Fourier Transform

EECS 442

Fall 2020, University of Michigan

Today

Fourier transforms

- Continuous and Discrete Fourier transforms
- Example with number
- Intuitions behind Fourier transforms
- Matching Game

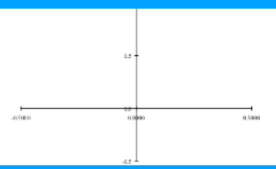
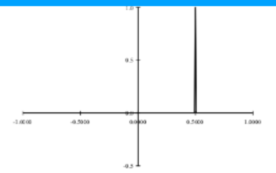
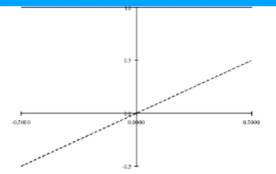
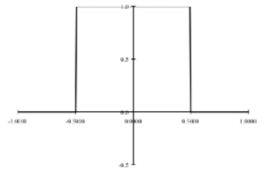
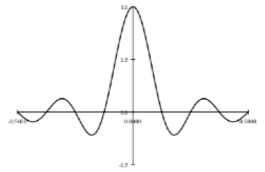
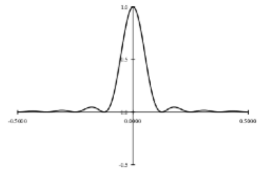
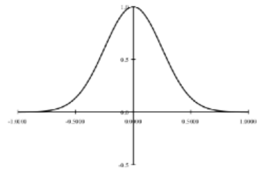
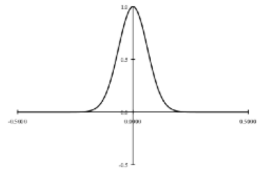
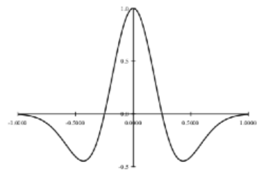
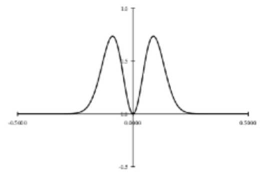
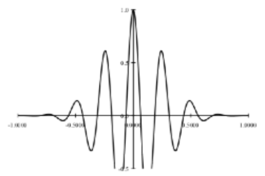
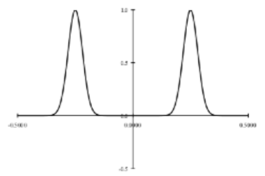
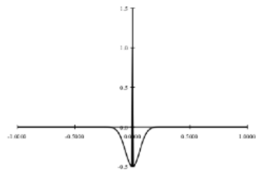
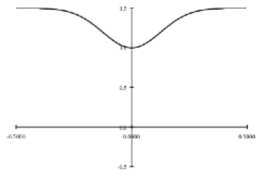
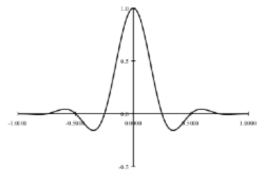
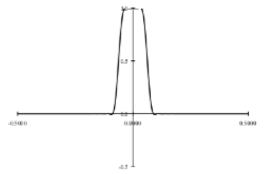
Fourier Transform- Continuous Signal

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$ **Fourier Transform**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$ **Inverse Fourier Transform**

- Fourier Transform decomposes a signal $x(t)$ into sinusoids of the form $e^{j\omega t}$, with associated magnitude $X(\omega)$

Examples

Name		Signal		Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\text{box}(x/a)$	$\Leftrightarrow$	$a\text{sinc}(a\omega)$	
tent		$\text{tent}(x/a)$	$\Leftrightarrow$	$a\text{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2G(\omega;\sigma^{-1})$	
Gabor		$\cos(\omega_0 x)G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega\pm\omega_0;\sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $-\gamma G(x;\sigma)$	$\Leftrightarrow$	$(1+\gamma)-$ $\frac{\sqrt{2\pi}\gamma}{\sigma}G(\omega;\sigma^{-1})$	
windowed sinc		$\text{rcos}(x/(aW))$ $\text{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

From Szeliski 3.4

The Discrete Fourier transform

The Discrete Fourier Transform can be expressed as

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi nk/N} \quad (k = 0, 1, 2 \ldots, N-1)$$

The relevant inverse Fourier Transform can be expressed as

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{i2\pi nk/N} \quad (n = 0, 1, 2 \ldots, N-1)$$

Examples

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi nk/N} \quad (k = 0, 1, 2 \ldots, N-1)$$

```
a = np.ones(4)
np.fft.fft(a)
```

→

```
array([4.+0.j, 0.+0.j,
0.+0.j, 0.+0.j]
```

```
a = np.array([1,0,0,0])
np.fft.fft(a)
```

→

```
array([1.+0.j, 1.+0.j,
1.+0.j, 1.+0.j]
```

Example

$$F[k] = f[0]\exp(-2\pi j \frac{k*0}{4}) + f[1]\ \exp(-2\pi j \frac{k*1}{4}) + f[2]\ \exp(-2\pi j \frac{k*2}{4}) + f[3]\exp\left(-2\pi j \frac{k*3}{4}\right)$$

$$= f[0]\exp\left(-2\pi j \frac{k*0}{4}\right) = f[0] = 1$$

Intuition behind Fourier Transform—Measure of similarity by inner product

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform

Q: How to measure the similarity of two functions?

Intuition behind Fourier Transform—Measure of similarity by inner product

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform

Q: How to measure the similarity of two functions?

Answer: One way: Inner product!

We have seen this idea used correlation in previous discussion section.

Q: How to measure how similar a function is to a function single frequency component, i.e., $\exp(j\omega t)$?

Answer: Inner product!

- So Fourier transform is just comparing the function $x(t)$ to every single frequency function $\exp(j\omega t)$, and gives you a similarity measure for each ω, as $X(\omega)$.

Intuition behind Fourier Transform—Change of basis

- Consider 3D space:
- A set of linearly independent vectors whose span is the whole 3D space, are called the basis for the 3D Plane
- Can you think of a basis for the 3D Plane?
- How many vectors are required to span the 3D Plane?

$$Example: \begin{bmatrix}3\\2\\5\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\2\\5\end{bmatrix}$$

Intuition behind Fourier Transform—Change of basis

The relevant inverse Fourier Transform can be expressed as

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)e^{i2\pi nk/N} \quad (n = 0, 1, 2 \ldots, N-1)$$

For N = 3:

$$f(n) = \frac{\mathbf{1}}{\mathbf{3}}\left(F(0)e^{i2\pi n*0/3} + F(1)e^{i2\pi n*1/3} + F(2)e^{i2\pi n*2/3}\right), n = 0, 1, 2$$

$$\vec{f} = \vec{e}_1 \cdot F(0) + \vec{e}_2 \cdot F(1) + \vec{e}_3 \cdot F(2)$$

where

$$\vec{f} \triangleq f(n), \vec{e}_1 \triangleq \tfrac{1}{3}e^{i2\pi n*0/3}, \vec{e}_2 \triangleq \tfrac{1}{3}e^{i2\pi n*1/3}, \vec{e}_3 \triangleq \tfrac{1}{3}e^{i2\pi n*2/3}$$

Intuition behind Fourier Transform—Change of basis

$$\vec{f} = \vec{e}_1 \cdot F(0) + \vec{e}_2 \cdot F(1) + \vec{e}_3 \cdot F(2)$$

where

$$\vec{f} \triangleq f(n), \vec{e}_1 \triangleq \frac{1}{3}e^{i2\pi n*0/3}, \vec{e}_2 \triangleq \frac{1}{3}e^{i2\pi n*1/3}, \vec{e}_3 \triangleq \frac{1}{3}e^{i2\pi n*2/3}$$

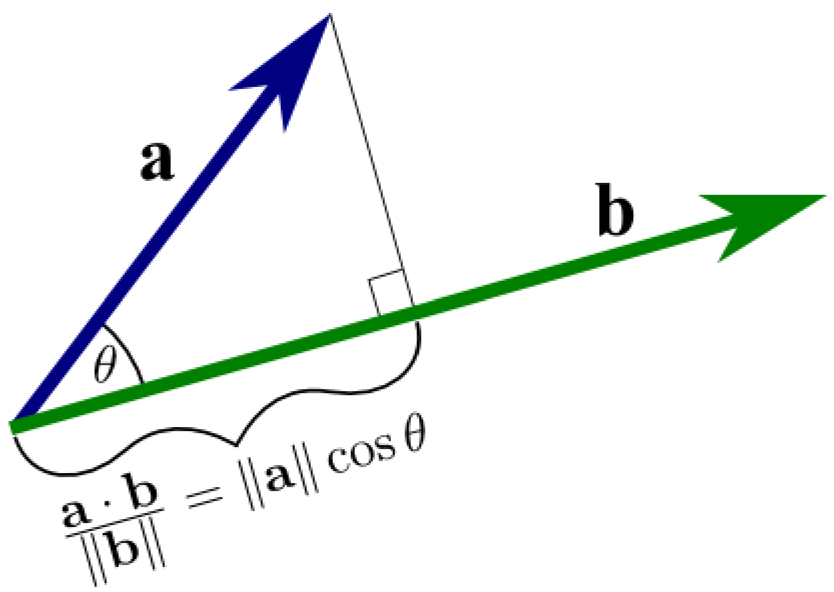
$$\vec{f} = \begin{bmatrix} | \\ f \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ F(2) \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Bottom Line:

- Fourier coefficients are coordinates in the Fourier basis defined by $\vec{e}_1\ \vec{e}_2\ \vec{e}_3$!
- Calculating Fourier coefficients is just about finding the projection of the vector $f(n)$ along the basis.

Intuition behind Fourier Transform—Change of basis

To find the coordinate value of the vector along each basis vector, use the inner product:



Here a is the function $f(n)$ we want to decompose and b is any one of the Fourier basis vector.

For N = 3:

$$\vec{e}_1 \triangleq \frac{1}{3}e^{i2\pi n*0/3}, \vec{e}_2 \triangleq \frac{1}{3}e^{i2\pi n*1/3}, \vec{e}_3 \triangleq \frac{1}{3}e^{i2\pi n*2/3}$$

$$F(k) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi nk/N} \quad (k = 0, 1, 2 \ldots, N-1)$$

Take away:

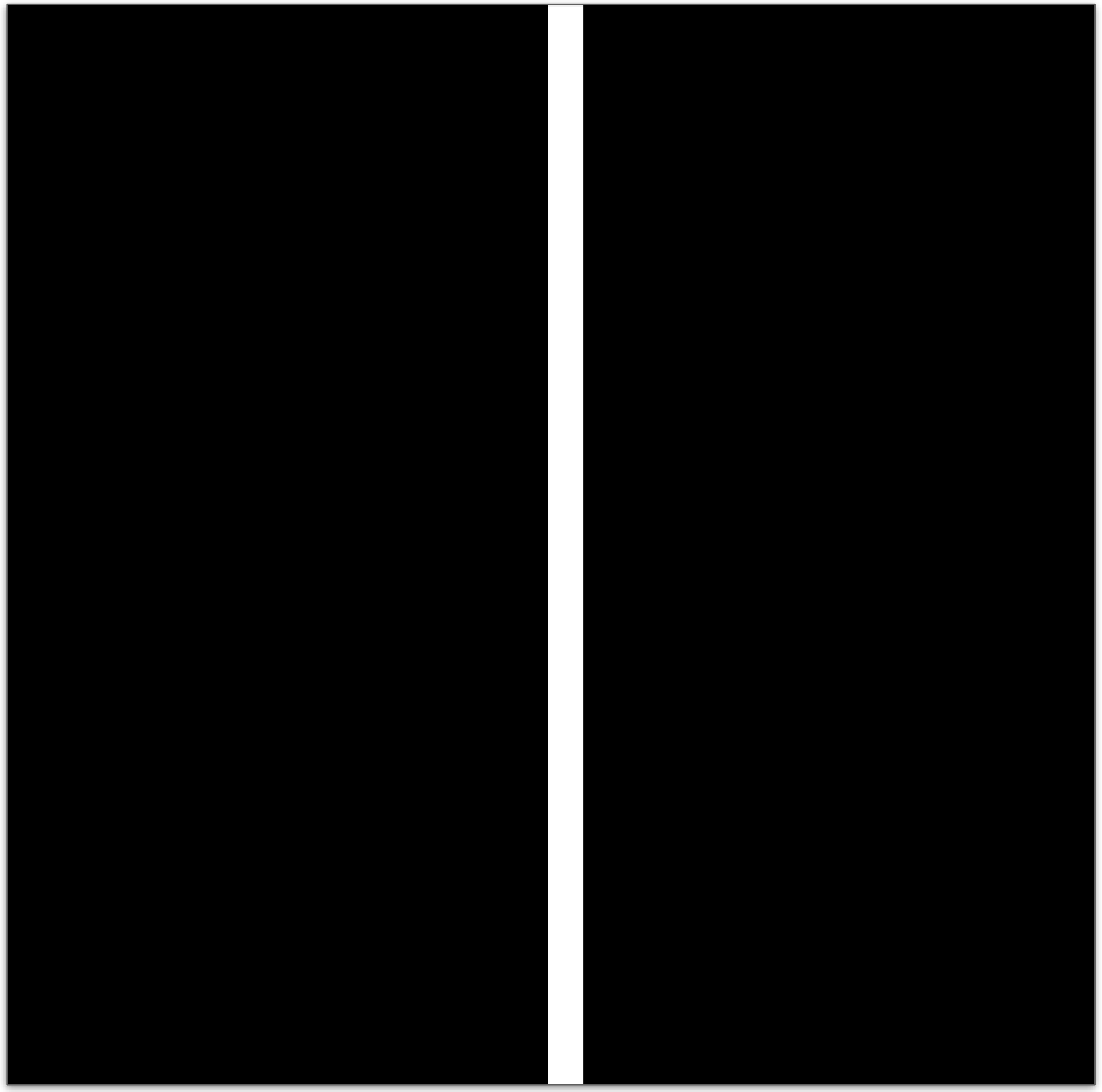
- **Fourier transform is finding its coordinate values $F(k)$ in the Fourier basis** $\vec{e}_1\ \vec{e}_2\ \vec{e}_3$
- **Inverse Fourier transform is synthesizing back the original vector $f(n)$ using the Fourier basis and corresponding coordinate values $F(k)$**

2D Discrete Fourier Transform

2D Discrete Fourier Transform (DFT) transforms an image $f[n,m]$ into $F[u,v]$ as:

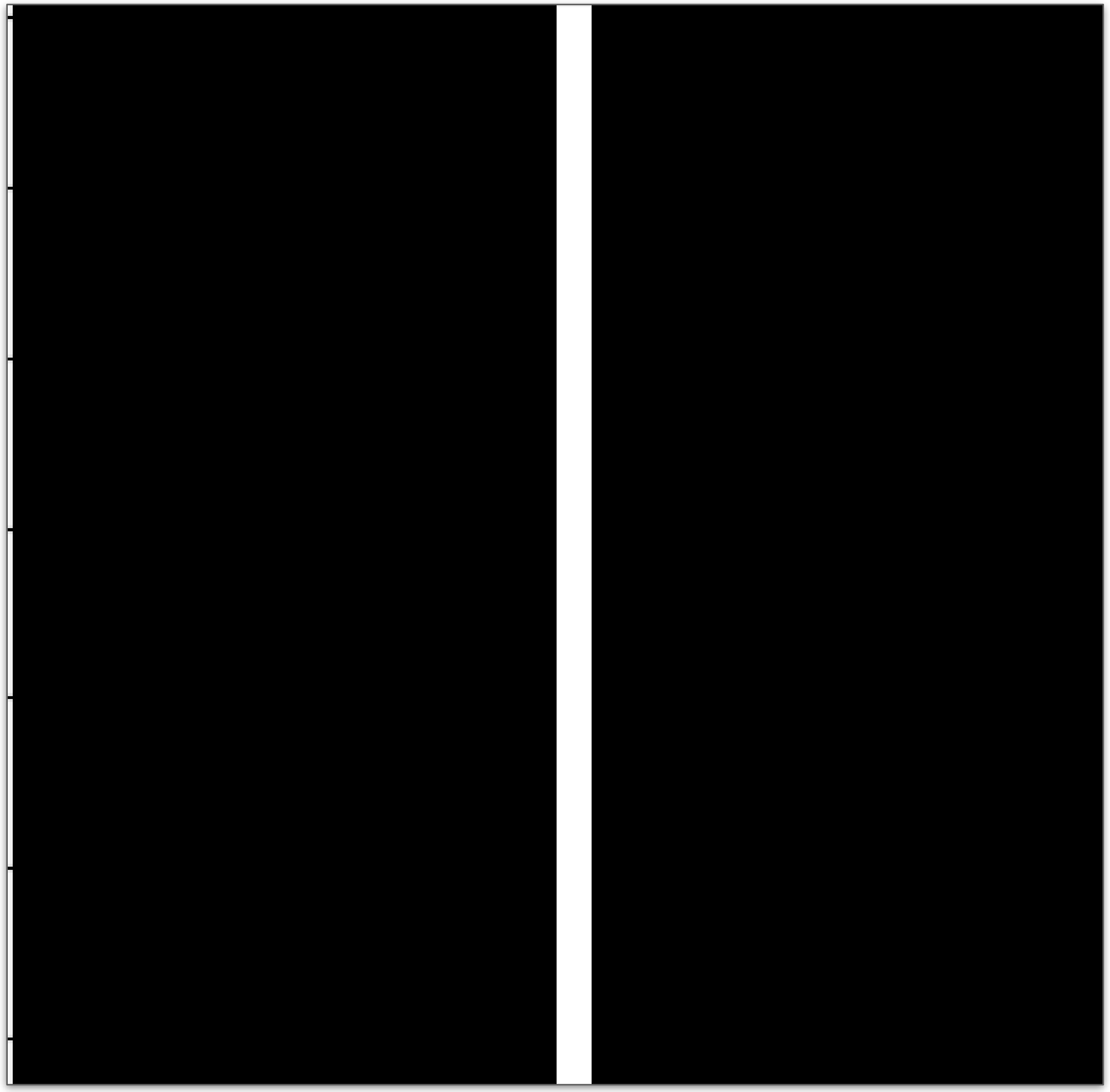
$$F[u,v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m] \exp\left(-2\pi j \left(\frac{un}{N} + \frac{vm}{M} \right) \right)$$

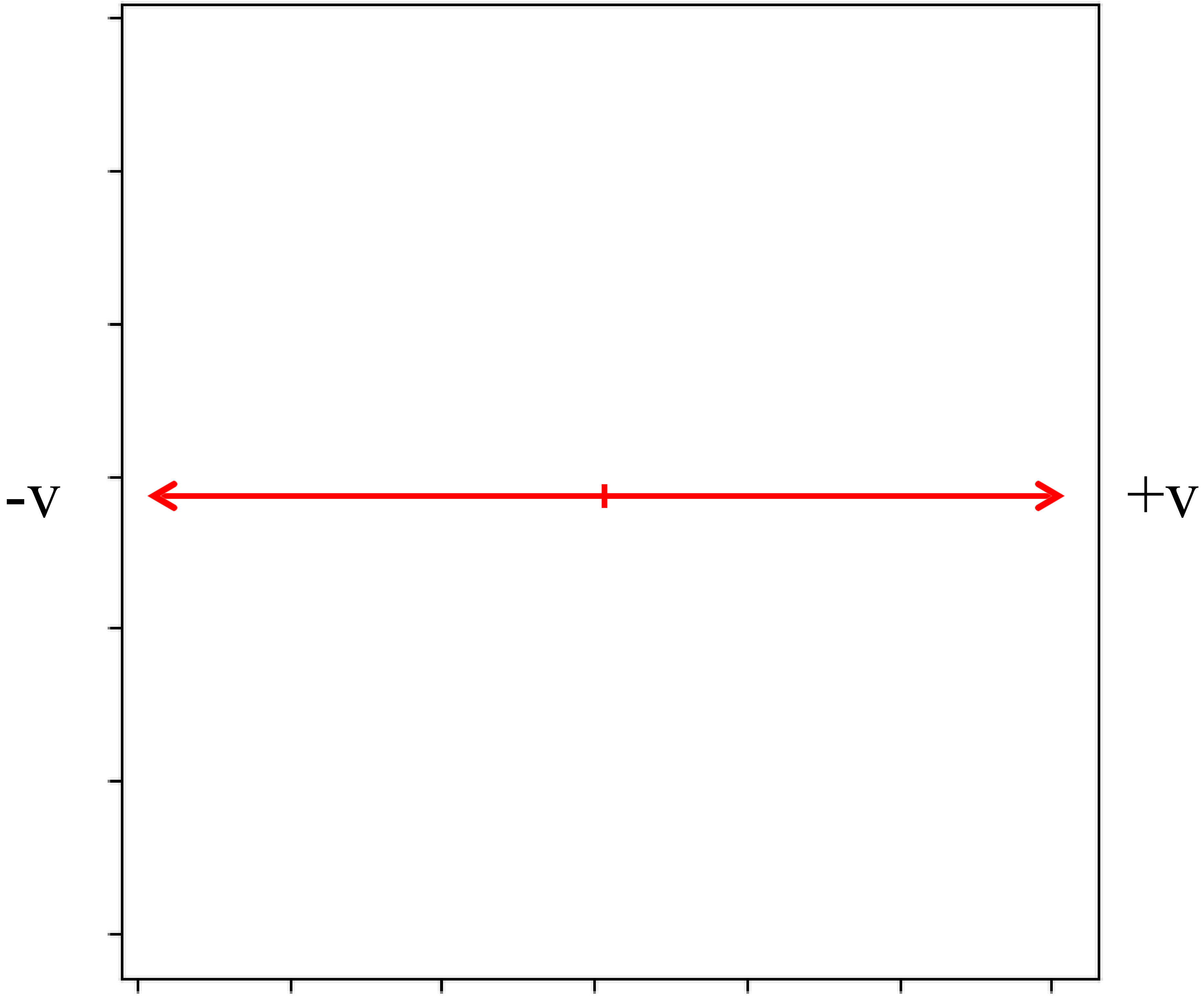
Examples



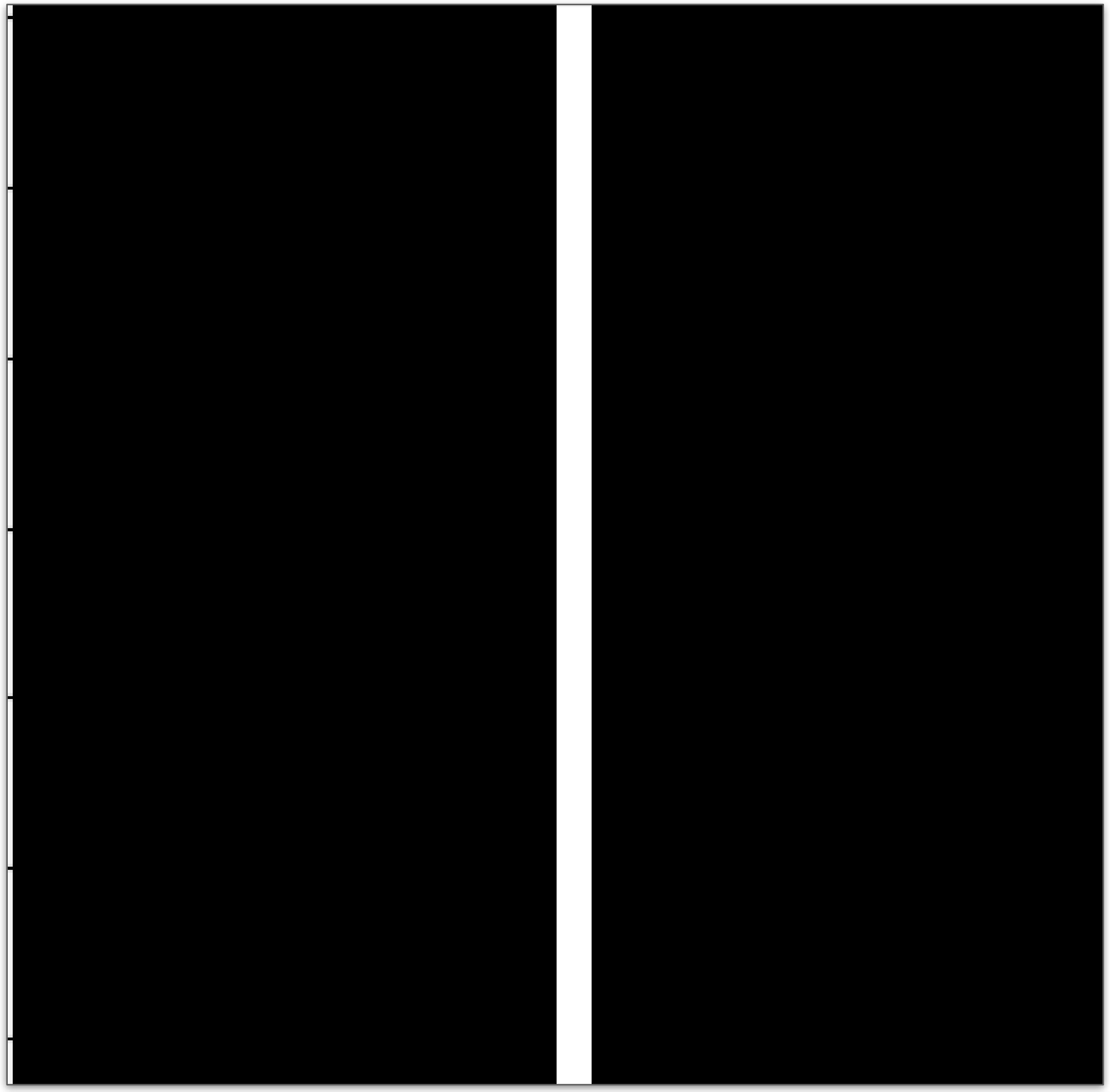
What is the 2D FFT of a line?

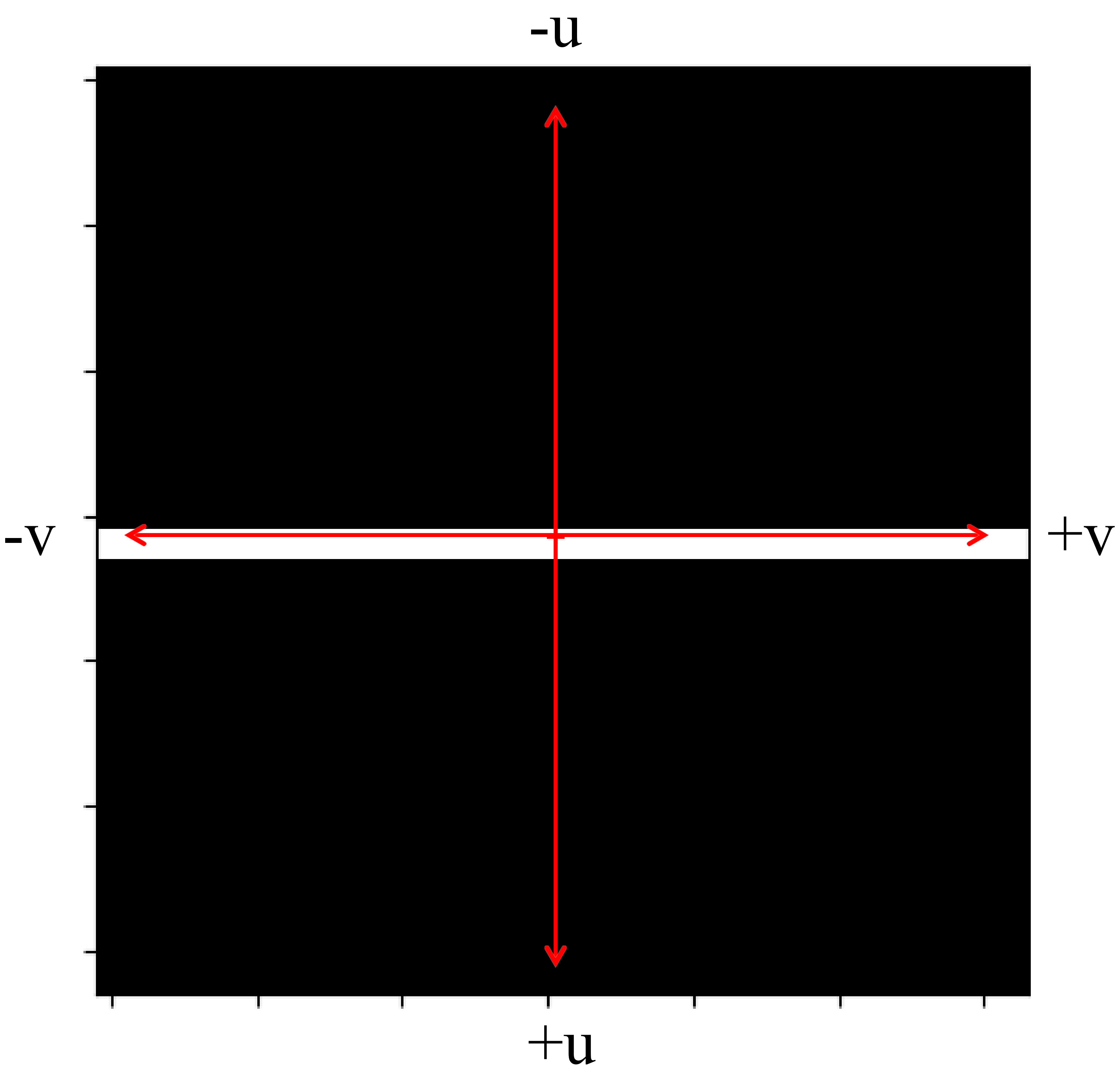
Fourier transform in *x* direction



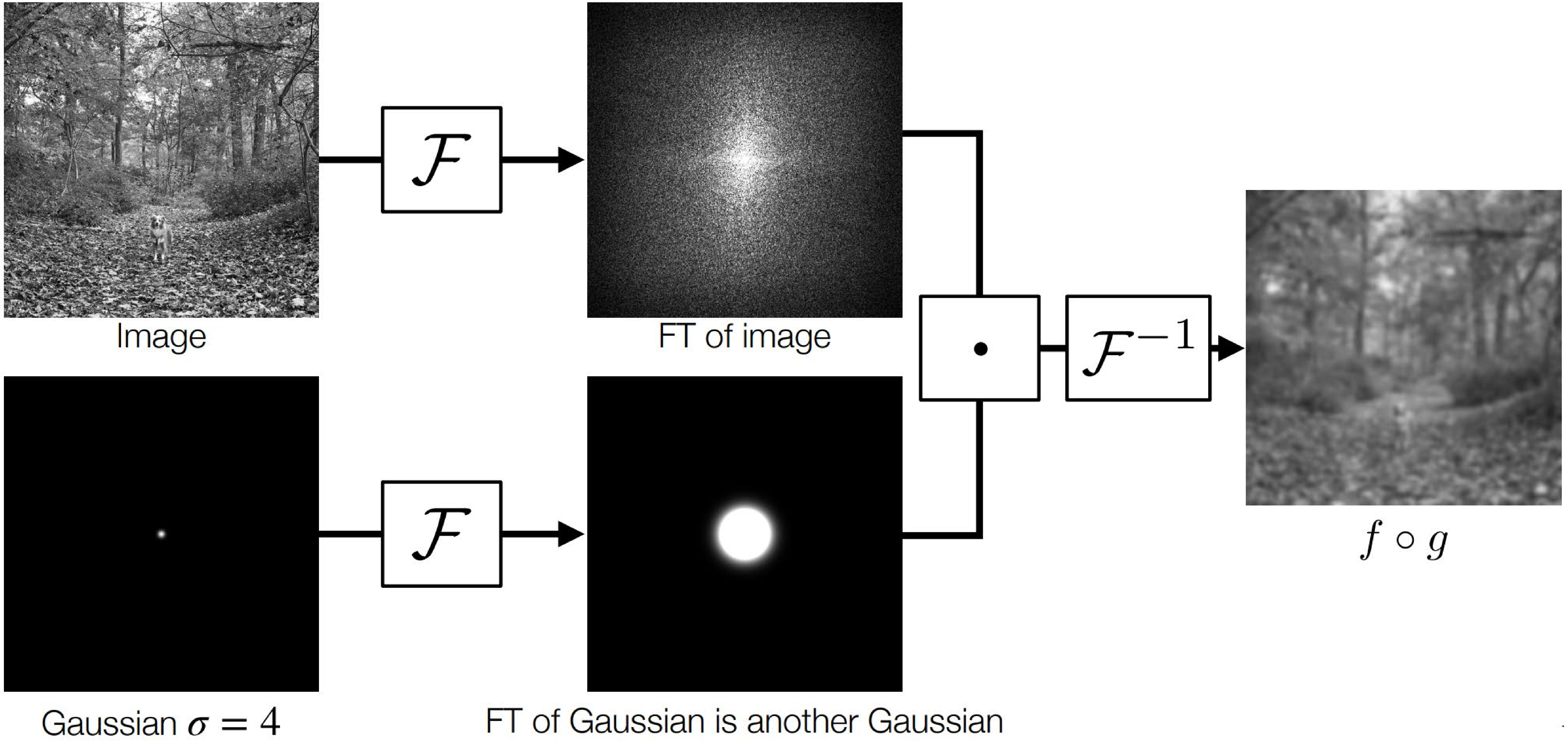


Full 2D Fourier transform





Convolution Theorem



Diffraction in optics is just Fourier Transform!

Draw a line matching the diffraction pattern to the aperture shape.