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## **Ch. 8: IIR Filters**

- Difference equation
- System function
- Frequency response
- Impulse response
  - Partial fraction expansion
- Poles/zeros
- Filter design

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### **Reading**

- Text Ch. 8

## 8.1

**Introduction**

In the preceding notch filter example we have seen that a system with poles that are not at the origin can be very useful.

Just as in the example that we worked out, such systems always have the property that the output signal value at any time  $n$  depends both on certain input signal values *as well as some previous output signal values*.

These systems are called **recursive**.

The general form for a (finitely-computable, causal, LTI) system that depends on both current and past inputs and past outputs is the following **difference equation**:

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k].$$

Are systems having the above diffeq are **causal**, **linear**, and **time invariant**?

Yes. Causality is easy to see; LTI is not hard to show.

The  $b_k$ 's are called **feed-forward** coefficients.

The  $a_l$ 's are called **feedback** coefficients.

To implement such a system, computing each output signal value requires  $N + M + 1$  multiplies. However, we usually say that the system is of  $N$ th order since the number of poles has great influence on the system properties.

When we design a filter, we get to pick  $N$ ,  $M$ , and the **filter coefficients**  $a_1, \dots, a_N$  and  $b_0, \dots, b_M$ .

Usually we design in the  $z$ -plane though and work backwards from the zeros and poles to find these filter coefficients.

The above form is the *most general* type of system that we will consider in 206, and is the bread-and-butter of DSP work.

We consider only the usual case of real filters, so the filter coefficients, the  $a_l$ 's and  $b_k$ 's, are real numbers.

Are FIR filters a special case? Yes, just choose  $N = 0$ .

**Example.**  $y[n] = \frac{1}{2}y[n-1] + x[n] + 3x[n-1]$ . Here,  $N = 1$  and  $M = 1$ .

**(Picture)** of block diagrams Direct Form I and Direct Form II.

Two delays combine into one just delay using Direct Form II.

## 8.2

**Impulse response: the hard way**

The chapter is called "IIR" so apparently such systems (for  $N \geq 1$ ) have an infinitely long impulse response, but that fact may not be immediately obvious.

Let us work out a concrete example now by "brute force." This is not the best way to find  $h[n]$  for an IIR system in general!

**Example.** Consider the **first-order** system described by  $y[n] = ay[n-1] + 2x[n]$ , where  $a$  is a real number. Find the **impulse response**  $h[n]$ .

Before proceeding, we must make a very important assumption, called **initial rest conditions**.

- We assume the input is zero prior to some starting time  $n_0$ , *i.e.*,  $x[n] = 0$  for  $n < n_0$ . These are called **suddenly applied** inputs.
- We assume the output signal is zero prior to the starting time of the signal, *i.e.*,  $y[n] = 0$  for  $n < n_0$ . We say that the system is **initially at rest**.<sup>1</sup>

Note that  $y[0] = ay[-1] + 2x[0]$  in this example. If the input signal  $x[n]$  is zero for  $n < 0$  (the usual case considered), then we assume  $y[-1] = 0$  so  $y[0] = 2x[0]$  for this example.

We assume **initial rest conditions** hereafter, both for simplicity of analysis and because that is the usual mode of operation of DSP systems. (All memory buffers are reset to zero when the system is powered up.)

<sup>1</sup>The word "rest" has its origins in mathematical models for mechanical systems: think of a ball at rest on a flat plane that is subsequently subject to forces such as a swift kick. The term "at rest" is not so natural for digital systems, but we use it anyway for historical reasons.

By definition, the impulse response is the system output when the input is the unit impulse  $x[n] = \delta[n]$ .

In other words, the impulse response function satisfies the following recursive relationship:  $h[n] = ah[n-1] + 2\delta[n]$ . Because it is recursive, it is *not* a “final answer” for  $h[n]$ . However, we can “execute” the recursion to find the impulse response.

Using the initial rest conditions, we have that  $h[n] = 0$  for  $n < 0$  since the unit impulse input is zero for  $n < 0$ .

$n$	$\delta[n]$	$h[n-1]$	$h[n] = ah[n-1] + 2\delta[n]$
$< 0$	0	0	0
0	1	0	2
1	0	2	$2a$
2	0	$2a$	$2a^2$
3	0	$2a^2$	$2a^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By inspection we see that the **impulse response** is:

$$h[n] = \begin{cases} 2a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} = 2a^n u[n], \text{ where } u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

We see that the impulse response decays to zero (if  $|a| < 1$ ) but never reaches zero (for  $a \neq 0$ ) so this is indeed an **IIR filter**.

### Stability

What happens to  $h[n]$  if  $a > 1$  in the preceding example?

The impulse response “blows up” as  $n \rightarrow \infty$ .

A system is called **bounded-input bounded-output (BIB) stable** if all bounded input signals (inputs where  $|x[n]| \leq c_1$  for some constant  $c_1 < \infty$ ) produce output signals that are also bounded:  $|y[n]| \leq c_2$  for some  $c_2 < \infty$ .

Is the unit impulse a bounded signal? Yes, with  $c_1 = 1$ .

In our example system, if  $|a| > 1$  then the (bounded) unit impulse input produces an unbounded output  $a^n$ , so that system would not be BIBO stable.

What if  $a = 1$ ? Would the system be stable in that case?

To show that the answer is “no” all we need is to find one bounded input signal that produces an unbounded output signal.

For this example, the step function input  $x[n] = u[n]$  suffices.

$n$	$u[n]$	$y[n-1]$	$y[n] = y[n-1] + 2x[n]$
$< 0$	0	0	0
0	1	0	2
1	1	2	4
2	1	4	6
3	1	6	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By inspection we see that the **step response** is:

$$y[n] = 2(n+1)u[n].$$

Is this bounded? No, so the system is unstable if  $|a| = 1$  as well as if  $|a| > 1$ .

One can show that the system is stable if  $|a| < 1$ ; we will return to that later when we discuss poles and zeros.

### Numerically finding the impulse and step response

The above brute-force method only works for simple cases. For more complicated cases one can either use the **partial-fraction expansion method** described later, if an analytical solution is needed, or simply use MATLAB’s `filter` function if a numerical solution is adequate.

Using `filter` can be convenient for checking your analytical work.

## 8.3

**System function**

Before we can find more general tools for finding the impulse response of IIR filters, we must examine the system function. Taking the  $z$ -transform of both sides of the general difference equation yields

$$Y(z) = \sum_{l=1}^N a_l z^{-l} Y(z) + \sum_{k=0}^M b_k z^{-k} X(z)$$

or equivalently

$$Y(z) \left[ 1 - \sum_{l=1}^N a_l z^{-l} \right] = \left[ \sum_{k=0}^M b_k z^{-k} \right] X(z).$$

Thus the **system function** is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{l=1}^N a_l z^{-l}} = \frac{z^N}{z^M} \frac{\sum_{k=0}^M b_k z^{M-k}}{z^N - \sum_{l=1}^N a_l z^{N-l}} = \frac{z^N}{z^M} \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M}{z^N - a_1 z^{N-1} - a_2 z^{N-2} \dots - a_{N-1} z - a_N}.$$

This is a ratio of polynomials so it is a **rational** system function.

Note that the *negatives* of the feed-forward coefficients appear in the denominator polynomial!

- If  $b_0 \neq 0$ , then the numerator polynomial is  $M$ th order and will have  $M$  roots called **zeros**.  
If  $b_0 = 0$  and  $b_1 \neq 0$ , then the numerator polynomial is of order  $M - 1$ , so there will be  $M - 1$  zeros.
- Because of the leading term  $z^N$ , the denominator polynomial is *always*  $N$ th order so it will have  $N$  roots called **poles**.
- There can also be additional poles or zeros at the origin due to the  $z^N/z^M$  term. These roots do not affect the magnitude response so are less important.
- If  $b_0 \neq 0$ , then the number of poles and zeros need not be identical.

In the usual case where  $b_0 \neq 0$ , the **factored form** of the system function is:

$$H(z) = \frac{B(z)}{A(z)} = b_0 \frac{z^N \prod_{i=1}^M (z - z_i)}{z^M \prod_{j=1}^N (z - p_j)} = b_0 \frac{z^N}{z^M} \frac{(z - z_1)(z - z_2) \dots (z - z_M)}{(z - p_1)(z - p_2) \dots (z - p_N)}.$$

Again we represent the system function graphically by its **pole-zero plot**.

Again, the relation between **frequency response** and **system function** is:

$$\boxed{\mathcal{H}(\hat{\omega}) = H(z) \Big|_{z=e^{j\hat{\omega}}} = H(e^{j\hat{\omega}})}$$

Example. For  $y[n] = ay[n - 1] + 2x[n]$  we have  $M = 0$ ,  $b_0 = 2$ ,  $N = 1$ ,  $a_1 = a$ , and the block diagram is (**Picture**).

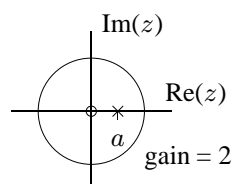
To find the system function, it is usually easier to return to first principles as follows:

$$y[n] = ay[n - 1] + 2x[n] \Rightarrow Y(z) = az^{-1}Y(z) + 2X(z) \Rightarrow (1 - az^{-1})Y(z) = 2X(z),$$

so the system function is (notice how  $b_0$  becomes the gain):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - az^{-1}} = 2 \frac{z}{z - a}.$$

So the pole-zero diagram is as follows.



The frequency response of this filter is

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = \frac{2}{1 - ae^{-j\hat{\omega}}},$$

so the magnitude response is

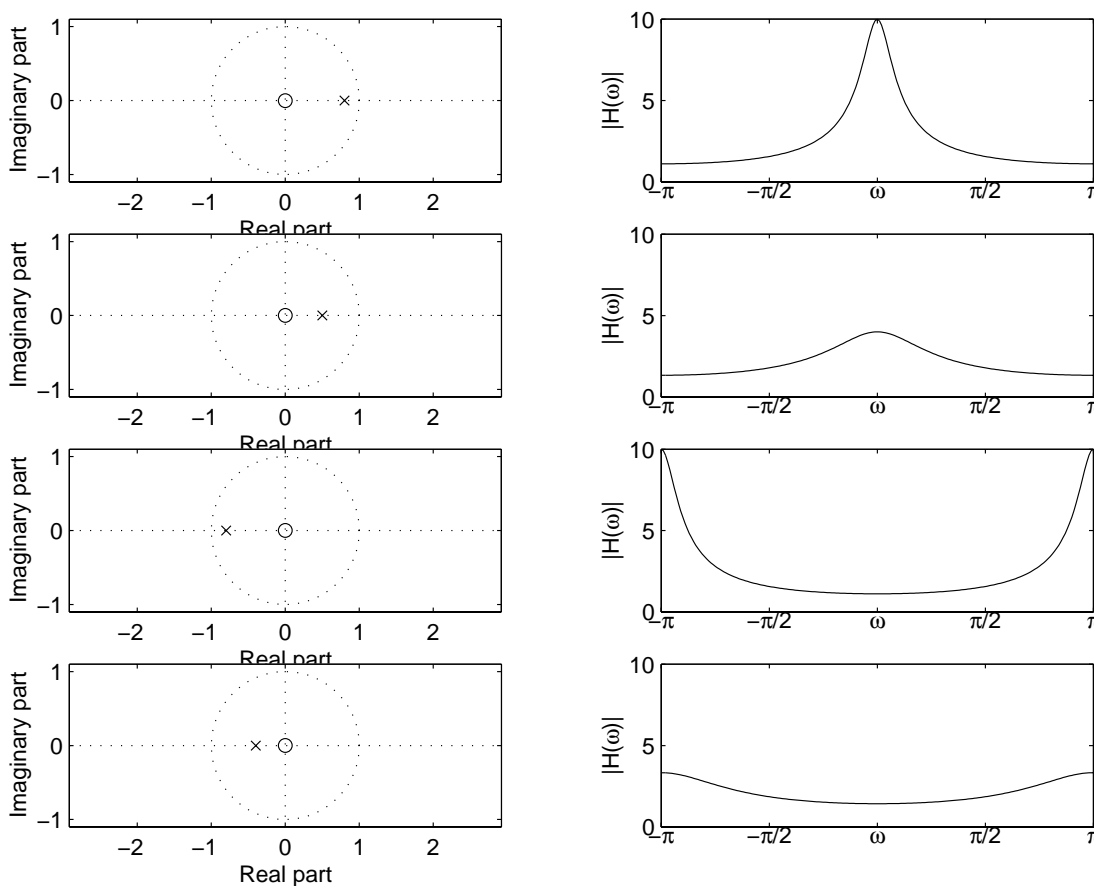
$$|\mathcal{H}(\hat{\omega})| = \frac{2}{\sqrt{(1 - ae^{-j\hat{\omega}})(1 - ae^{j\hat{\omega}})}} = \frac{2}{\sqrt{1 + a^2 - 2a \cos(\hat{\omega})}}.$$

It is difficult to visualize the frequency response from the formula for  $\mathcal{H}(\hat{\omega})$ , particularly for IIR filters.

However, we can easily enough make a rough sketch of the magnitude response using the relationship derived previously:

$$|\mathcal{H}(\hat{\omega})| = |b_0| \frac{|e^{j\hat{\omega}} - z_1| \cdots |e^{j\hat{\omega}} - z_M|}{|e^{j\hat{\omega}} - p_1| \cdots |e^{j\hat{\omega}} - p_N|}.$$

For a precise plot, we turn to `freqz`.



As expected from our previous analyses, frequencies that correspond to positions along the unit circle that are closer to the pole have a larger magnitude response.

Positioning the pole closer to the unit circle induces a more “peaked” magnitude response.

What happens to the magnitude response as we position the pole closer and closer to the center of the unit circle?

**Impulse response revisited**

Given the system function  $H(z) = \frac{2}{1-az^{-1}}$  for this example, can we work backwards to find the **impulse response**  $h[n]$ ?

Recall the geometric series formula given in the lecture notes:

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}, \quad \text{if } |b| < 1.$$

Identify  $b = az^{-1}$  in this example and we have

$$H(z) = \frac{2}{1-az^{-1}} = 2 \left[ \sum_{n=0}^{\infty} (az^{-1})^n \right] = \sum_{n=0}^{\infty} 2a^n z^{-n}.$$

Since in general

$$H(z) = \sum_n h[n] z^{-n}$$

we see immediately that

$$h[n] = \begin{cases} 2a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} = 2a^n u[n].$$

This is valid only if  $|b| = |az^{-1}| < 1$ , *i.e.*,  $|z| > |a|$ . This is called the **region of convergence** of the z-transform, an important topic discussed in detail in EECS 451 but not in 206.

Of course we already knew this particular impulse response from the brute-force method used earlier, but using the z-transform will be more general.

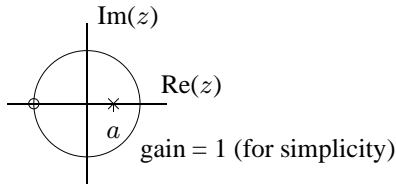
However, the specific **series expansion** approach used above is convenient only in special cases. We still need to find a more general approach.

Nevertheless, we have established a particularly important z-transform pair that should be memorized:

$$a^n u[n] \iff \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

Example. The preceding filter was a fairly poor lowpass filter.

Where would we move the zero to improve this lowpass filter? To  $z = -1$ .



Now let us analyze everything about this filter. As always, the starting point is the system function.

$$H(z) = \frac{z+1}{z-a} = \frac{1+z^{-1}}{1-az^{-1}}$$

Since  $H(z) = Y(z)/X(z)$ , cross multiplying yields

$$Y(z) - az^{-1}Y(z) = X(z) + z^{-1}X(z) \Rightarrow y[n] = ay[n-1] + x[n] + x[n-1].$$

So  $N = 1$  and  $M = 1$ . This illustrates the easiest way to find the **difference equation** given the system function.

We now find the impulse response in two different ways.

Method 1.

$$H(z) = \frac{1+z^{-1}}{1-az^{-1}} = \frac{1}{1-az^{-1}} + z^{-1}\frac{1}{1-az^{-1}} \Rightarrow h[n] = a^n u[n] + \underbrace{a^{n-1}u[n-1]}_{\text{shift property}},$$

using linearity.

Method 2.

$$H(z) = \frac{1+z^{-1}}{1-az^{-1}} = \frac{1-az^{-1}+az^{-1}+z^{-1}}{1-az^{-1}} = 1 + (a+1)z^{-1}\frac{1}{1-az^{-1}} \Rightarrow h[n] = \delta[n] + (a+1)a^{n-1}u[n-1].$$

This answer is a bit easier to visualize, but the two answers are mathematically equal.

To show that they are equal, recall that  $\delta[n] = u[n] - u[n-1]$ .

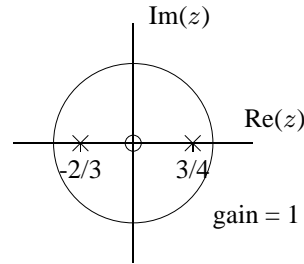
### Inverse $z$ -transform by partial fraction expansion

In the preceding first-order examples, we found the impulse response  $h[n]$  from the system function  $H(z)$  by simply using the following approach to the inverse  $z$ -transform:

- “table lookup” of the pair  $a^n u[n] \iff \frac{1}{1-az^{-1}}$ ,
- linearity,
- shift property.

For higher order systems (more than 1 pole), we need a more general approach, since most higher order cases are not in tables. The **partial fraction expansion** method is the systematic approach, and is illustrated in the following example.

Example. Consider the system with the following pole-zero plot.



$$H(z) = \frac{z}{(z + 2/3)(z - 3/4)} = \frac{A_1}{z + 2/3} + \frac{A_2}{z - 3/4}$$

where the coefficients are given by :

$$A_1 = (z + 2/3)H(z) \Big|_{z=-2/3} = \frac{z}{z - 3/4} \Big|_{z=-2/3} = \frac{-2/3}{-2/3 - 3/4} = \frac{8}{17}$$

$$A_2 = (z - 3/4)H(z) \Big|_{z=3/4} = \frac{z}{z + 2/3} \Big|_{z=3/4} = \frac{3/4}{3/4 + 2/3} = \frac{9}{17}$$

Thus the system function has the following **expanded form**:

$$H(z) = \frac{8/17}{z + 2/3} + \frac{9/17}{z - 3/4} = z^{-1} \frac{8/17}{1 + 2/3z^{-1}} + z^{-1} \frac{9/17}{1 - 3/4z^{-1}}$$

(It is a good idea to cross multiply to check the algebra.)

Again using table lookup, linearity, and the shift property, we find that the impulse response by taking the inverse  $z$ -transform:

$$h[n] = \frac{8}{17}(-2/3)^{n-1}u[n-1] + \frac{9}{17}(3/4)^{n-1}u[n-1].$$

To find the diffeq, note that

$$H(z) = \frac{z}{z^2 - (1/12)z - 1/2} = \frac{z^{-1}}{1 - (1/12)z^{-1} - 1/2z^{-2}}$$

So the diffeq is

$$y[n] = \frac{1}{12}y[n-1] + \frac{1}{2}y[n-2] + x[n-1].$$

So  $N = 2$  and  $M = 1$ . Since  $b_0 \neq 0$ , the number of zeros differs from the number of poles.



Alternative derivation of impulse response in terms of  $z^{-1}$ .

$$H(z) = \frac{z^{-1}}{(1 + 2/3z^{-1})(1 - 3/4z^{-1})} = \frac{C_1}{1 + 2/3z^{-1}} + \frac{C_2}{1 - 3/4z^{-1}}$$

where

$$C_1 = (1 + 2/3z^{-1})H(z) \Big|_{z=-2/3} = \frac{z^{-1}}{1 - 3/4z^{-1}} \Big|_{z=-2/3} = \frac{-3/2}{1 - 3/4(-3/2)} = \frac{-3/2}{1 + 9/8} = -\frac{12}{17}$$

$$C_2 = (1 - 3/4z^{-1})H(z) \Big|_{z=3/4} = \frac{z^{-1}}{1 + 2/3z^{-1}} \Big|_{z=3/4} = \frac{4/3}{1 + 2/3(4/3)} = \frac{4/3}{1 + 8/9} = \frac{12}{17}.$$

So the expanded system function is:

$$H(z) = \frac{z^{-1}}{(1 + 2/3z^{-1})(1 - 3/4z^{-1})} = \frac{-12/17}{1 + 2/3z^{-1}} + \frac{12/17}{1 - 3/4z^{-1}}.$$

Thus the impulse response is

$$h[n] = -\frac{12}{17}(-2/3)^n u[n] + \frac{12}{17}(3/4)^n u[n].$$

The two expressions for  $h[n]$  look slightly different, but in fact are identical! In particular, note that  $h[0] = 0$ .

### Application: Channel Equalization

The following example illustrates many concepts: cascaded systems, PFE with complex poles, channel equalization.

**Example.** Assume that the signal received by a mobile phone base station is the sum of the transmitted signal plus an attenuated and delayed version of the transmitted signal due to reflection off of a building:

$$y_r[n] = x[n] + \frac{1}{4}x[n-2].$$

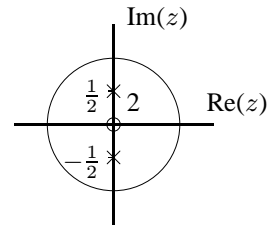
We would like to process the received signal  $y_r[n]$  to recover the original signal  $x[n]$ , using a “channel equalizing” filter with impulse response  $h_e[n]$ .

$$x[n] \rightarrow \boxed{\text{channel: } h_c[n] = \delta[n] + \frac{1}{4}\delta[n-2]} \xrightarrow{y_r[n]} \boxed{\text{equalizer: } h_e[n]} \rightarrow \hat{x}[n] \approx x[n].$$

Design the equalizing filter and determine its impulse response  $h_e[n]$ .

As usual, look in the  $z$  domain: we want  $H_c(z)H_e(z) = 1$ , so

$$H_c(z) = 1 + \frac{1}{4}z^{-2} \Rightarrow H_e(z) = \frac{1}{H_c(z)} = \frac{1}{1 + \frac{1}{4}z^{-2}} = \frac{z^2}{z^2 + 1/4} = \frac{z^2}{(z - j/2)(z + j/2)}.$$



$$H_e(z) = \frac{1}{(1 - \frac{j}{2}z^{-1})(1 + \frac{j}{2}z^{-1})} = \frac{C_1}{1 - \frac{j}{2}z^{-1}} + \frac{C_2}{1 + \frac{j}{2}z^{-1}}$$

$$C_1 = (1 - \frac{j}{2}z^{-1})H_e(z) \Big|_{z=j/2} = \frac{1}{1 + \frac{j}{2}z^{-1}} \Big|_{z=j/2} = \frac{1}{1+1} = \frac{1}{2}$$

$$C_2 = (1 + \frac{j}{2}z^{-1})H_e(z) \Big|_{z=-j/2} = \frac{1}{1 - \frac{j}{2}z^{-1}} \Big|_{z=-j/2} = \frac{1}{1+1} = \frac{1}{2}$$

$$H_e(z) = \frac{1/2}{1 - \frac{j}{2}z^{-1}} + \frac{1/2}{1 + \frac{j}{2}z^{-1}}$$

Now we can find the impulse response:

$$h_e[n] = \frac{1}{2}\left(-\frac{j}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{j}{2}\right)^n u[n].$$

Note that there is one **mode** associated with each pole. FIR channel  $\Rightarrow$  IIR equalizer!

However, this form is unacceptable; we must manipulate it to put it in a form that is real:

$$\begin{aligned} h_e[n] &= \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n u[n] + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n u[n] = \frac{1}{2} (1/2)^n (e^{-j\pi/2^n} + e^{j\pi/2^n}) u[n] \\ &= \frac{1}{2} (1/2)^n \left[ 2 \cos\left(\frac{\pi}{2}n\right) \right] u[n] = (1/2)^n \cos\left(\frac{\pi}{2}n\right) u[n]. \end{aligned}$$

In general, a pair of complex-conjugate poles located at  $re^{\pm j\omega_0}$  yields a **mode** in the impulse response of the form  $r^n \cos(\omega_0 n)$ .

To implement this equalization filter, we need its diffeq:

$$H_e(z) = \frac{1}{1 + \frac{1}{4}z^{-2}} = \frac{\hat{X}(z)}{Y_r(z)} \Rightarrow \hat{X}(z) + \frac{1}{4}z^{-2}\hat{X}(z) = Y_r(z) \Rightarrow \hat{x}[n] = -\frac{1}{4}\hat{x}[n-2] + y_r[n].$$

This filter would be very difficult to design with only time-domain tools!

### Input signals to output signals examples

Given  $x[n]$  and a filter, how do we find the output  $y[n]$ ?

The best approach usually depends on the form of the input signal!

The type of input signal will influence which type of system description will be most useful ( $h[n]$ ,  $\mathcal{H}(\hat{\omega})$ , or  $H(z)$ ).

We give several examples using the following 1st-order system:

$$x[n] \rightarrow \boxed{h[n] = \left(\frac{1}{2}\right)^n u[n]} \rightarrow y[n] = h[n] * x[n] \Rightarrow \begin{aligned} H(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}} \\ \mathcal{H}(\hat{\omega}) &= \frac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}} \end{aligned} \Rightarrow \begin{array}{c} \text{Im}(z) \\ \circ \\ \times \\ \text{Re}(z) \end{array}$$

Convolution is rarely the best way!

Example1.  $x[n]$  is impulses. Use  $h[n]$ .

$$x[n] = 3\delta[n] + 7\delta[n-4] \xrightarrow{\mathcal{T}} y[n] = 3h[n] + 7h[n-4] = 3\left(\frac{1}{2}\right)^n u[n] + 7\left(\frac{1}{2}\right)^{n-4} u[n-4].$$

Example2.  $x[n]$  has a rational  $z$ -transform. Use  $H(z)$ .

$$x[n] = a^n u[n] \xrightarrow{\mathcal{T}} Y(z) = H(z) X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - az^{-1})} = \frac{C_1}{1 - \frac{1}{2}z^{-1}} + \frac{C_2}{1 - az^{-1}}$$

$$C_1 = \left(1 - \frac{1}{2}z^{-1}\right)H(z) \Big|_{z=1/2} = \frac{1}{1 - az^{-1}} \Big|_{z=1/2} = \frac{1}{1 - 2a}$$

$$C_2 = \left(1 - az^{-1}\right)H(z) \Big|_{z=a} = \frac{1}{1 - \frac{1}{2}z^{-1}} \Big|_{z=a} = \frac{-2a}{1 - 2a}$$

So

$$y[n] = \underbrace{\frac{1}{1 - 2a} \left(\frac{1}{2}\right)^n u[n]}_{\text{natural response}} + \underbrace{\frac{-2a}{1 - 2a} a^n u[n]}_{\text{forced response}}.$$

Example3. Eternal sinusoid. Use  $\mathcal{H}(\hat{\omega})$ .

$$x[n] = \cos(\pi n) \xrightarrow{\mathcal{T}} y[n] = |\mathcal{H}(\pi)| \cos(\pi n + \angle \mathcal{H}(\pi)), \quad \mathcal{H}(\pi) = H(-1) = 2/3. \Rightarrow y[n] = \frac{2}{3} \cos(\pi n).$$

Example4. Suddenly applied sinusoid.

- Use  $\mathcal{H}(\hat{\omega})$  if only the steady-state response is needed.
- Use  $X(z)$  and  $H(z)$  otherwise.

$$x[n] = \cos(\pi n)u[n] = (-1)^n u[n] \xrightarrow{\mathcal{T}} y[n] = \underbrace{\frac{1}{3} \left(\frac{1}{2}\right)^n u[n]}_{\text{transient response}} + \underbrace{\frac{2}{3} (-1)^n u[n]}_{\text{steady-state response}},$$

where we used Example2 with  $a = -1$ .

For a sinusoid of frequency  $\omega_0$ , in general we have two terms, one at  $e^{j\omega_0 n} = (e^{j\omega_0})^n$  and one at its conjugate. So Example2 covers suddenly applied complex exponential signals too, and hence, using linearity, sinusoids.

In general:

$$x[n] = A \cos(\omega_0 n + \phi)u[n] \rightarrow \boxed{\mathcal{H}(\hat{\omega})} \rightarrow y[n] = y_{\text{transient}}[n] + |\mathcal{H}(\omega_0)| A \cos(\omega_0 n + \angle \mathcal{H}(\omega_0)).$$

So the “sine in / sine out” property holds asymptotically as  $n \rightarrow \infty$  (i.e., in steady state) for IIR systems, assuming all of the poles are within the unit circle so the transient response decays towards zero.

Example2'. Step response. Use  $H(z)$  since  $u[n] \iff \frac{1}{1-z^{-1}}$ .

$$x[n] = u[n] \xrightarrow{\mathcal{T}} y[n] = -(1/2)^n u[n] + 2u[n],$$

using  $a = 1$  in Example2. Note that  $\mathcal{H}(0) = H(1) = 2$ .

Again we see a transient + steady-state response.

For steady-state response only, use  $\mathcal{H}(0)$ .

Example6. Periodic input signal. Use DFT and  $\mathcal{H}(\hat{\omega})$ .

$x[n] = 1, 0, 1, 0, \dots$  (**Picture**) Analysis:

$$X[k] = \frac{1}{2} \sum_{n=0}^1 x[n] e^{-j \frac{2\pi}{2} kn} = \frac{1}{2},$$

so by synthesis:

$$x[n] = \frac{1}{2} + \frac{1}{2} e^{j\pi n} = \frac{1}{2} + \frac{1}{2} (-1)^n \xrightarrow{\mathcal{T}} y[n] = \frac{1}{2} \mathcal{H}(0) + \mathcal{H}(\pi) \frac{1}{2} \cos(\pi n + \angle \mathcal{H}(\pi)) = 1 + \frac{1}{3} \cos(\pi n) = 4/3, 2/3, \dots$$

Example4'. Suddenly applied sinusoid. Use  $H(z)$ .

$$\begin{aligned} x[n] &= \cos(\omega_0 n) u[n] = \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n] \\ &= \frac{1}{2} (e^{j\omega_0})^n u[n] + \frac{1}{2} (e^{-j\omega_0})^n u[n] \\ \iff X(z) &= \frac{1}{2} \frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\omega_0} z^{-1}} \\ &= \frac{1}{2} \frac{1 - e^{j\omega_0} z^{-1} + 1 - e^{-j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})} \\ &= \frac{1 - \cos \omega_0 z^{-1}}{1 - 2 \cos \omega_0 z^{-1} + z^{-2}} \end{aligned}$$

$$Y(z) = X(z) H(z) = \frac{1 - \cos \omega_0 z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})} \cdot \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{C_1}{1 - \frac{1}{2} z^{-1}} + \frac{C_2}{1 - e^{j\omega_0} z^{-1}} + \frac{C_2^*}{1 - e^{-j\omega_0} z^{-1}}$$

$$C_1 = \left. (1 - \frac{1}{2} z^{-1}) Y(z) \right|_{z=\frac{1}{2}} = \frac{1 - 2 \cos \omega_0}{5 - 4 \cos \omega_0}$$

$$C_2 = \left. (1 - e^{j\omega_0} z^{-1}) Y(z) \right|_{z=e^{j\omega_0}} = \frac{1 - \cos \omega_0 e^{j\omega_0}}{1 - e^{-j2\omega_0}} \frac{1}{1 - \frac{1}{2} e^{-j\omega_0}} = \frac{1}{2} \frac{1}{1 - \frac{1}{2} e^{-j\omega_0}} = \frac{1}{2} \mathcal{H}(\omega_0).$$

So the output signal is:

$$\begin{aligned} y[n] &= \frac{1 - 2 \cos \omega_0}{5 - 4 \cos \omega_0} (1/2)^n u[n] + \frac{1}{2} \mathcal{H}(\omega_0) e^{j\omega_0 n} u[n] + \frac{1}{2} \mathcal{H}^*(\omega_0) e^{-j\omega_0 n} u[n] \\ &= \underbrace{\frac{1 - 2 \cos \omega_0}{5 - 4 \cos \omega_0} (1/2)^n u[n]}_{\text{transient response}} + \underbrace{|\mathcal{H}(\omega_0)| \cos(\omega_0 n + \angle \mathcal{H}(\omega_0)) u[n]}_{\text{steady-state response}}. \end{aligned}$$

**Discrete-time systems described by difference equations (FIR and IIR)**

Difference equation:

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

System function (in expanded polynomial and in factored polynomial forms):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{l=1}^N a_l z^{-l}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 - a_1 z^{-1} - \dots - a_M z^{-N}} = z^{N-M} b_0 \frac{\prod_{k=1}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

Relationships:

