t} \big\}, \quad \text{where } \big(A\mathrm{e}^{\jmath\phi}\big) \text{ is the } \textbf{phasor}.$$

- Viewpoint 2:

$$\begin{aligned} x(t) = A\cos(2\pi f_0 t + \phi) &= \frac{A}{2}\mathrm{e}^{\jmath(2\pi f_0 t + \phi)} + \frac{A}{2}\mathrm{e}^{-\jmath(2\pi f_0 t + \phi)} \\ &= \frac{1}{2}\big(A\mathrm{e}^{\jmath\phi}\big)\,\mathrm{e}^{\jmath 2\pi f_0 t} + \frac{1}{2}\big(A\mathrm{e}^{-\jmath\phi}\big)\,\mathrm{e}^{-\jmath 2\pi f_0 t}. \end{aligned}$$

  Note that the phasor and its complex conjugate appear!
  So a sinusoidal signal is the sum of two **rotating phasors**.
  Why? Because of **inverse Euler** identity: $\cos\theta = \frac{1}{2}\mathrm{e}^{\jmath\theta} + \frac{1}{2}\mathrm{e}^{-\jmath\theta}$.

Note that there is a **negative frequency** for the second complex exponential.
This corresponds to a **rotating phasor** that has **clockwise** rotation in the complex plane.
We need the combination of the two rotating phasors having opposite directions of rotation so that when added together, the imaginary parts cancel out and we are left with the real part which is the cosine part.
We never need a negative frequency for sinusoidal signals, only for complex exponential signals.

Plotting complex exponential signals

There are three ways to plot a complex exponential signal.

$$\bar{x}(t) = A\mathrm{e}^{\jmath(2\pi f_0 t + \phi)} = A\cos(2\pi f_0 t + \phi) + \jmath A\sin(2\pi f_0 t + \phi) = \mathrm{Re}\{\bar{x}(t)\} + \jmath\,\mathrm{Im}\{\bar{x}(t)\}.$$

1. Separate plots of real and imaginary parts
   (Picture) of two sinusoids
2. Plot in complex plane (**rotating phasor**)
   - magnitude $|\bar{x}(t)| = A$
   - angle $\angle\bar{x}(t) = 2\pi f_0 t + \phi$

   (Picture) of counter-clockwise rotation (for positive f_0)
3. 3D plot: real and imaginary vs time ***(Picture)*** of **helix**

Complex signals

We began the course defining simple signal characteristics and simple signal operations. Those definitions were for **real signals**, although many apply to **complex signals** too.

A complex signal has a **real part** and an **imaginary part** as follows:

$$z(t) = x(t) + \jmath\, y(t)\,.$$

Most signal characteristics are easy generalizations of those defined for real signals and are described at end of Part 1 lecture notes.

Example. Mean of complex signal.

$$\begin{aligned} \mathrm{M}(z) &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} z(t)\,\mathrm{d}t = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [x(t) + \jmath\, y(t)]\,\mathrm{d}t \\ &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x(t)\,\mathrm{d}t + \jmath\,\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(t)\,\mathrm{d}t = \mathrm{M}(x) + \jmath\,\mathrm{M}(y)\,. \end{aligned}$$

An important difference is that for complex signal properties, anywhere we had the **squared value** $x^2(t)$ before, we replace it with the **magnitude squared** $|z(t)|^2 = z(t)\,z^*(t) = x^2(t) + y^2(t)$.

Example. The **energy** of a complex signal $z(t)$ is $\boxed{\mathrm{E}(z) = \int_{t_1}^{t_2} |z(t)|^2\,\mathrm{d}t.}$

Another related difference is that we define correlation for complex signals as follows:

$$C(z_1, z_2) = \int_{t_1}^{t_2} z_1(t)\,z_2^*(t)\,\mathrm{d}t.$$

One reason for this choice is that it satisfies $\mathrm{E}(z) = C(z, z)$.

Example. Find the correlation between $z_1(t) = \mathrm{e}^{-\jmath\,(2\pi 7t + \pi/3)}$ and $z_2(t) = \mathrm{e}^{\jmath\,(2\pi 21t - \pi/4)}$ over the interval $[0, 1/7]$.

$$\begin{aligned} C(z_1, z_2) &= \int_0^{1/7} z_1(t)\,z_2^*(t)\,\mathrm{d}t = \int_0^{1/7} \mathrm{e}^{-\jmath\,(2\pi 7t + \pi/3)}\mathrm{e}^{-\jmath\,(2\pi 21t - \pi/4)}\,\mathrm{d}t \\ &= \mathrm{e}^{-\jmath\,\pi/12}\int_0^{1/7} \mathrm{e}^{-\jmath\,2\pi 28t}\,\mathrm{d}t = \mathrm{e}^{-\jmath\,\pi/12}\int_0^{1/7} [\cos(2\pi 28t) + \jmath\,\sin(2\pi 28t)]\,\mathrm{d}t = 0, \end{aligned}$$

since the integral is over 4 periods of the sinusoids.

(More generally, such **harmonically-related** complex exponential signals are uncorrelated.)

The **signal operations** like time scaling, time shift, etc. all apply to both the real part and the imaginary part.

Similar considerations for **discrete-time** signals.

Example. Find the energy of the signal $z(t) = \begin{cases} \mathrm{e}^{\jmath\,5t}\mathrm{e}^{-2t}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$

$$\mathrm{E}(z) = \int_0^{\infty} |z(t)|^2\,\mathrm{d}t = \int_0^{\infty} \left|\mathrm{e}^{\jmath\,5t}\right|^2 \left|\mathrm{e}^{-2t}\right|^2\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-4t}\,\mathrm{d}t = \frac{1}{4}.$$

Note that the frequency (5 rad/s) had no effect on the energy!

Beat frequencies (Ch. 3.2)

Are complex exponential signals useful for summing sinusoidal signals with *different* frequencies? Sometimes!

Ch. 3 on **spectra** is all about sinusoids of different frequencies!

Example. Sum of two "nearly same" frequencies. (Same amplitude for simplicity, not necessity.)

$$x(t) = A\cos(2\pi f_1 t) + A\cos(2\pi f_2 t)$$

where $|f_2 - f_1|$ is "small."

Define the center frequency $\bar{f} = \frac{f_1+f_2}{2}$ and $\Delta = f_2 - \bar{f} = \bar{f} - f_1$ for $f_2 > f_1$.

(Picture) .

This type of $x(t)$ has a notable auditory property. Can we describe it mathematically?

$$\begin{aligned}
x(t) &= \mathrm{Re}\left\{A e^{\jmath 2\pi f_1 t} + A e^{\jmath 2\pi f_2 t}\right\} \\
&= A\mathrm{Re}\left\{e^{\jmath 2\pi(\bar{f}-\Delta)t} + e^{\jmath 2\pi(\bar{f}+\Delta)t}\right\} \\
&= A\mathrm{Re}\left\{e^{\jmath 2\pi\bar{f}t}\left(e^{-\jmath 2\pi\Delta t} + e^{\jmath 2\pi\Delta t}\right)\right\} \\
&= A\mathrm{Re}\left\{e^{\jmath 2\pi\bar{f}t} 2\cos(2\pi\Delta t)\right\} \\
&= 2A\cos(2\pi\Delta t)\mathrm{Re}\left\{e^{\jmath 2\pi\bar{f}t}\right\} = 2A\cos(2\pi\Delta t)\cos(2\pi\bar{f}t).
\end{aligned}$$

No need to remember and use and trigonometry identities. Using complex exponential signals provides a systematic approach.

Alternatively, remembering that $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$ we have

$$\begin{aligned}
x(t) &= 2A\left[\frac{1}{2}\cos\big(2\pi(\bar{f}+\Delta)t\big) + \frac{1}{2}\cos\big(2\pi(\bar{f}-\Delta)t\big)\right] \\
&= 2A\left[\frac{1}{2}\cos\left(\underbrace{2\pi\bar{f}t}_{\alpha} + \underbrace{2\pi\Delta t}_{\beta}\right) + \frac{1}{2}\cos\big(2\pi\bar{f}t - 2\pi\Delta t\big)\right] \\
&= 2A\cos(2\pi\Delta t)\cos\big(2\pi\bar{f}t\big)\,.
\end{aligned}$$

If $\Delta \ll \bar{f}$, then we have the product of a slowly changing sinusoidal signal times a higher frequency sinusoidal signal.

(Picture) of signals and their product.

Demo of closely spaced case and harmonically-related case.

Also try summing square wave and triangular wave.
Why similar? Sum of sinusoids!

Sinusoids? Enough already!

Yes, the real world has many signals that are far more interesting than sinusoidal signals.

Joseph Fourier showed in 1807 that most any signal can be expressed as the sum of (a lot of) sinusoidal signals (not of the same frequency though!), simply by carefully choosing the frequencies, amplitudes, and phases.

"Joe" did it almost 200 years ago without calculators or MATLAB...