Linear Algebra Tutorial II

EECS 442

Fall 2020, University of Michigan

Announcements

- Extra Office Hours for PS1
- Professor Andrew Owens
  - Sunday: 7pm - 8:30pm
- Hansal Shah
  - Friday: 5pm - 6:30pm
  - Monday: 6pm - 7pm

Brief recap from Last Session

$$\boldsymbol{y}_{2x1} = \boldsymbol{A}_{2x3}\boldsymbol{x}_{3x1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = [v_1, \cdots, v_n] = \begin{bmatrix} v_{1_1} & v_{2_1} & v_{3_1} \\ v_{1_2} & v_{2_2} & v_{3_2} \end{bmatrix}$$

$$\boldsymbol{y} = x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} + x_3\boldsymbol{v_3}$$

Linear combination of columns of ***A***

$$\boldsymbol{AB} = \begin{bmatrix} - & \boldsymbol{a_1^T} & - \\ & \vdots & \\ - & \boldsymbol{a_m^T} & - \end{bmatrix} \begin{bmatrix} | & & | \\ \boldsymbol{b_1} & \cdots & \boldsymbol{b_p} \\ | & & | \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1^T b_1} & \cdots & \boldsymbol{a_1^T b_p} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a_m^T b_1} & \cdots & \boldsymbol{a_m^T b_p} \end{bmatrix}$$

Linear Independence

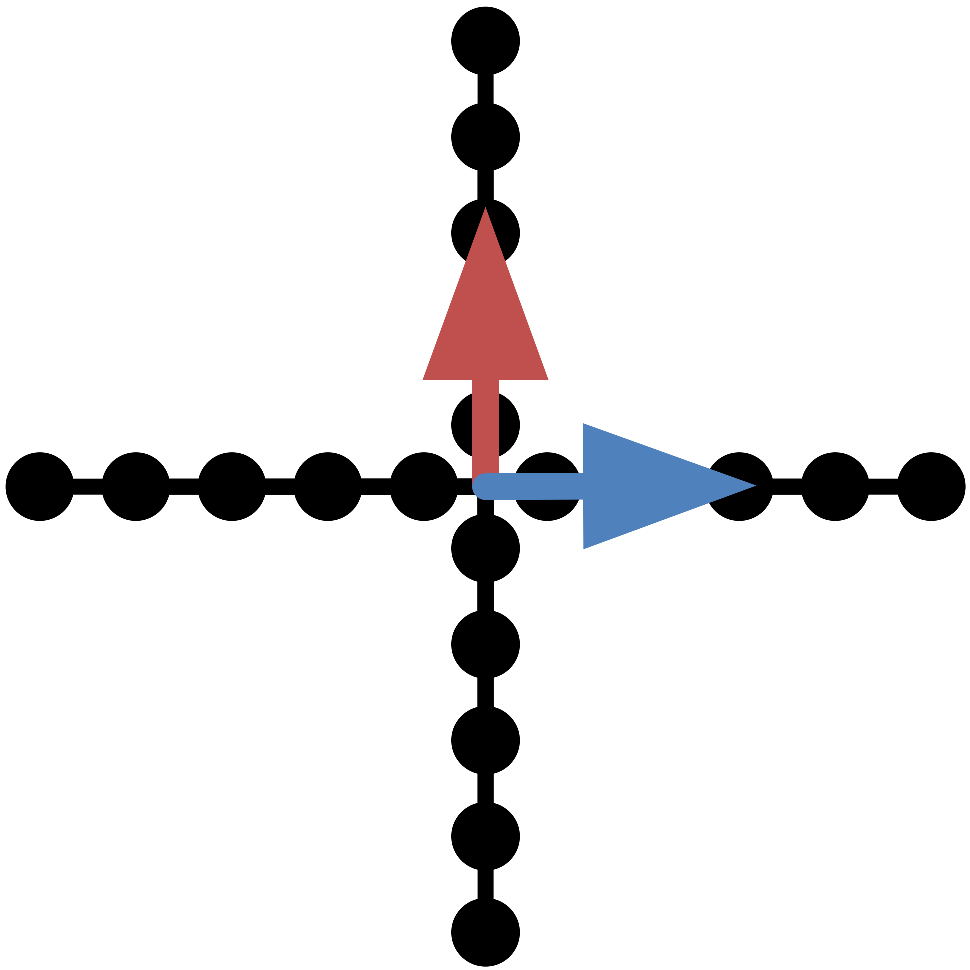
A set of vectors are linearly independent if you can't write one as a linear combination of the others.

Suppose: $\boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$ $\boldsymbol{c} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

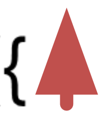
$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 2\boldsymbol{a} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$$

- **Is the set {a,b,c} linearly independent?**
- **Is the set {a,b,x} linearly independent?**
- **Max # of independent 3D vectors?**

Span

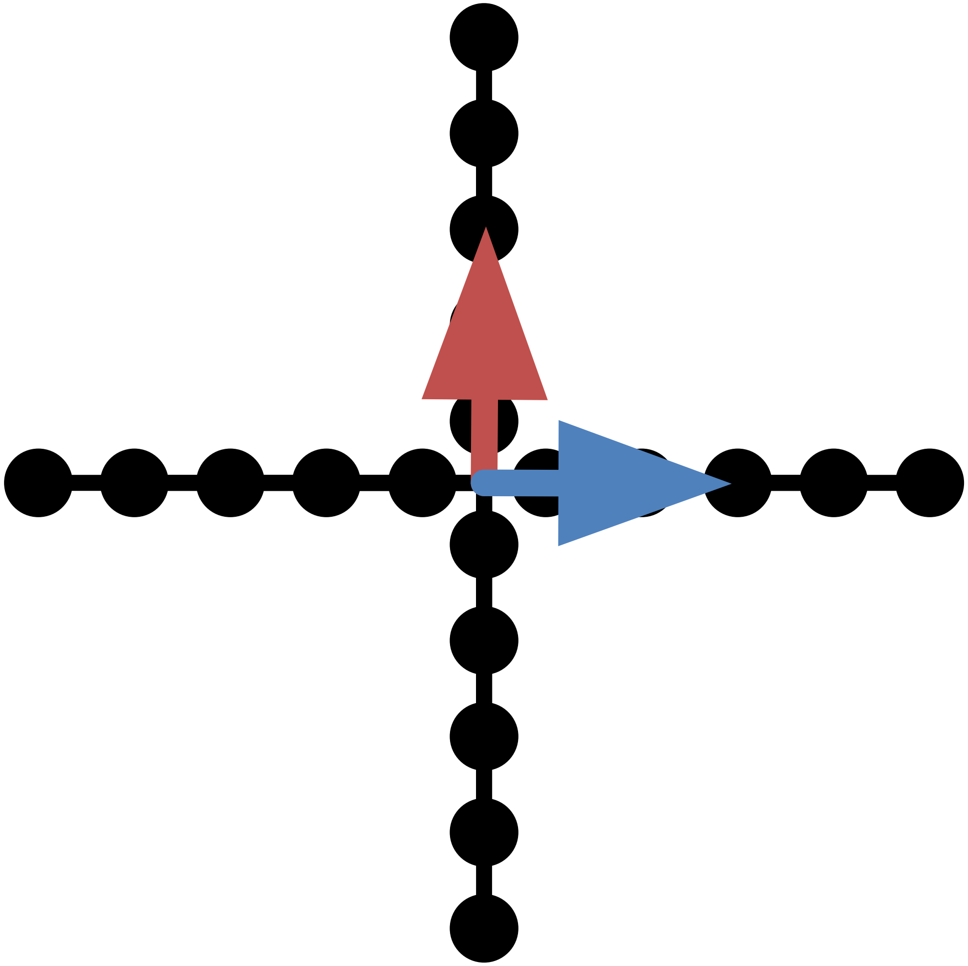


Span: all linear combinations of a set of vectors

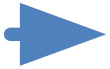
Span({}) =
Span({[0,2]}) = ?
All vertical lines through origin =
$\{\lambda[0,1] : \lambda \in R\}$

Is blue in {red}'s span?

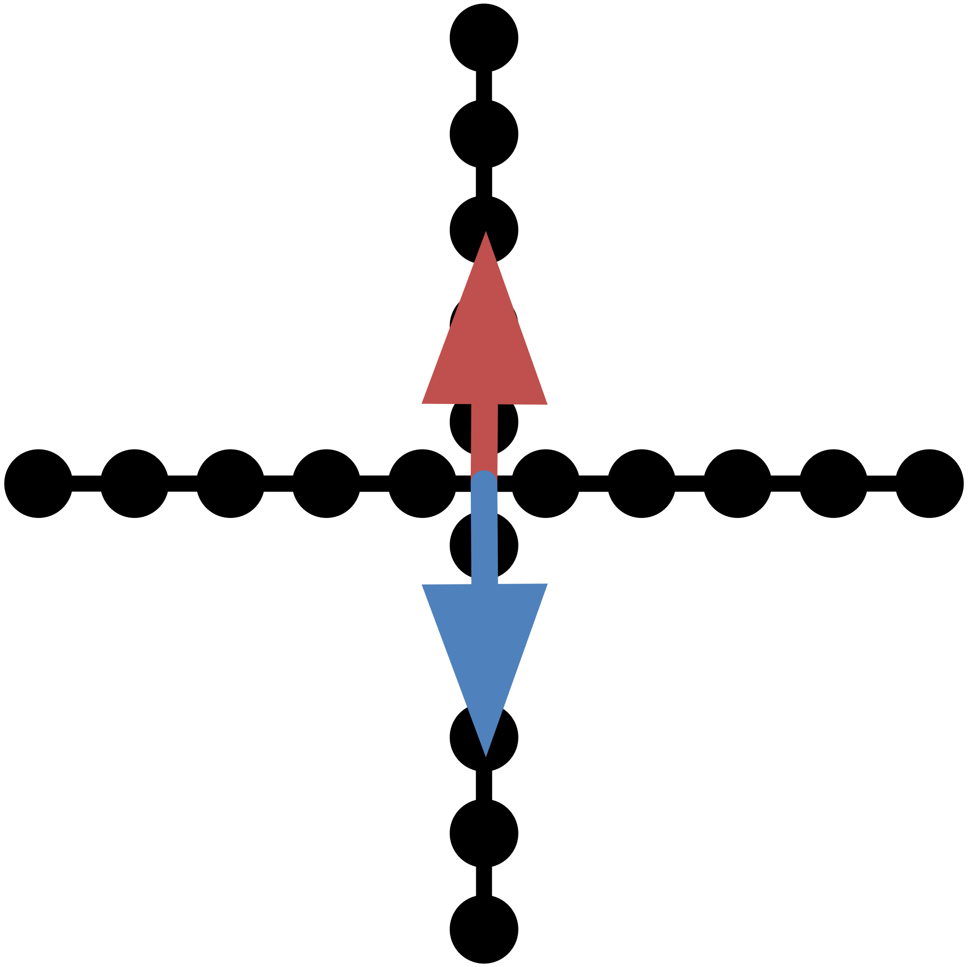
Span



Span: all linear combinations of a set of vectors

Span({ , }) = **?**

Span



Span: all linear combinations of a set of vectors

Span({,  }) = **?**

Basis

- Consider all vectors in $\mathbb{R}^3$ (3D Plane)
- A set of linearly independent vectors whose span is the whole 3D Plane, are called the basis for the 3D Plane
- Basis are defined on a subspace
- Can you think of a basis for the 3D Plane?
- How many vectors are required to span the 3D Plane?
- Remember, a set of basis vectors should span the subspace and also be linearly independent.

Using Basis for expressing vectors

$$Example: \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

- So any vector in 3D can be written as a linear combination of the basis vectors.
- We could decompose the vector in terms of some other basis as well.
- In the above example, what will change in that case?

Matrix-Vector Product

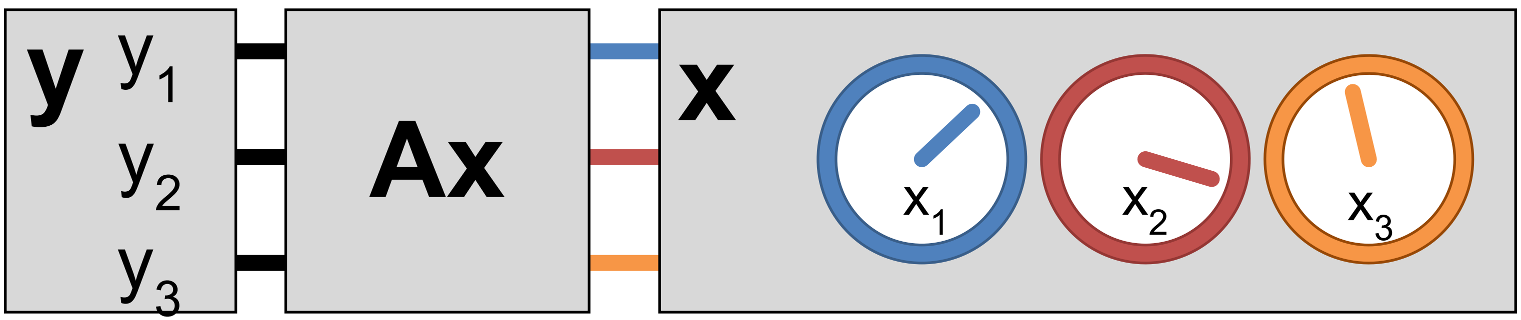
$$\boldsymbol{Ax} = \begin{bmatrix} | & & | \\ \boldsymbol{c_1} & \cdots & \boldsymbol{c_n} \\ | & & | \end{bmatrix} \boldsymbol{x}$$

Right-multiplying **A** by **x** mixes columns of **A** according to entries of **x**

- The output space of f(**x**) = **Ax** is constrained to be the *span* of the columns of **A**.
- Can’t output things you can’t construct out of your columns

An Intuition

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} | & | & | \\ \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_n} \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



x – knobs on machine (e.g., fuel, brakes)
y – state of the world (e.g., where you are)
A – machine (e.g., your car)

Linear Independence

Suppose the columns of 3x3 matrix **A** are *not* linearly independent (c_1, αc_1, c_2 for instance)

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} | & | & | \\ \boldsymbol{c_1} & \alpha\boldsymbol{c_1} & \boldsymbol{c_2} \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

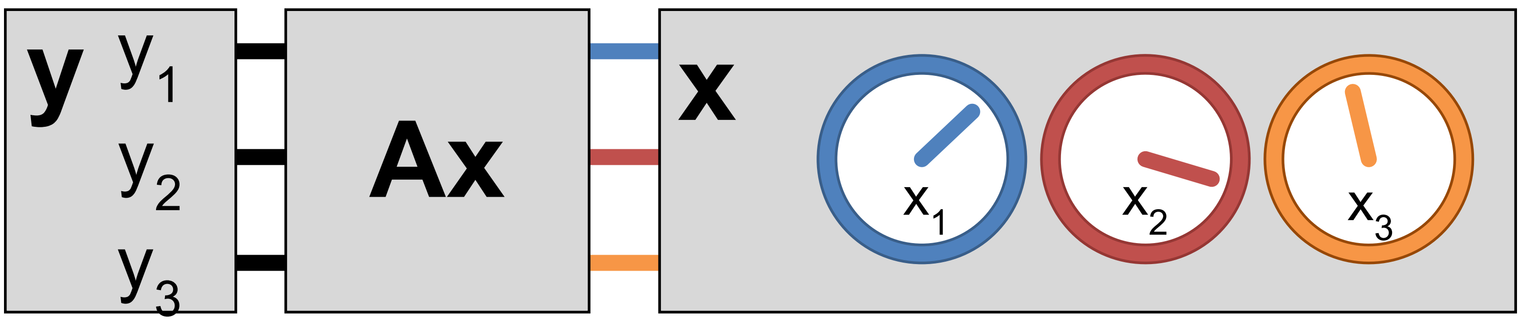
$$\boldsymbol{y} = x_1\boldsymbol{c_1} + \alpha x_2\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

$$\boldsymbol{y} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

Linear Independence Intuition

Knobs of **x** are redundant. Even if **y** has 3 outputs, you can only control it in two directions

$$\boldsymbol{y} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$



Linear Independence

Recall: $$\boldsymbol{Ax} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

$$\boldsymbol{y} = \boldsymbol{A}\begin{bmatrix} x_1 + \beta \\ x_2 - \beta/\alpha \\ x_3 \end{bmatrix} = \left(x_1 + \beta + \alpha x_2 - \alpha\frac{\beta}{\alpha}\right)c_1 + x_3 c_2$$

- Can write **y** an infinite number of ways by adding β to $\mathbf{x_1}$ and subtracting $\frac{\beta}{\alpha}$ from $\mathbf{x_2}$
- Or, given a vector **y** there's not a unique vector **x** s.t. **y** =**Ax**
- Not all **y** have a corresponding **x** s.t. **y=Ax**

Rank

- Rank of a nxn matrix **A** – number of linearly independent columns (**or rows**) of A / the dimension of the span of the columns
- Matrices with *full rank* (n x n, rank n) behave nicely: can be inverted, span the full output space, are one-to-one.
- Matrices with *full rank* are machines where every knob is useful and every output state can be made by the machine

Inverses

- Given **y** = **Ax**, **y** is a linear combination of columns of **A** proportional to **x**. If A is full-rank, we should be able to invert this mapping.
- Given some **y** (output) and **A**, what **x** (inputs) produced it?
- $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$

Special Matrices - Symmetric Matrices

- Symmetric: $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}$ or $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}$
- Have **lots** of special properties

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Any matrix of the form $\boldsymbol{A} = \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}$ is symmetric.

Quick check:

$$\boldsymbol{A}^{\boldsymbol{T}} = \left(\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}\right)^{\boldsymbol{T}}$$
$$\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{X}^{\boldsymbol{T}}\left(\boldsymbol{X}^{\boldsymbol{T}}\right)^{\boldsymbol{T}}$$
$$\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}$$

Special Matrices – Rotations

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Rotation matrices **R** rotate vectors and **do not change vector L2 norms** ($||\mathbf{Rx}||_2 = ||\mathbf{x}||_2$)
- Every row/column is unit norm
- Every row is linearly independent
- Transpose is inverse $\mathbf{RR}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$
- Determinant is 1

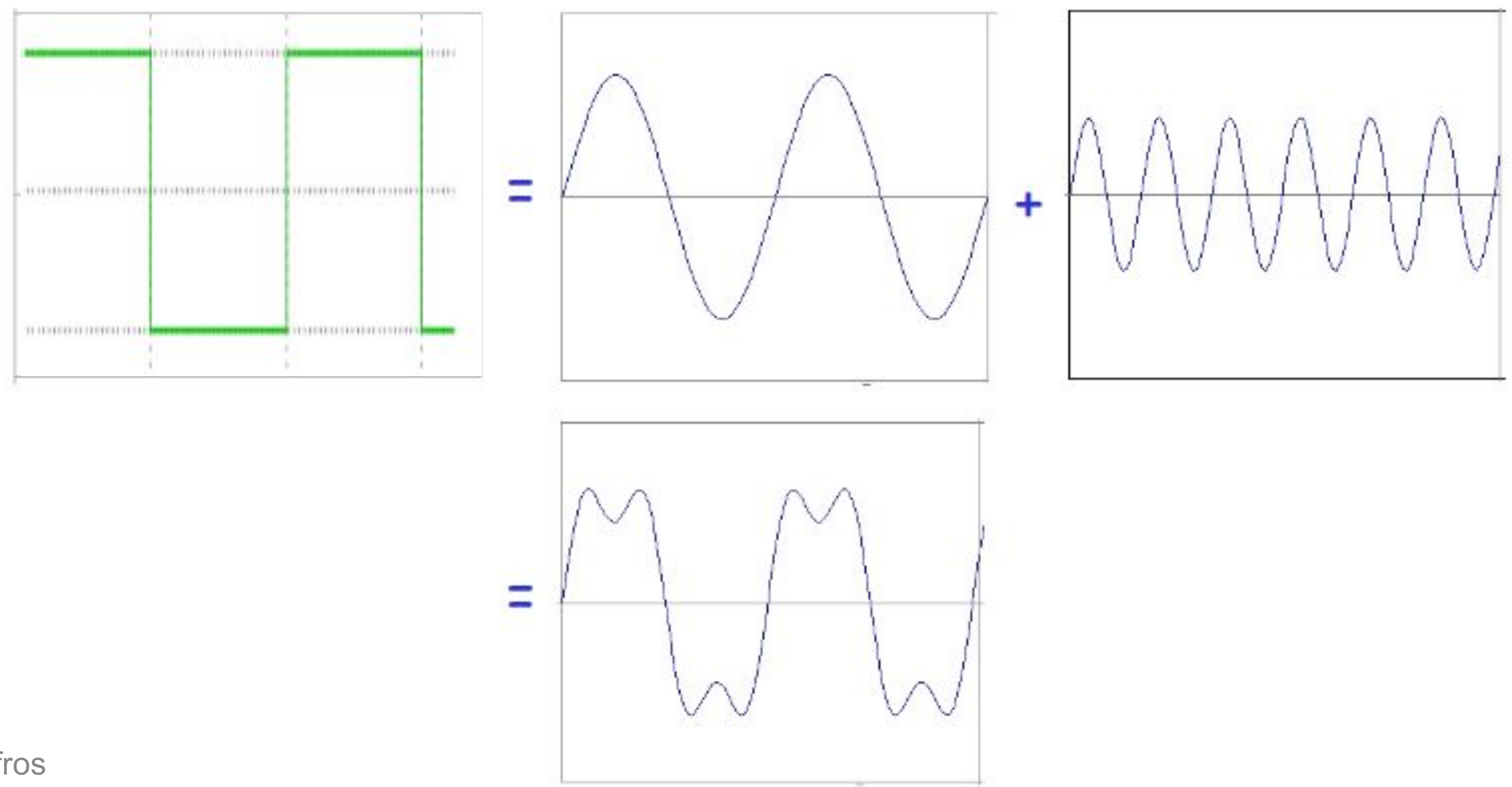
Fourier Transform

(Named after Jean Baptiste Joseph Fourier)

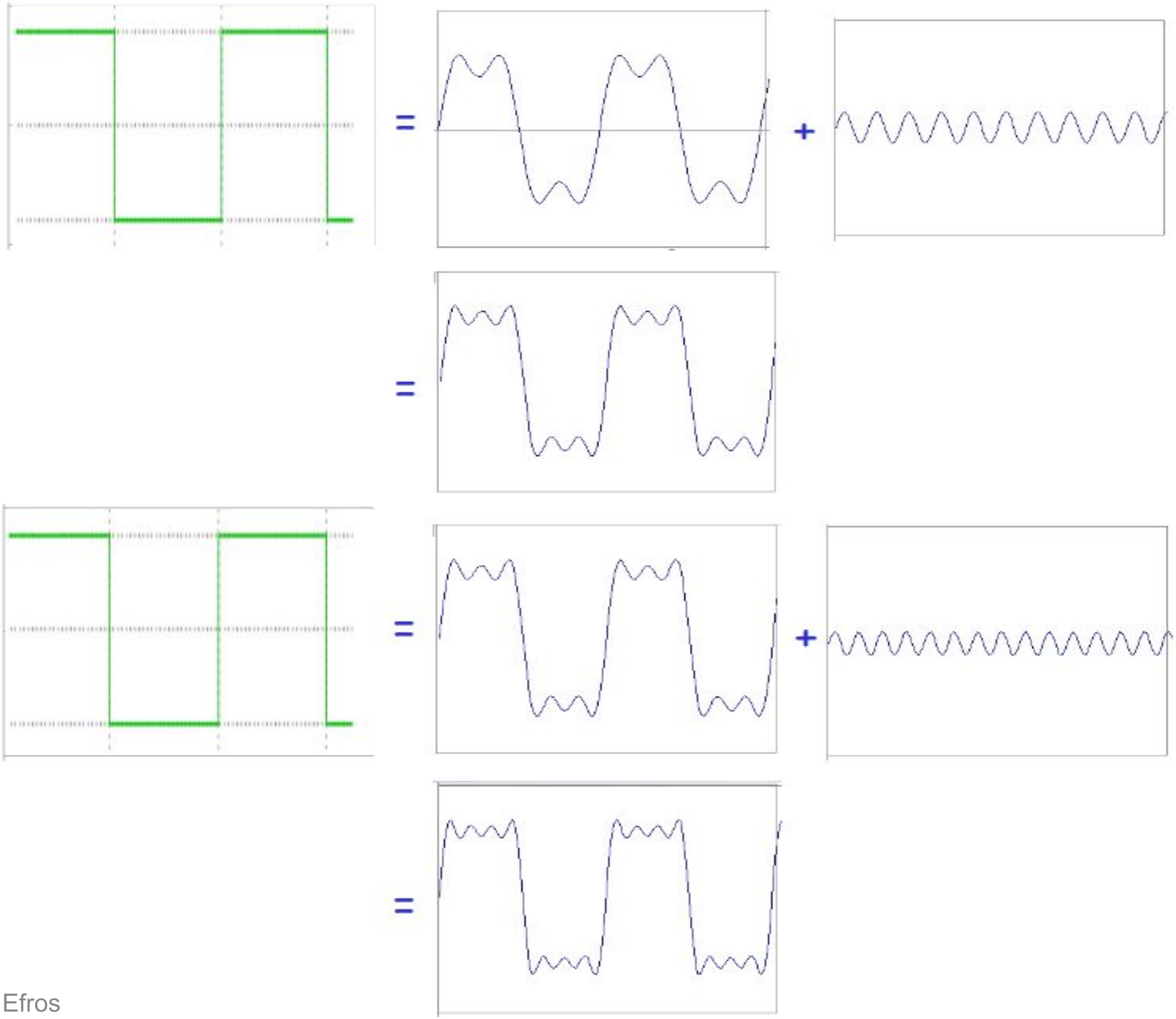
- Early in the Nineteenth century, Fourier studied heat and conceived the idea (in 1807) that **any** univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies. Called Fourier Series.
- He presented this idea to a committee including Lagrange and Laplace, but they wouldn't believe it!
- Not translated to English until 1878!
- Popular today for applications in various fields: Data Compression, Music, astronomy, etc

Rectangular Signal as Sum of Sines

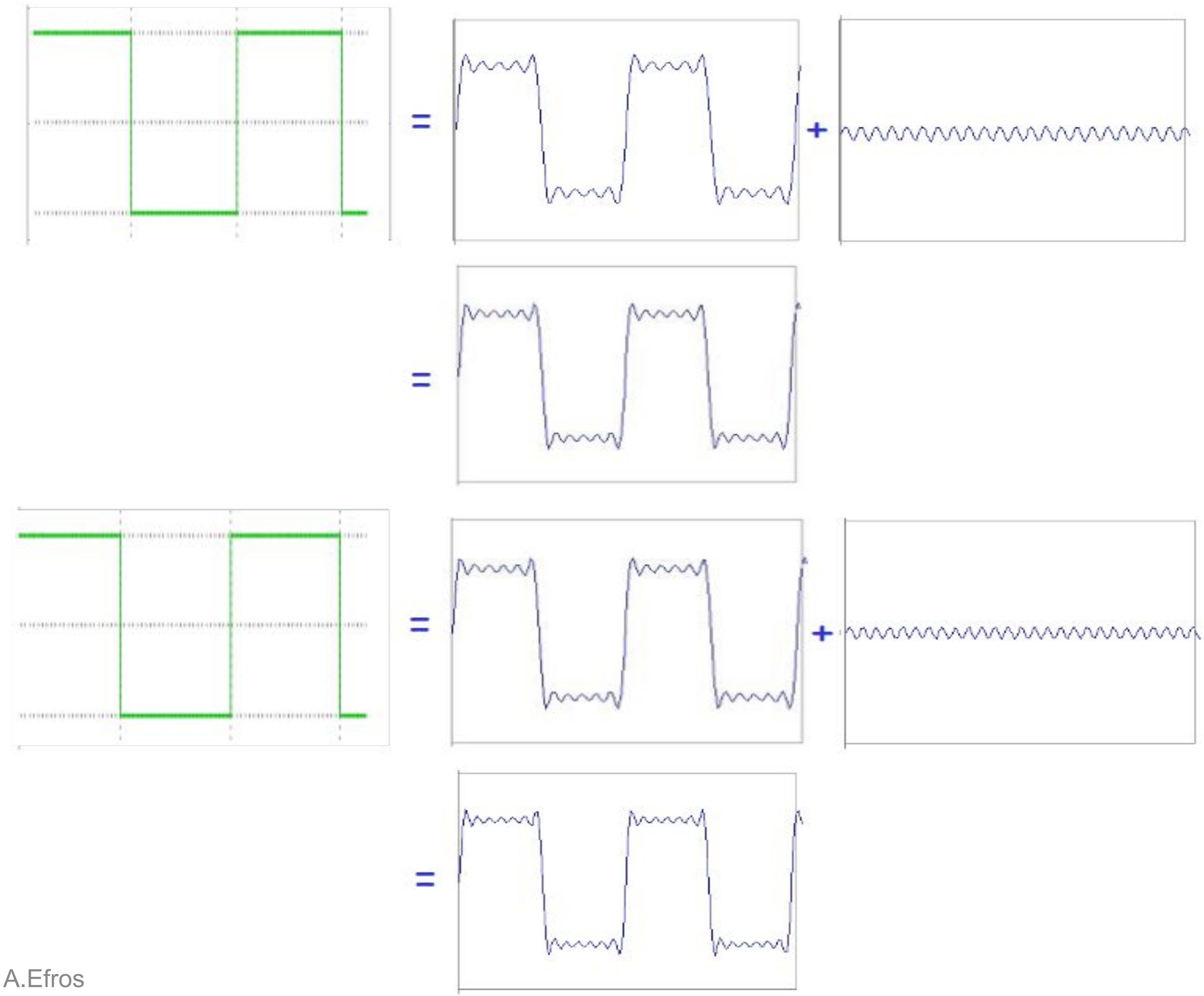
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.



Rectangular signal as Sum of Sines continued...



Rectangular signal as Sum of Sines continued...

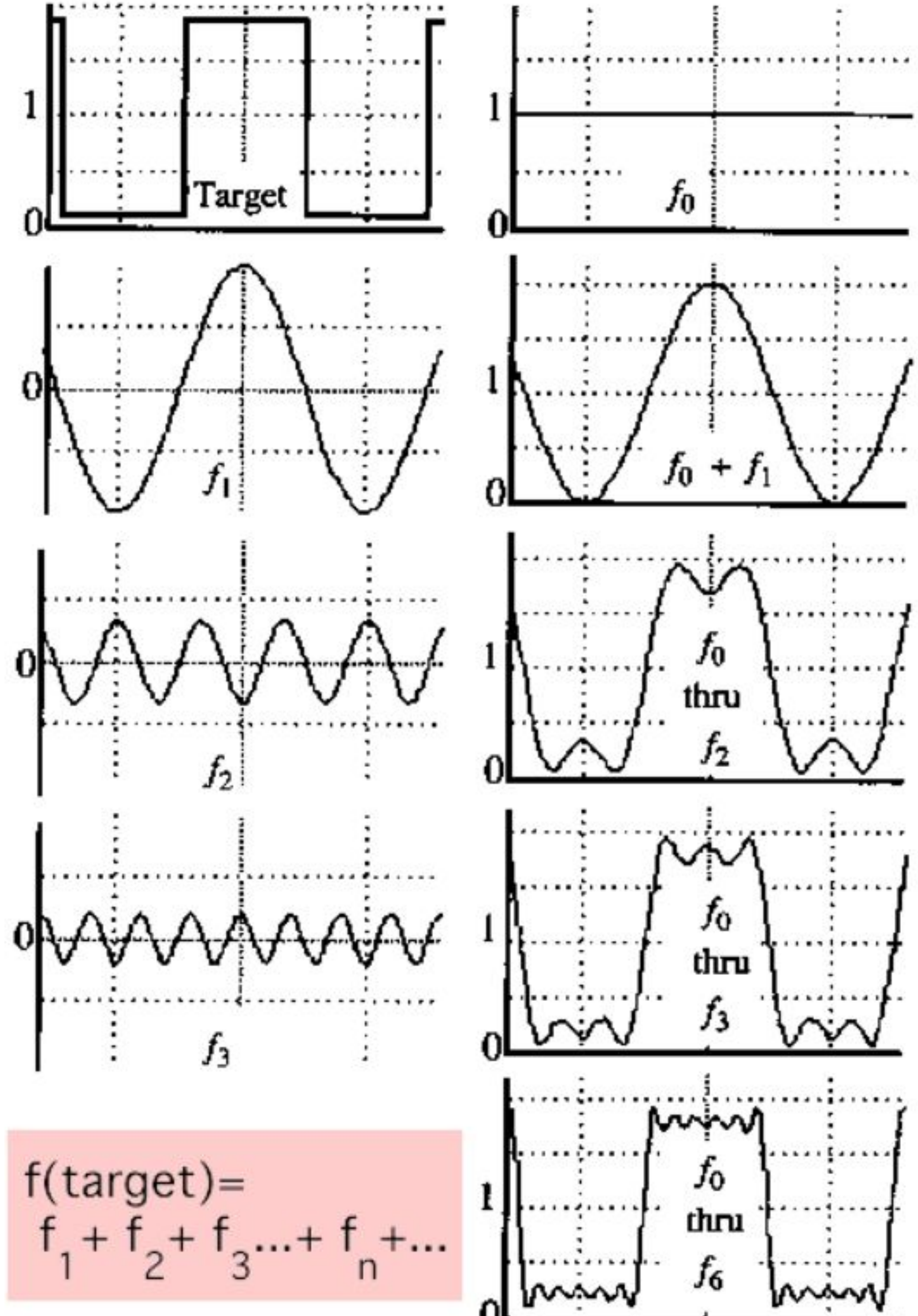


Source: A.Efros

Another Visualization

See an intuitive explanation of fourier transform here:

[An Interactive Introduction to Fourier Transforms]()

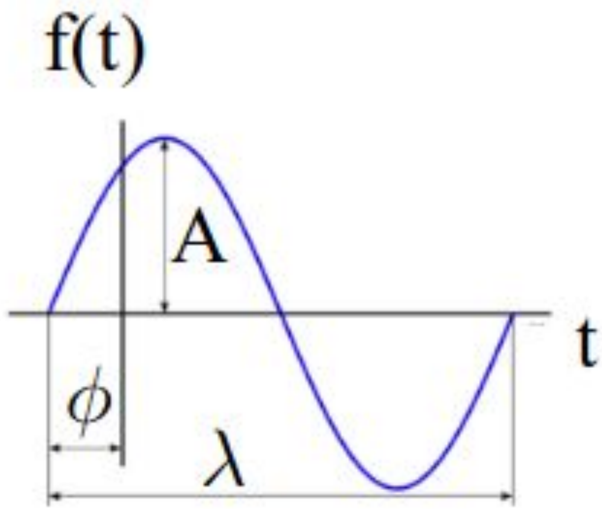


$$f(\text{target}) = f_1 + f_2 + f_3 \ldots + f_n + \ldots$$

Image Source: A.Efros

Background for Fourier Transform

Sinusoids



$$f(t) = A\sin(2\pi f t + \phi) = A\sin(\omega x + \phi)$$

A: amplitude
ϕ: phase
f: frequency

ω: angular frequency
λ: Wavelength

Background: **Complex Numbers**

$$z = x + iy = re^{i\varphi}$$
$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$r = \sqrt{x^2 + y^2}$$
$$\varphi = \tan^{-1}\frac{y}{x}$$
$$Re(z) = x = r\cos\varphi$$
$$Im(z) = y = r\sin\varphi$$

$$z^* = x - iy = re^{-i\varphi}$$ (Complex Conjugate)
$$e^{-i\varphi} = \cos\varphi - isin\varphi$$

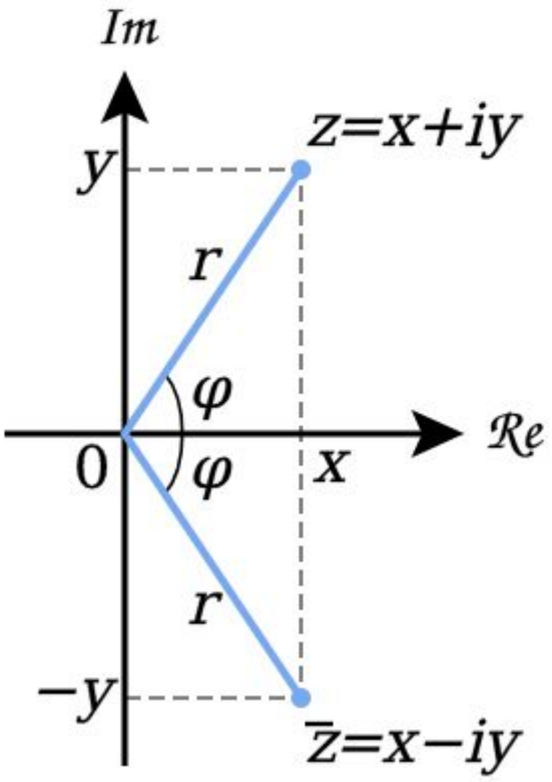


Image adapted from Wikipedia, Oleg Alexandrov - Vectorized version of http://ja.wikipedia.org/wiki/画像:Complex.png

The Discrete Fourier Transform

$$F[u] = \sum_{n=0}^{N-1} f[n] exp(-2\pi j \frac{un}{N})$$

$$exp(-2\pi j \frac{un}{N}) = \cos 2\pi j \frac{un}{N} - j \sin 2\pi j \frac{un}{N}$$

- Output F is a weighted sum of Sines and Cosines with the weights governed by input f.
- We can think of the exponentials as basis functions, and the function F is expressed in terms of those basis.
- Note that this again is a linear transformation!
- More on Fourier Transform in next class on Wednesday and the next session on Friday!