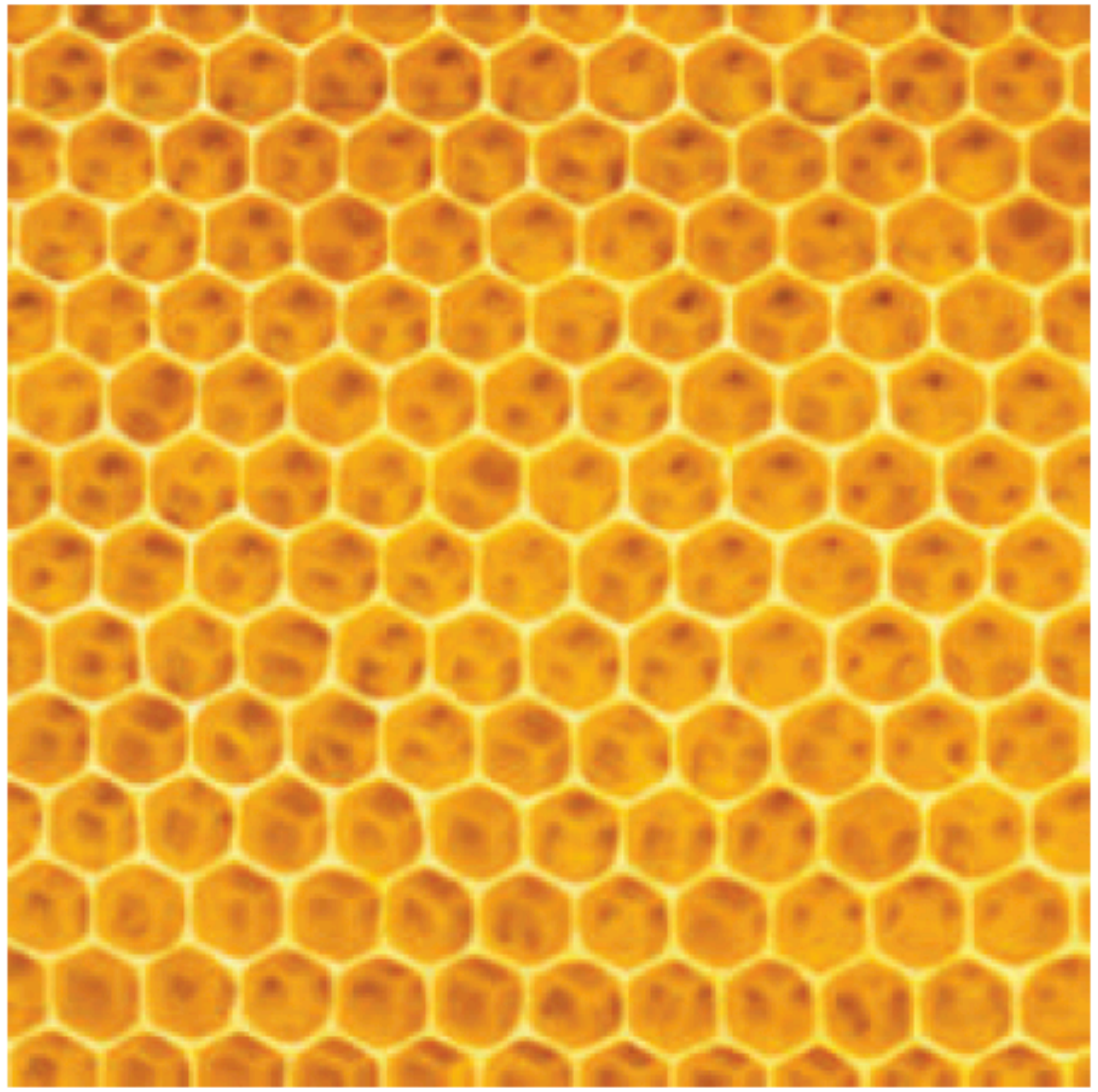
Lecture 4: Frequency

Today

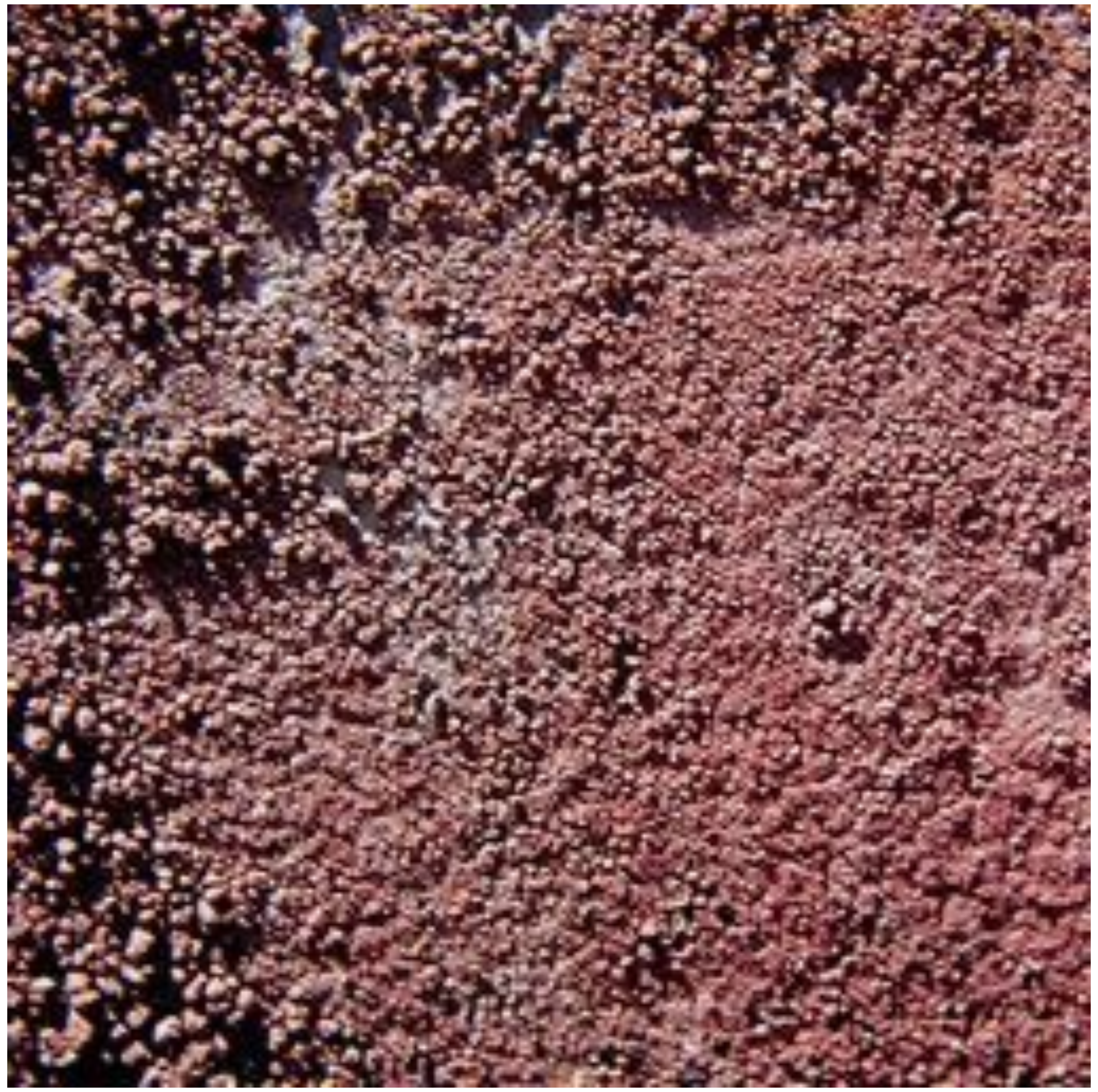
- Reminder: PS1 due Weds.
- Section this week: pyramids and Fourier trnasform
- Suggested reading:
  - Szeliski 3.4
  - Torralba, Freeman, Isola manuscript chapter

Thinking in frequency

The visual world contains:



Repetitive structures



…that repeat quickly

…or slowly

1D Fourier Transform

The 1D Discrete Fourier Transform (DFT) transforms an N-dimensional signal *f[n]* into *F[u]* as:

Fourier coefficient for **frequency** *u*

1D image

A complex exponential

$$F[u] = \sum_{n=0}^{N-1} f[n] \exp\left(-2\pi i \frac{un}{N}\right)$$

Recall that:

$$\exp(i\omega x) = \cos(\omega x) + i\sin(\omega x)$$

Adapted from: Torralba, Freeman, Isola

1D Fourier Transform

The 1D Discrete Fourier Transform (DFT) transforms an N-dimensional signal *f[n]* into *F[u]* as:

Fourier coefficient for **frequency** *u*

1D image

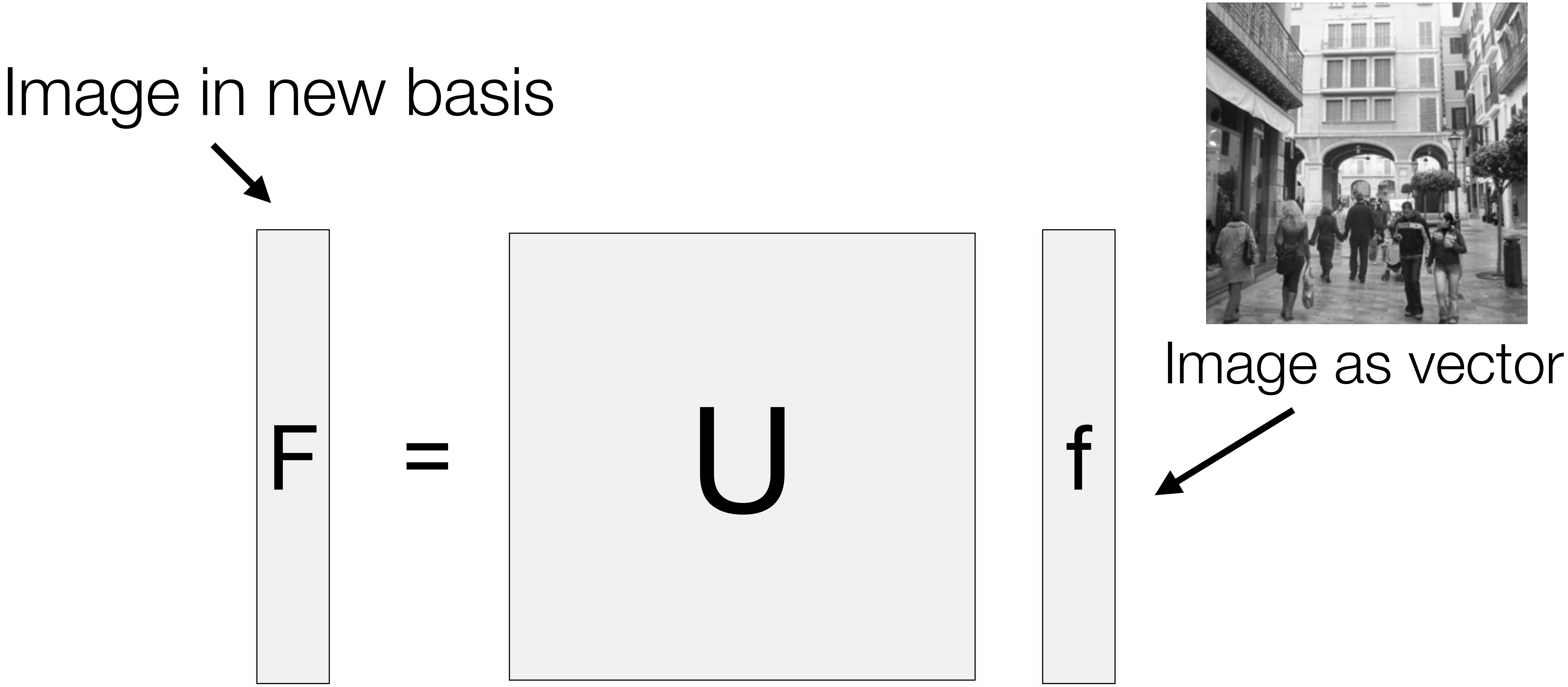
$$F[u] = \sum_{n=0}^{N-1} f[n] \left[\cos\left(-2\pi \frac{un}{N}\right) + i \sin\left(-2\pi \frac{un}{N}\right) \right]$$

Recall that:

$$\exp(i\omega x) = \cos(\omega x) + i \sin(\omega x)$$

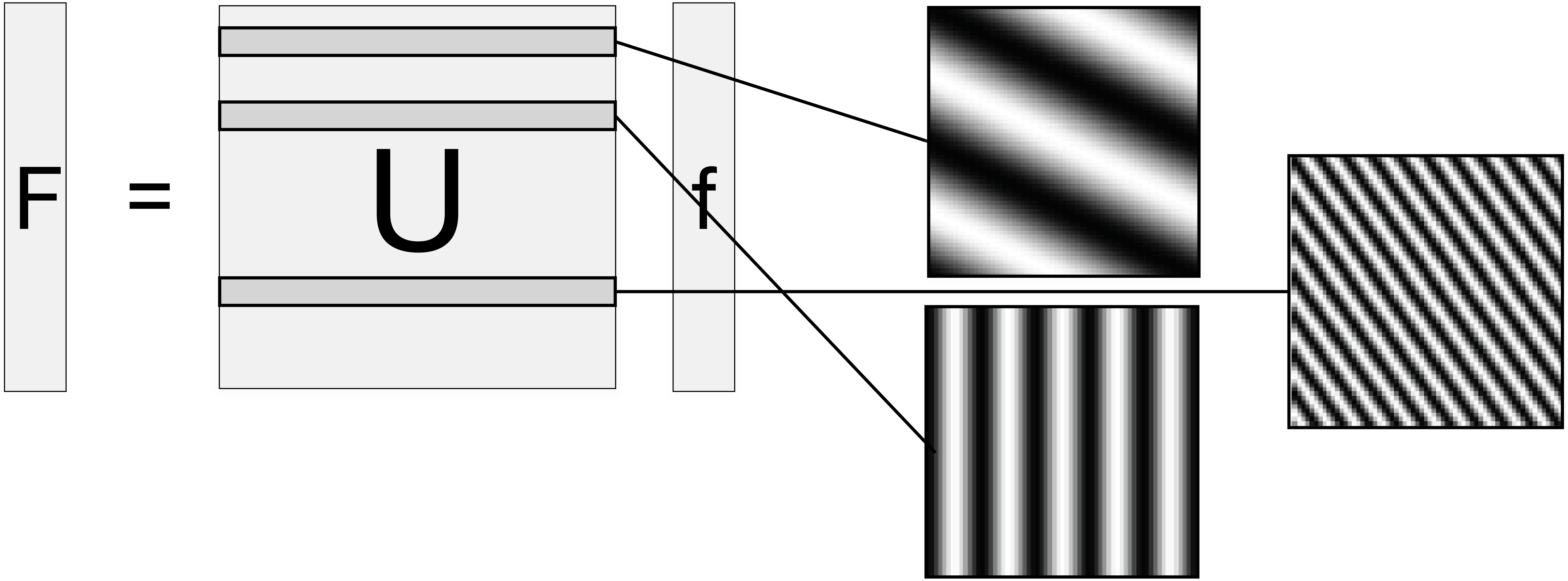
Adapted from: Torralba, Freeman, Isola

Change of basis



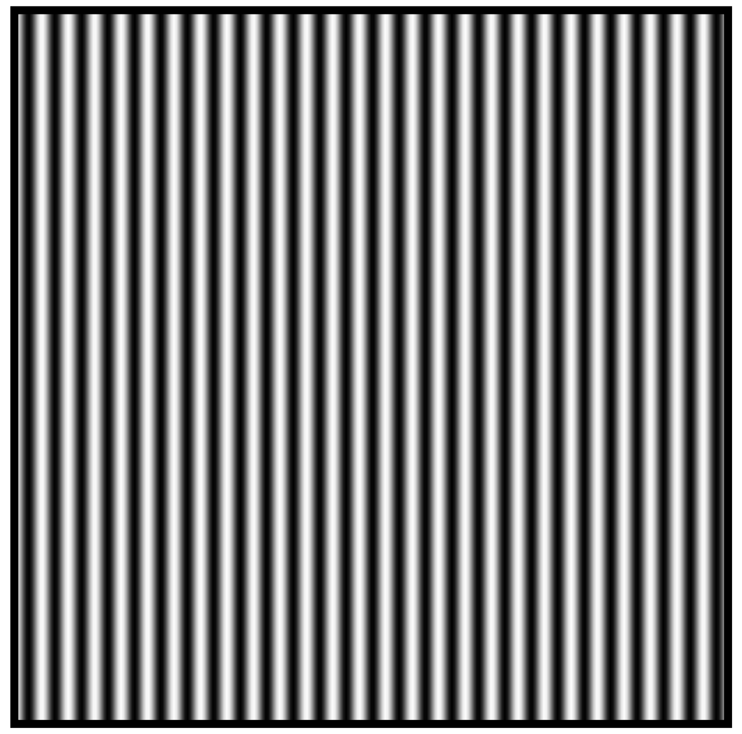
Frequency basis

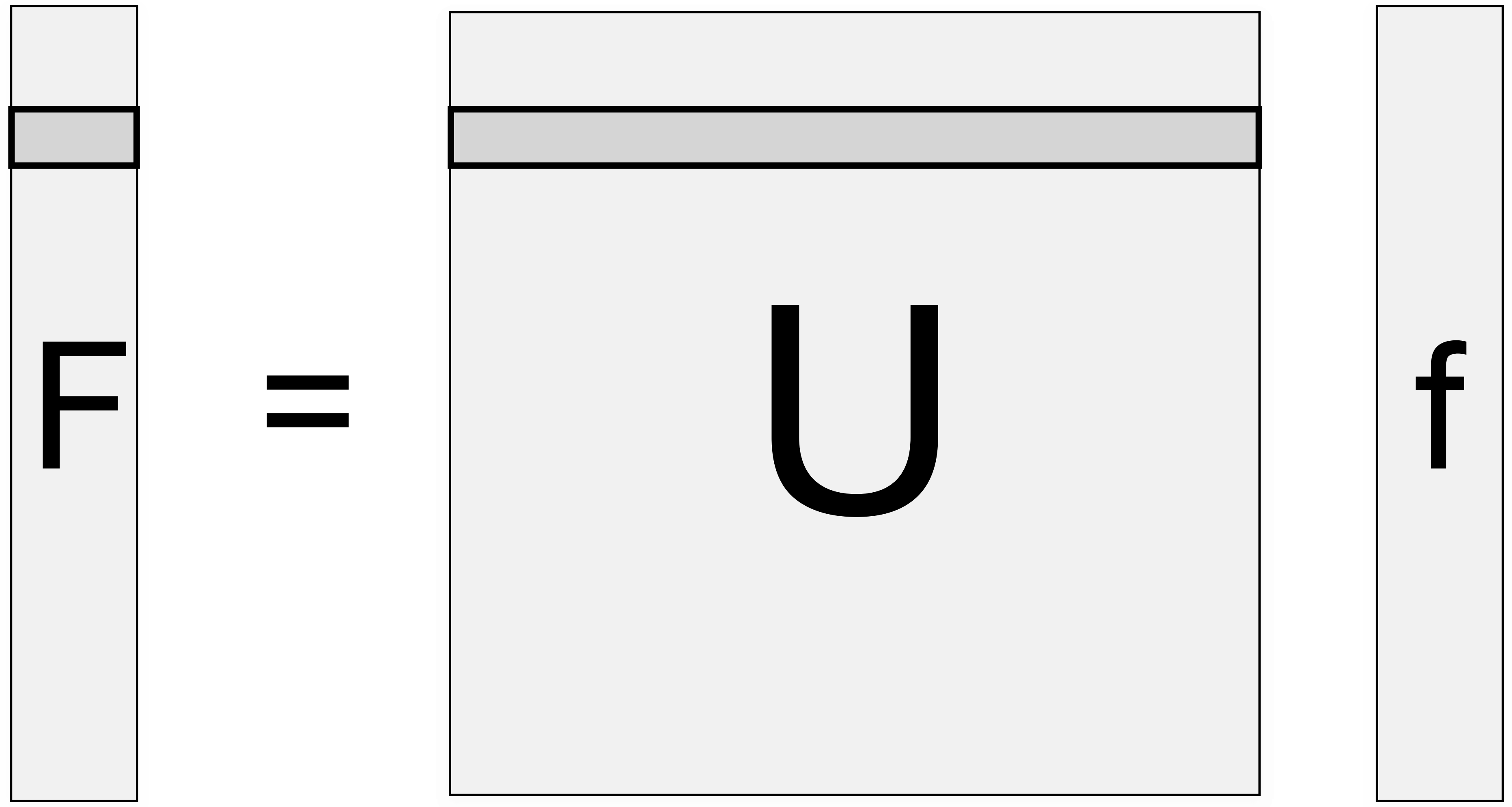
We're using a basis of sinusoids with different frequencies.



Frequency basis

Dot product

$F_i =$  $\bullet$ 

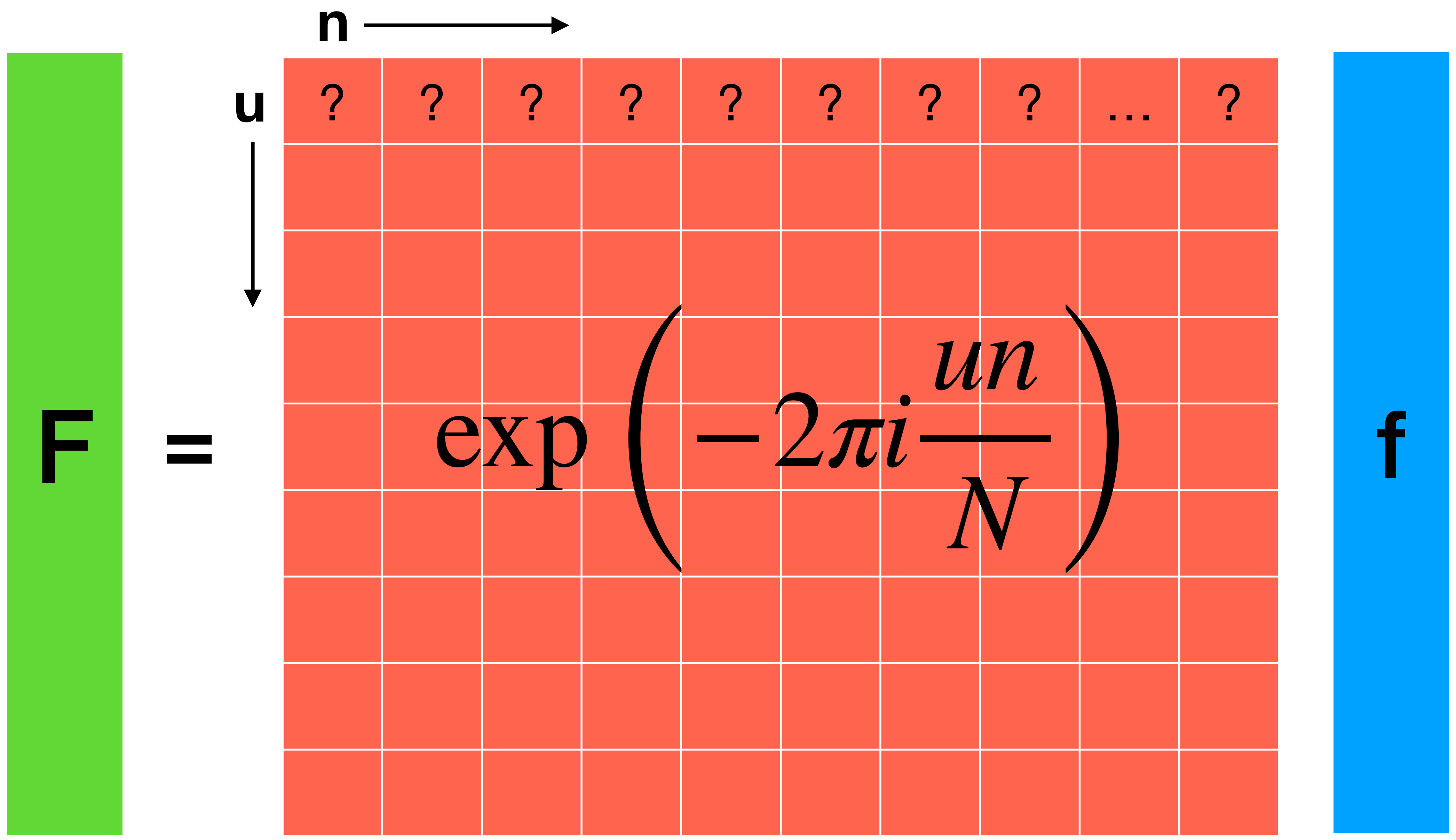


1D Fourier Transform

Discrete Fourier Transform (DFT) transforms a signal *f[n]* into *F[u]* as:

$$F[u] = \sum_{n=0}^{N-1} f[n] \exp\left(-2\pi i \frac{un}{N}\right)$$

Discrete Fourier Transform (DFT) is a linear operator. In matrix form:



U_F : NxN matrix

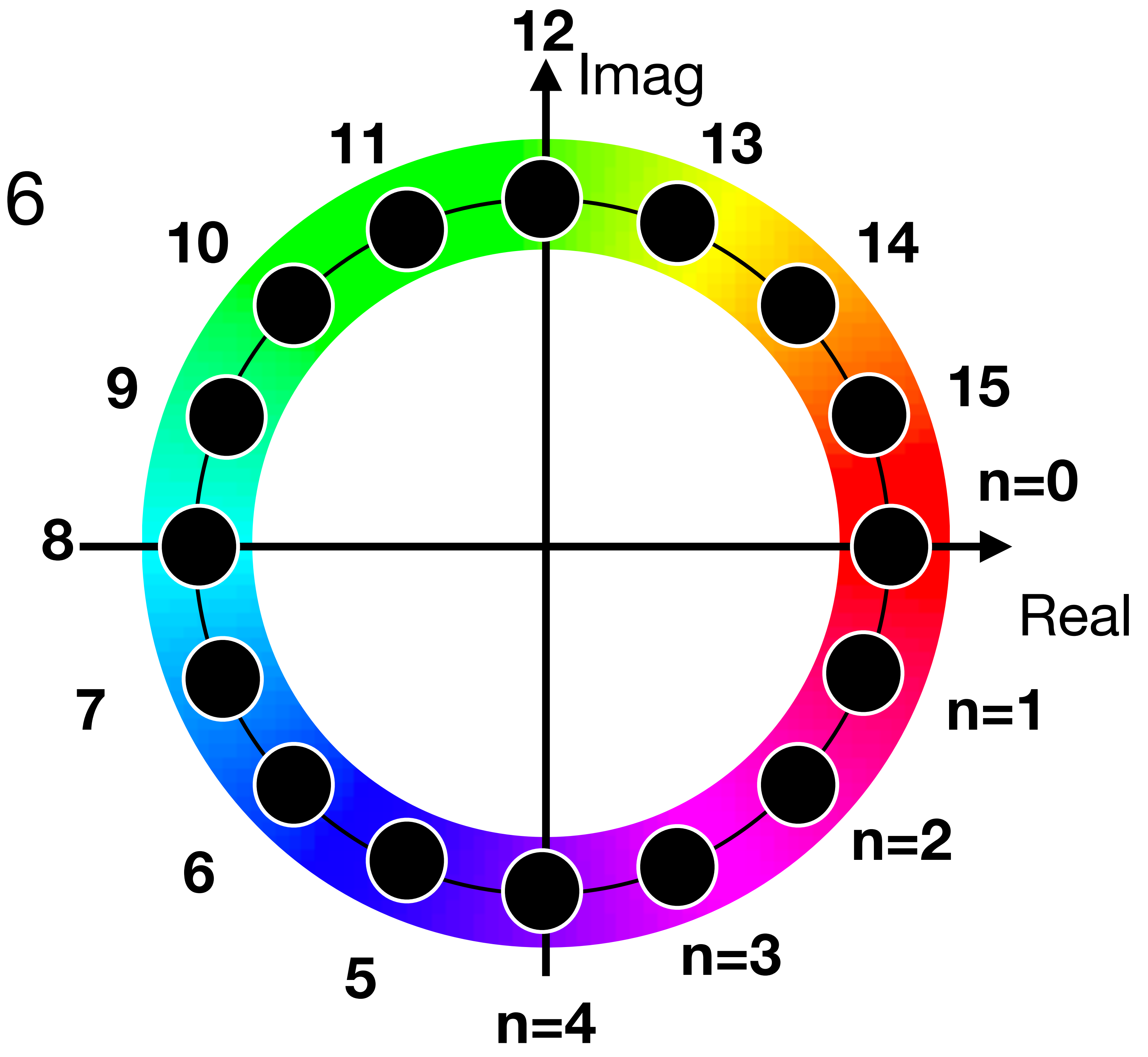
Source: Torralba, Freeman, Isola

Visualizing the Fourier transform

$$F[u] = \sum_{n=0}^{N-1} f[n] \exp\left(-2\pi i \frac{un}{N}\right)$$

$$\cos\left(2\pi \frac{un}{N}\right) - i \;\sin\left(2\pi \frac{un}{N}\right)$$

For:
u=1
N=16

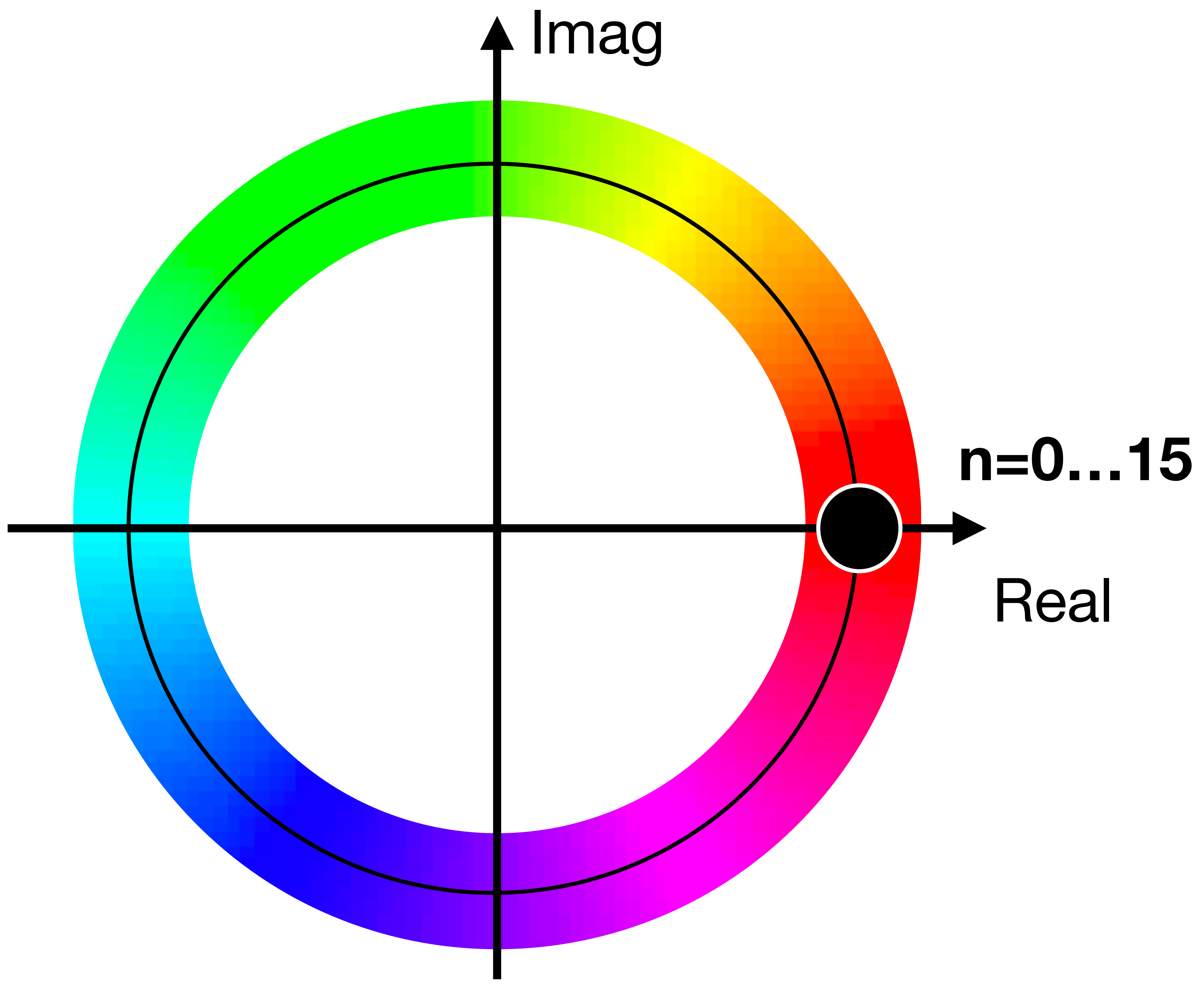


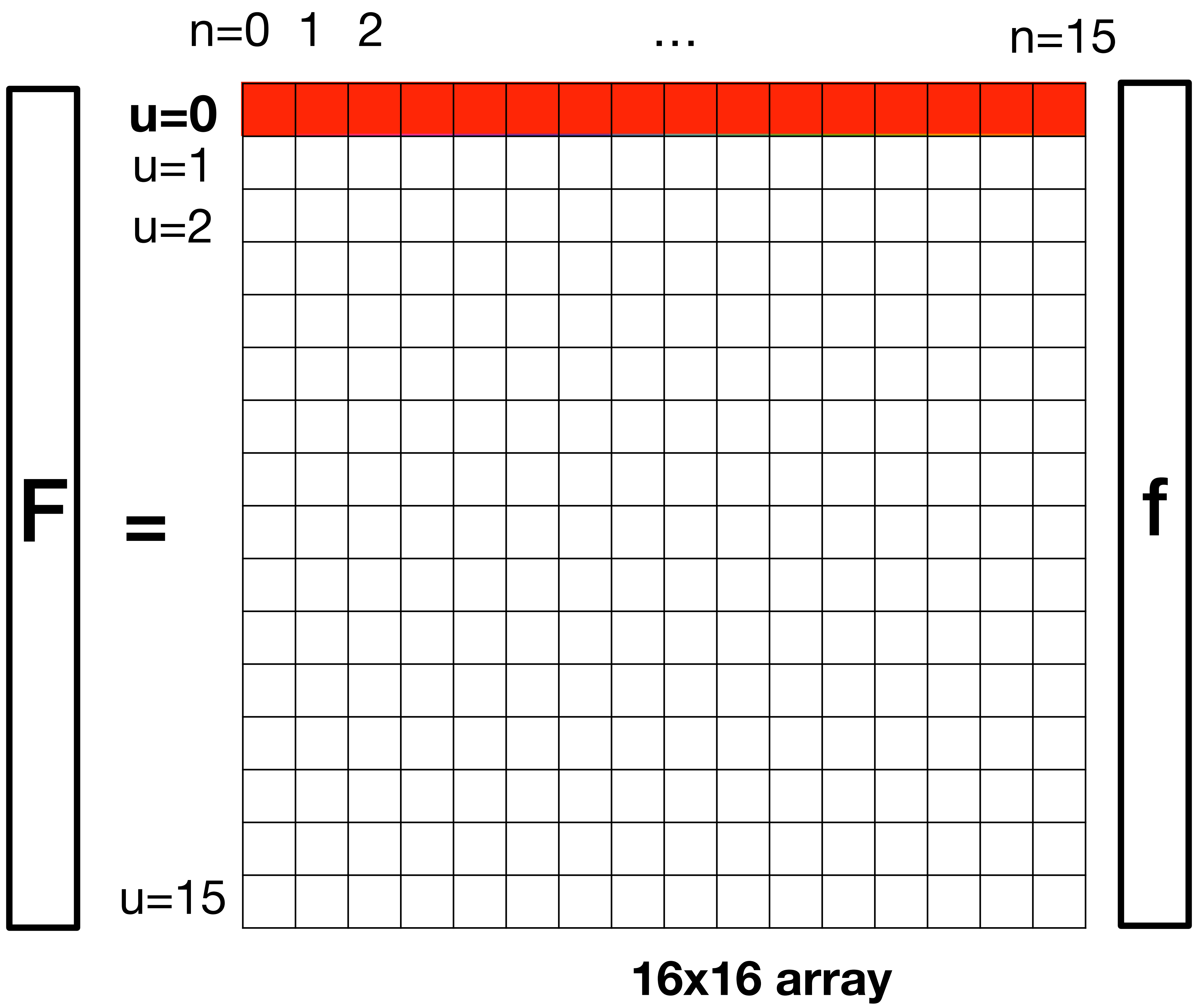
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



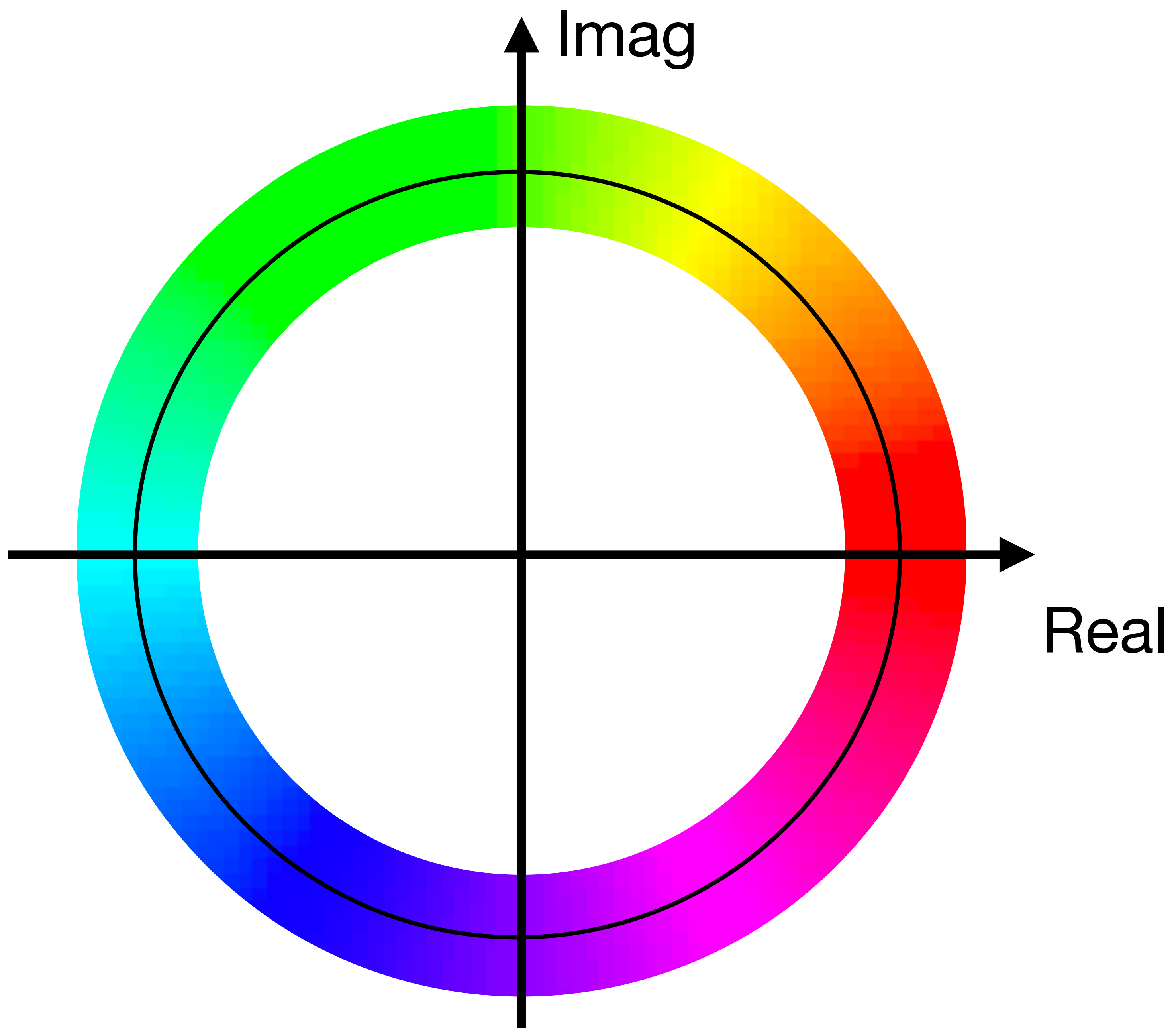


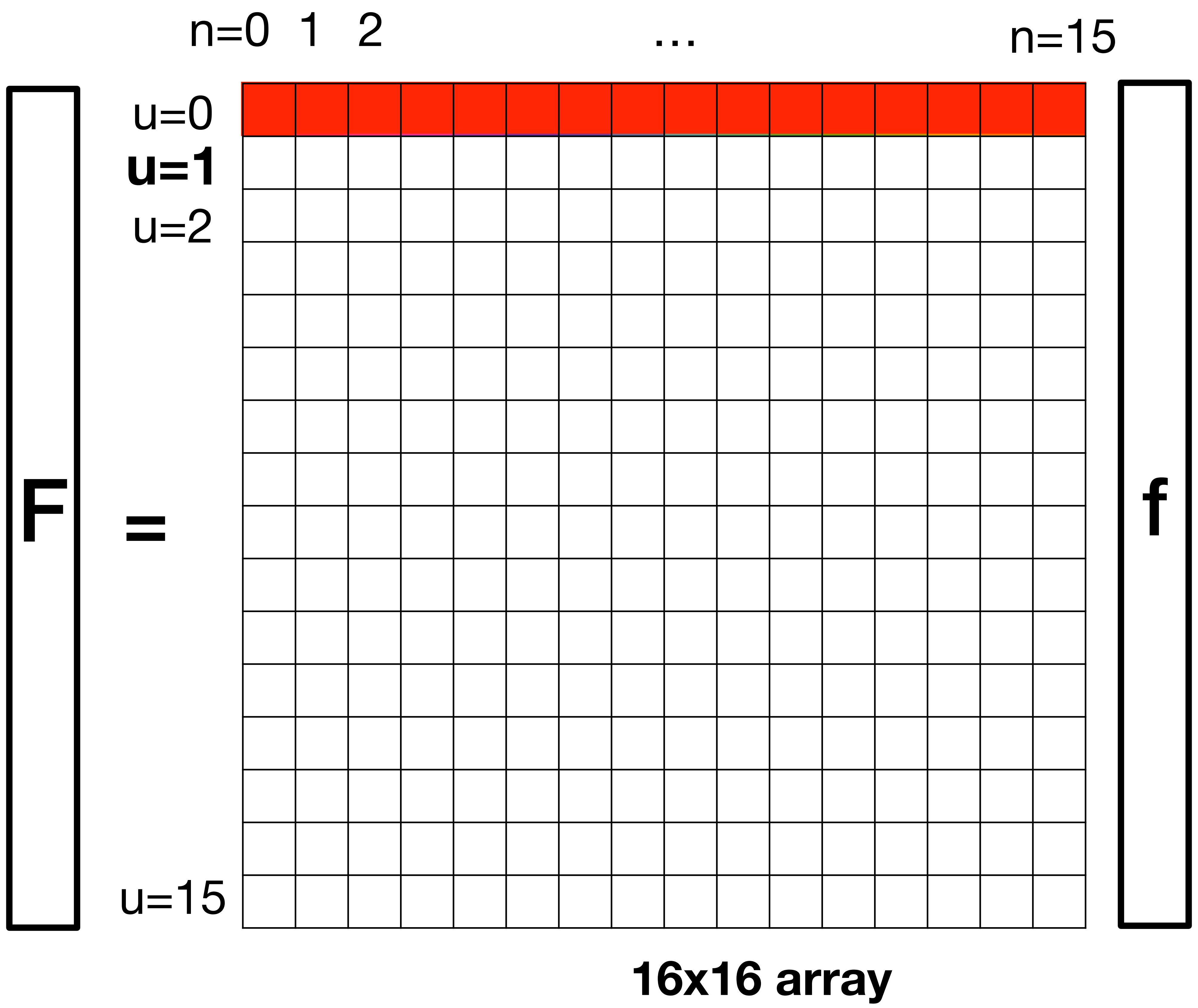
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



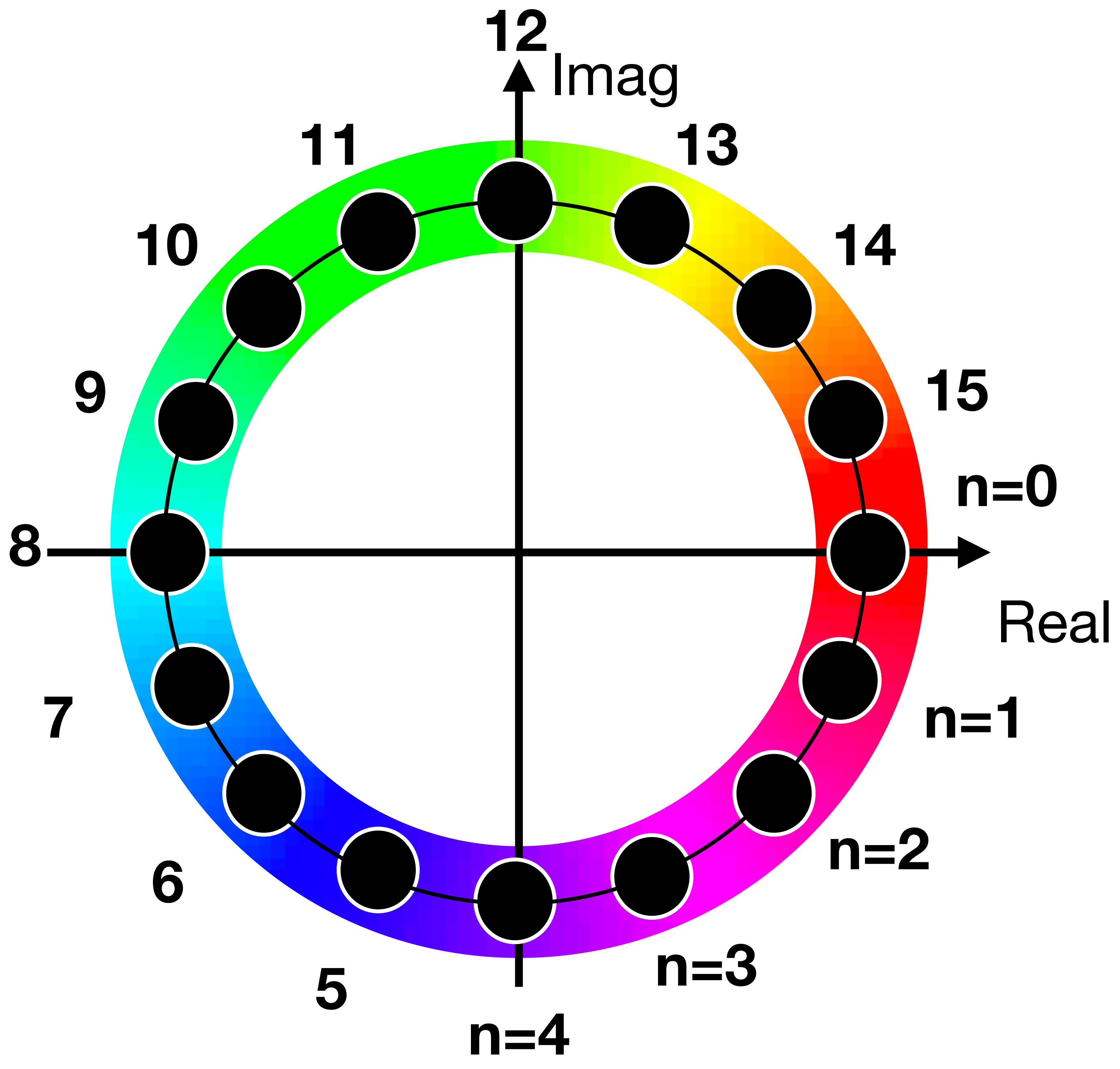


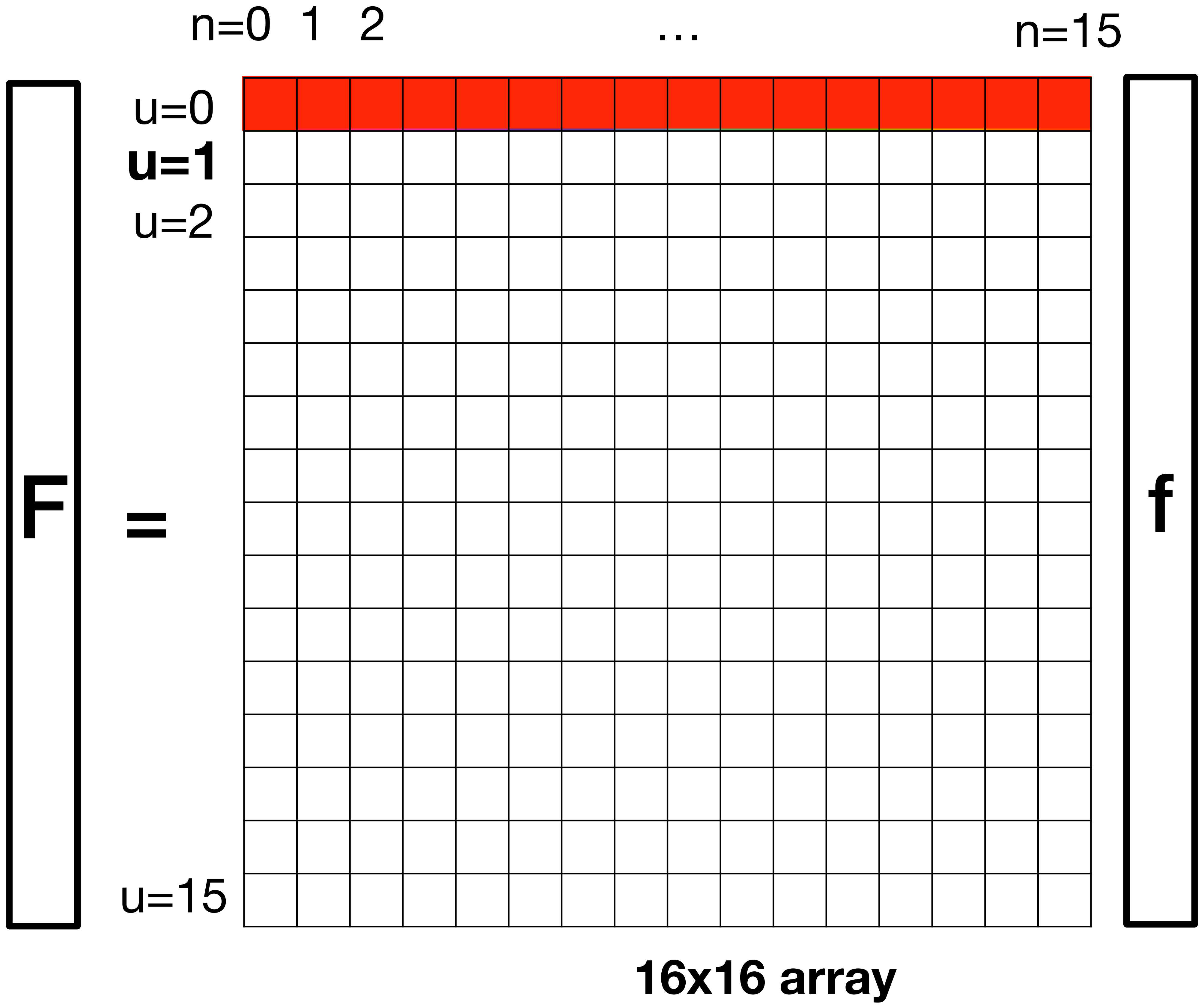
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



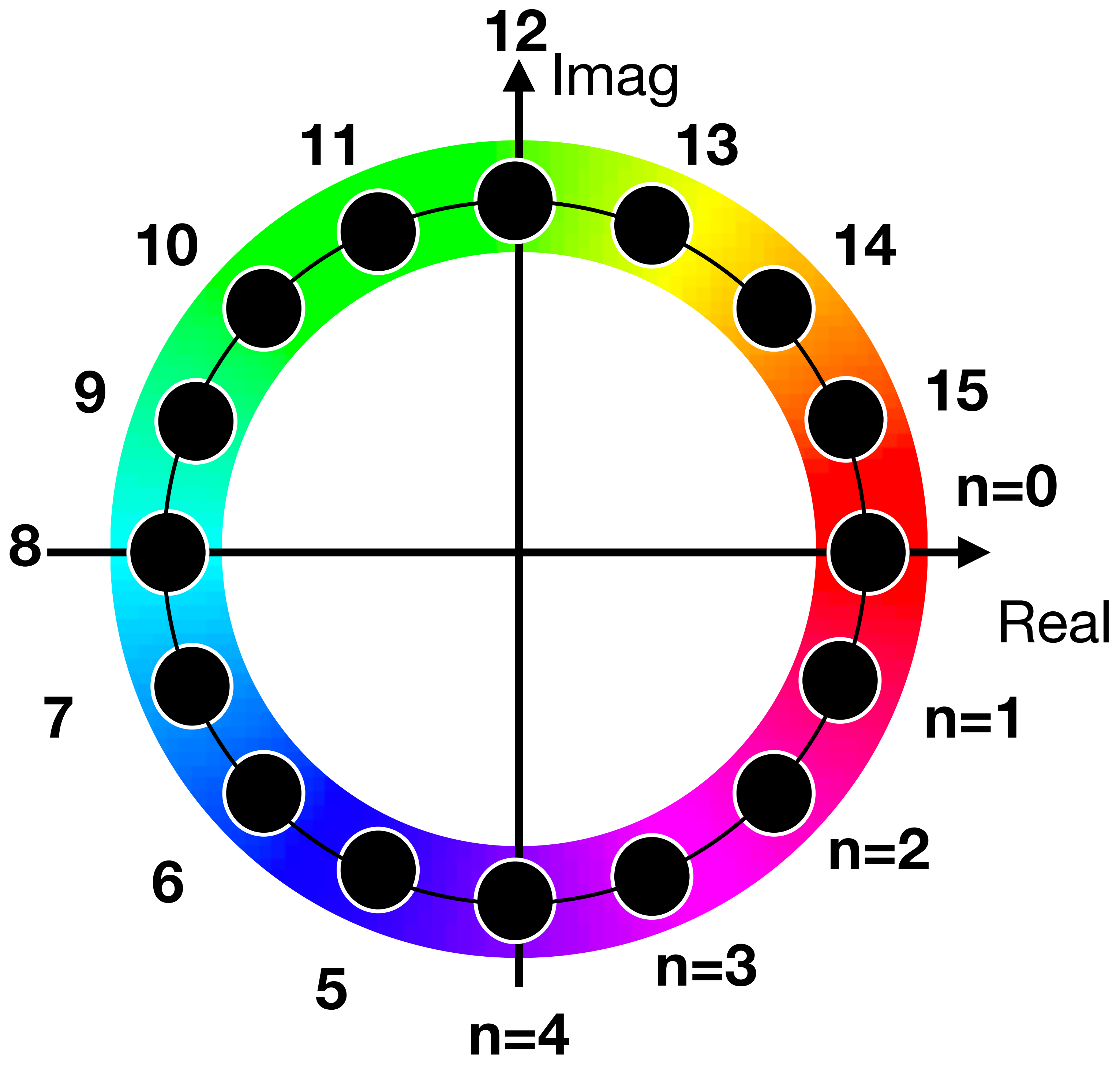


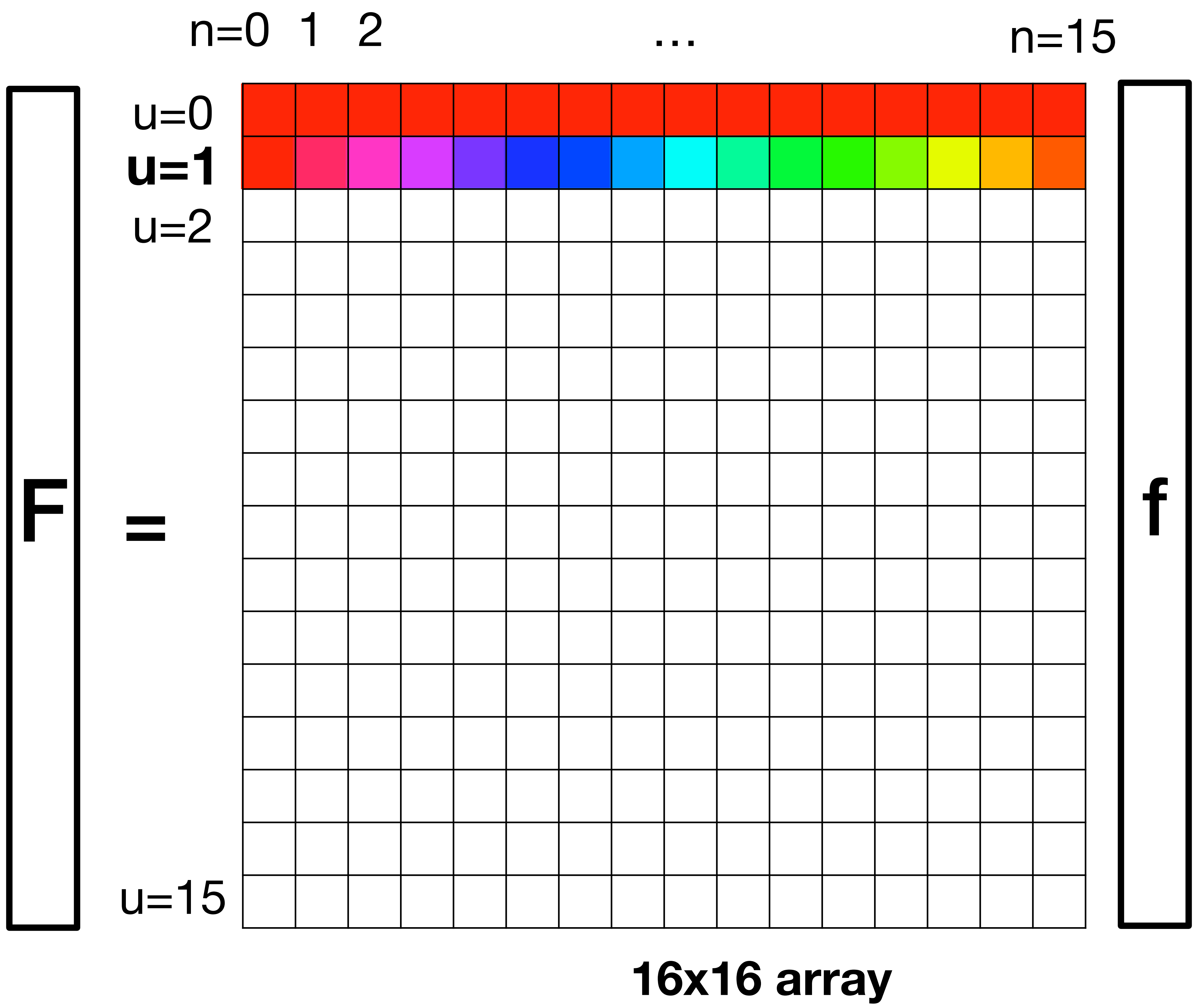
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



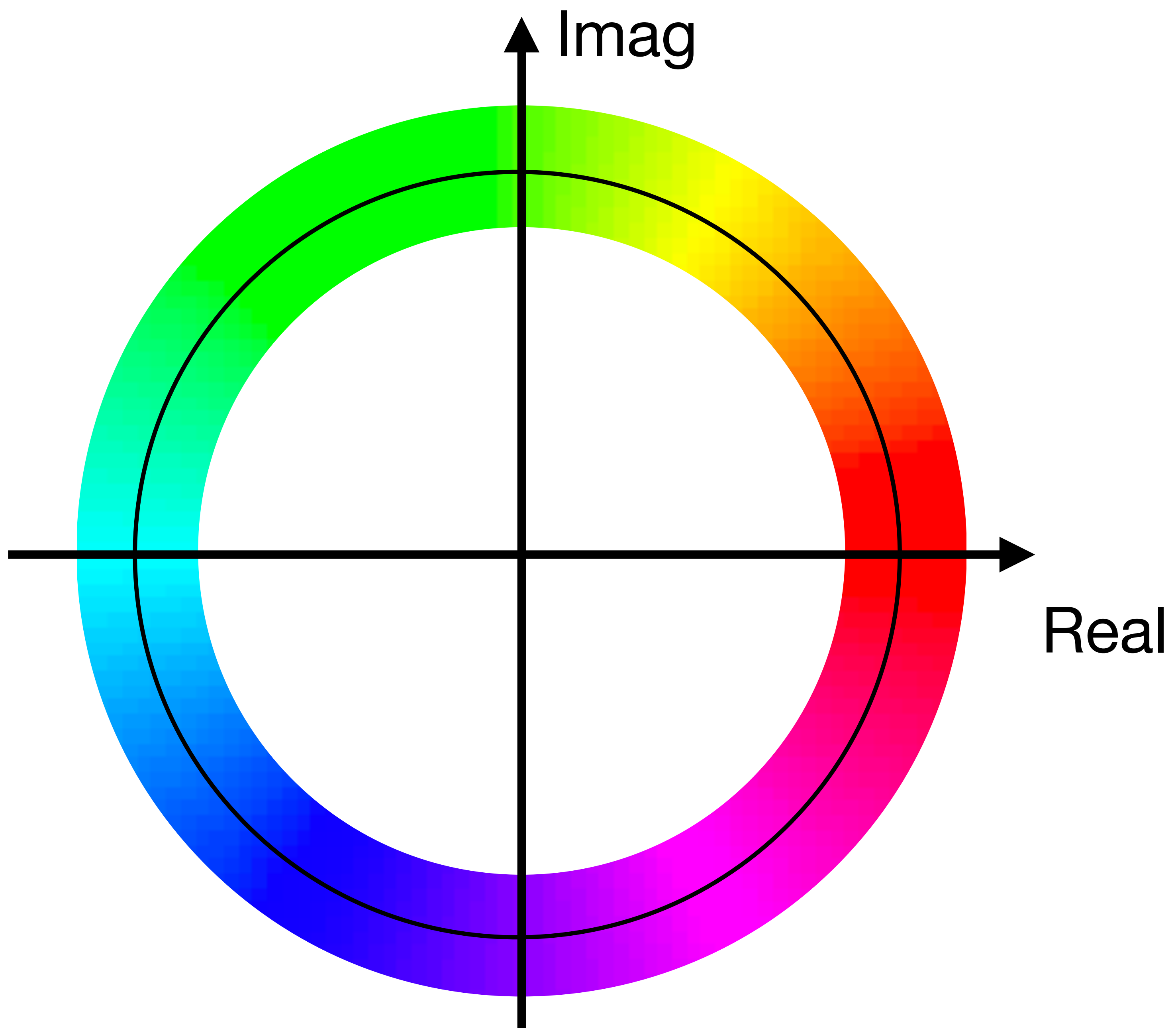


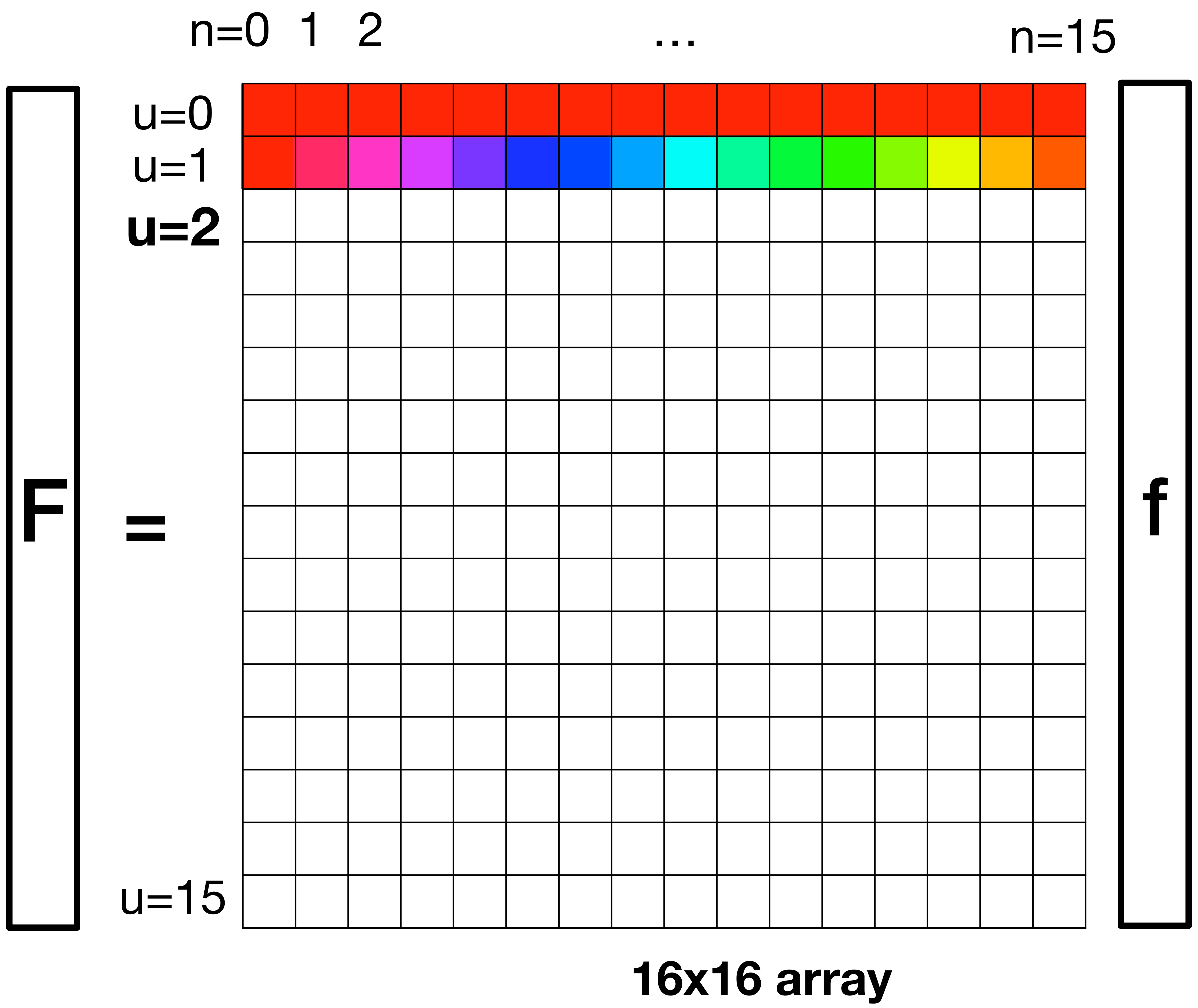
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



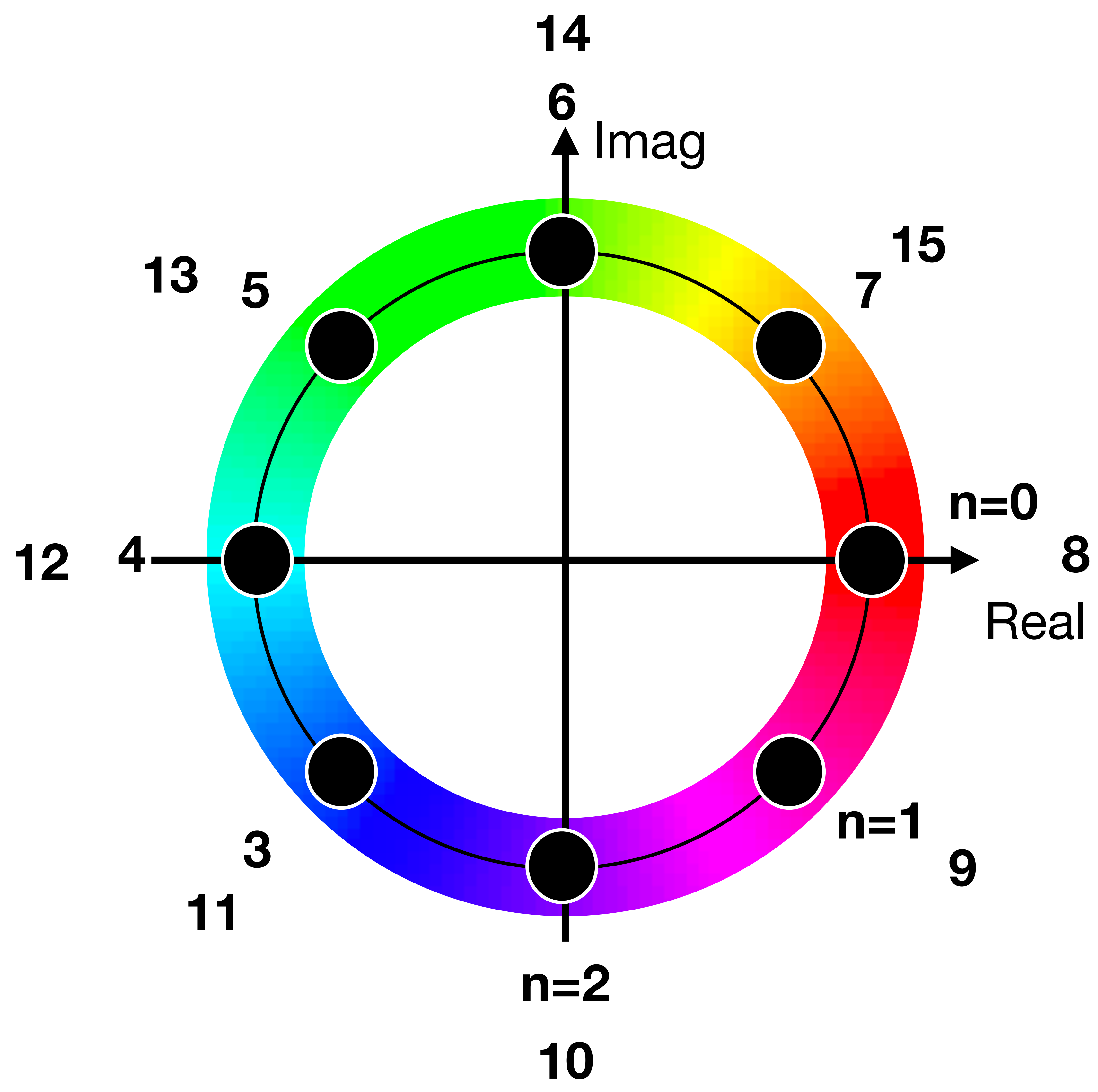


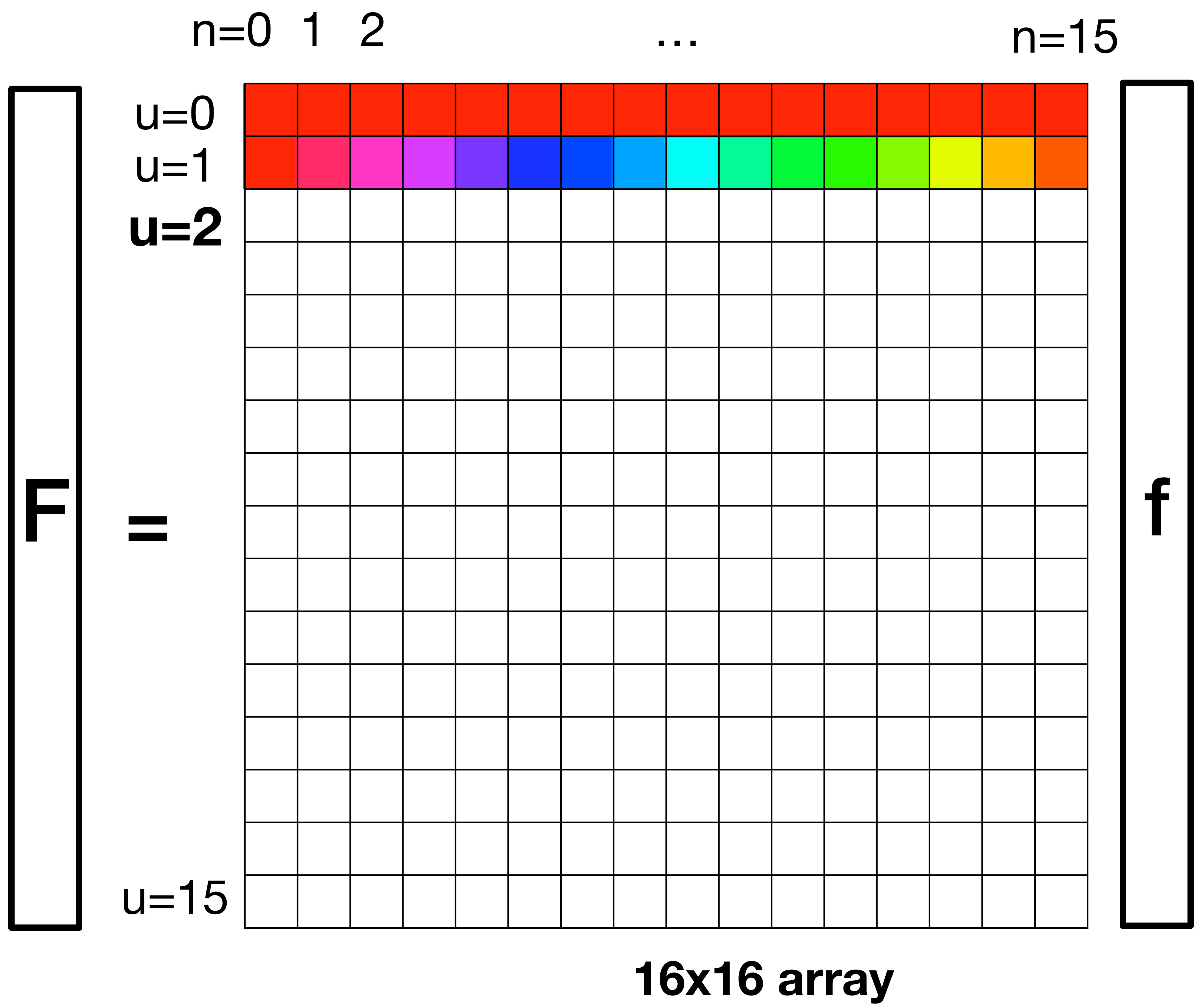
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16



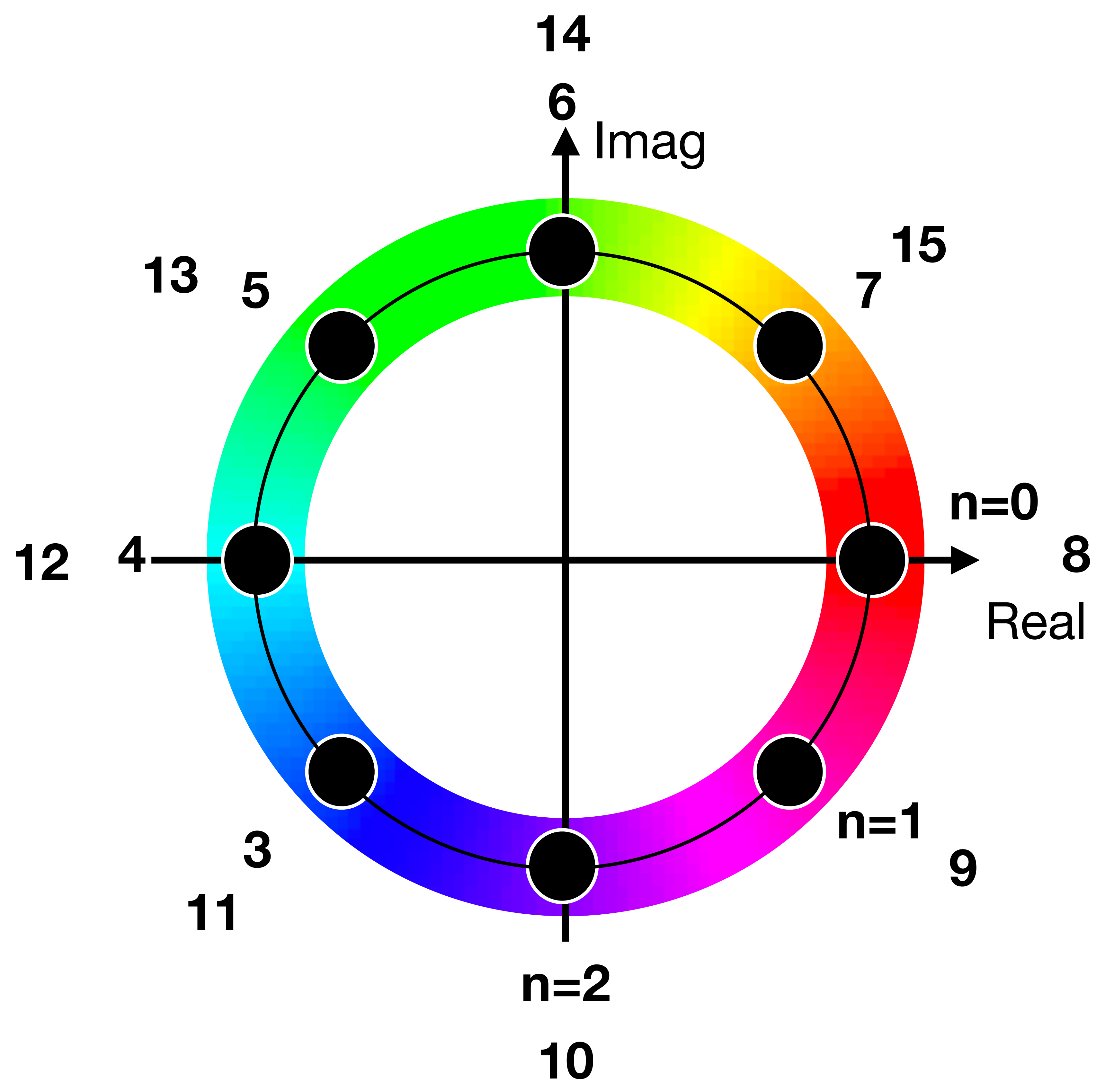


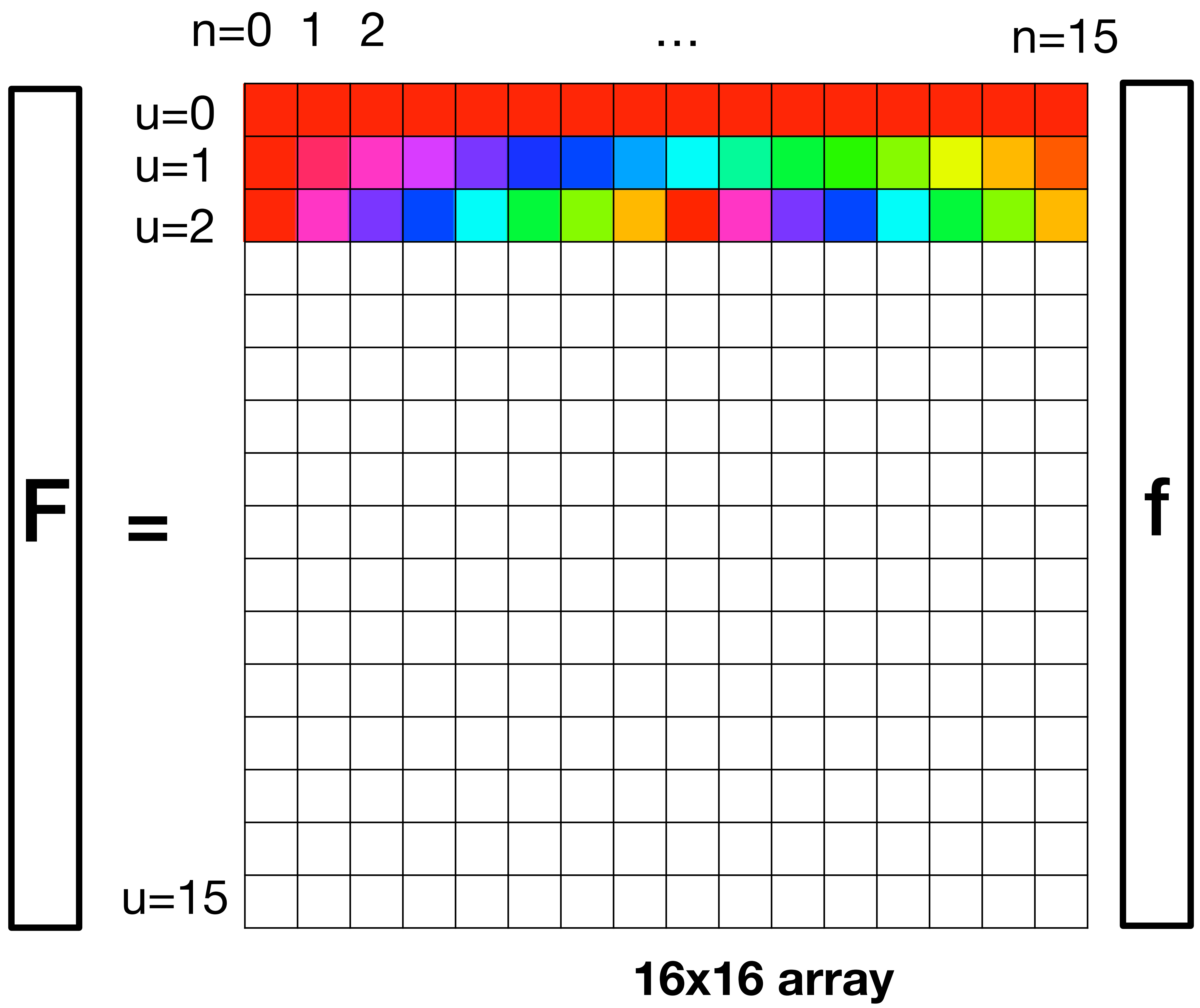
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i\frac{un}{N}\right)$$

For N=16



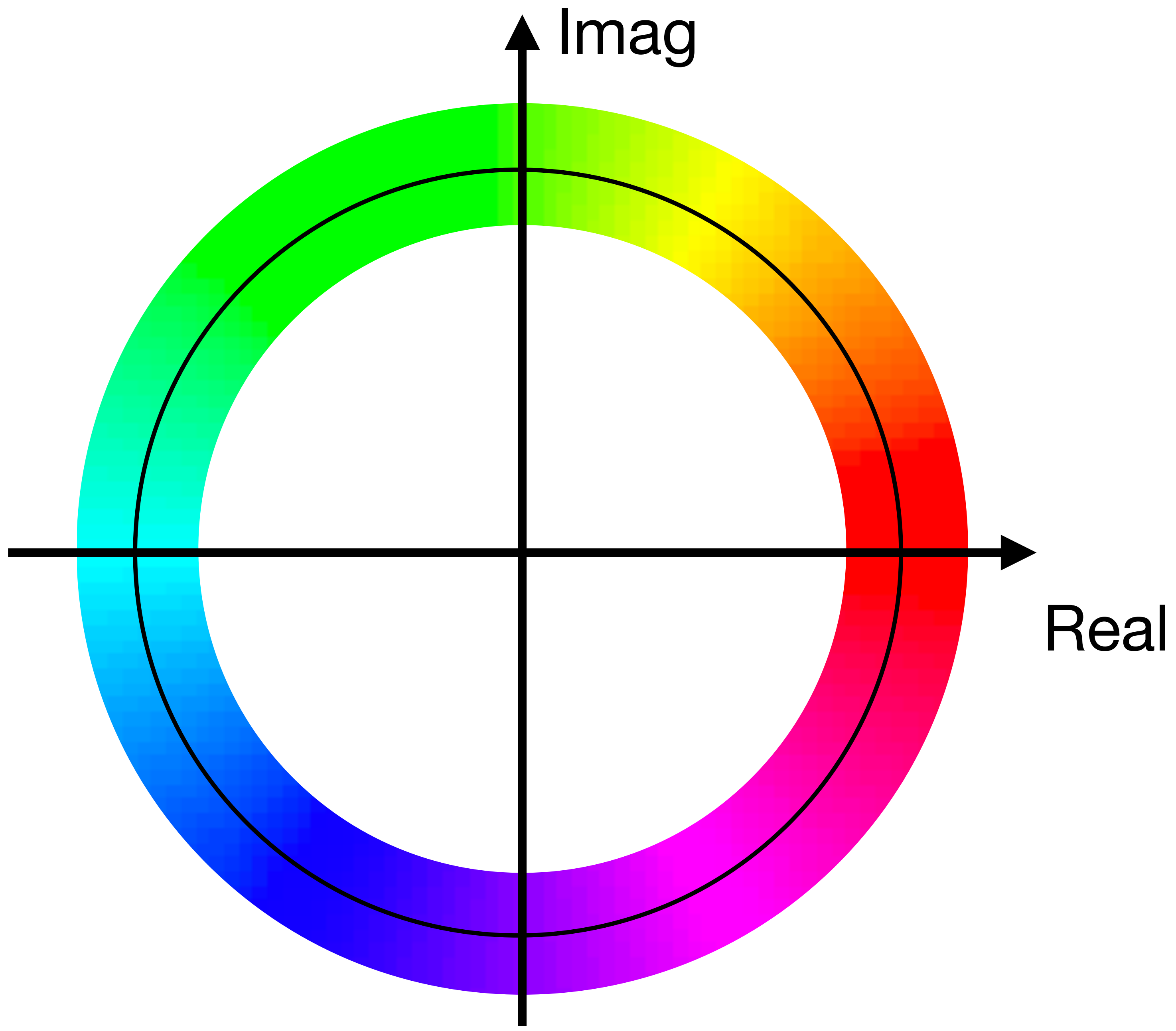


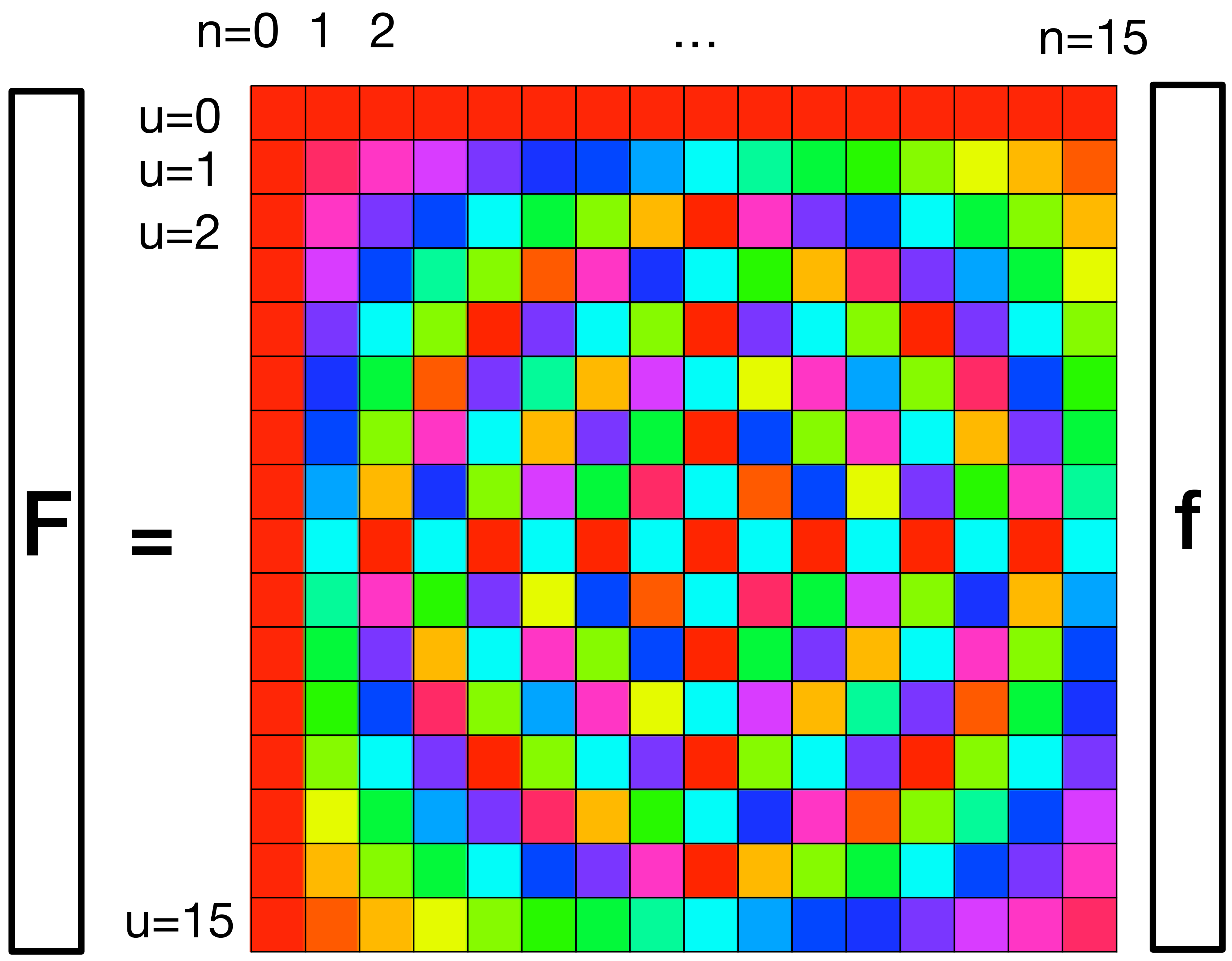
Source: Torralba, Freeman, Isola

Visualizing the transform coefficients

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16





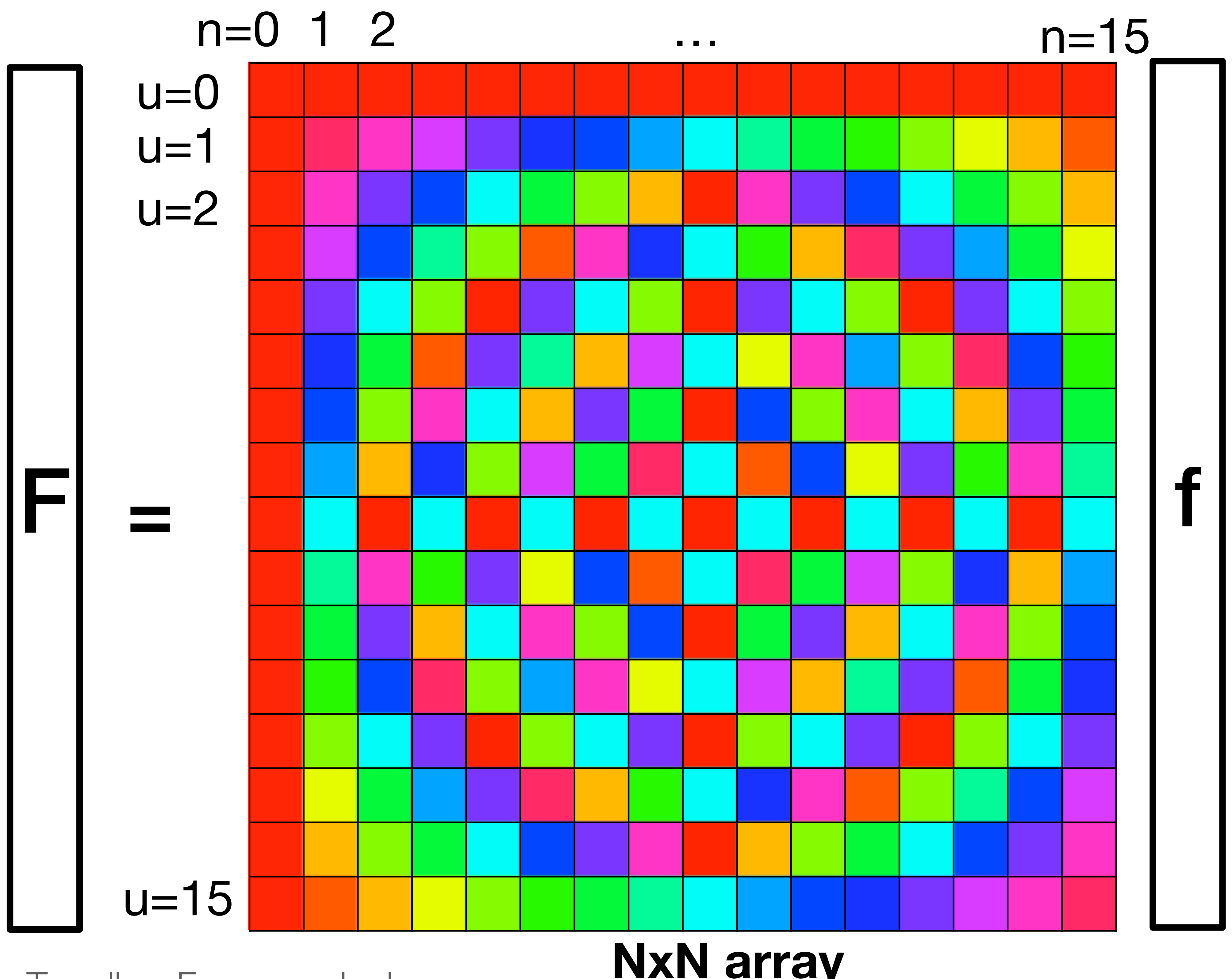
16x16 array

Source: Torralba, Freeman, Isola

The inverse of the Discrete Fourier transform

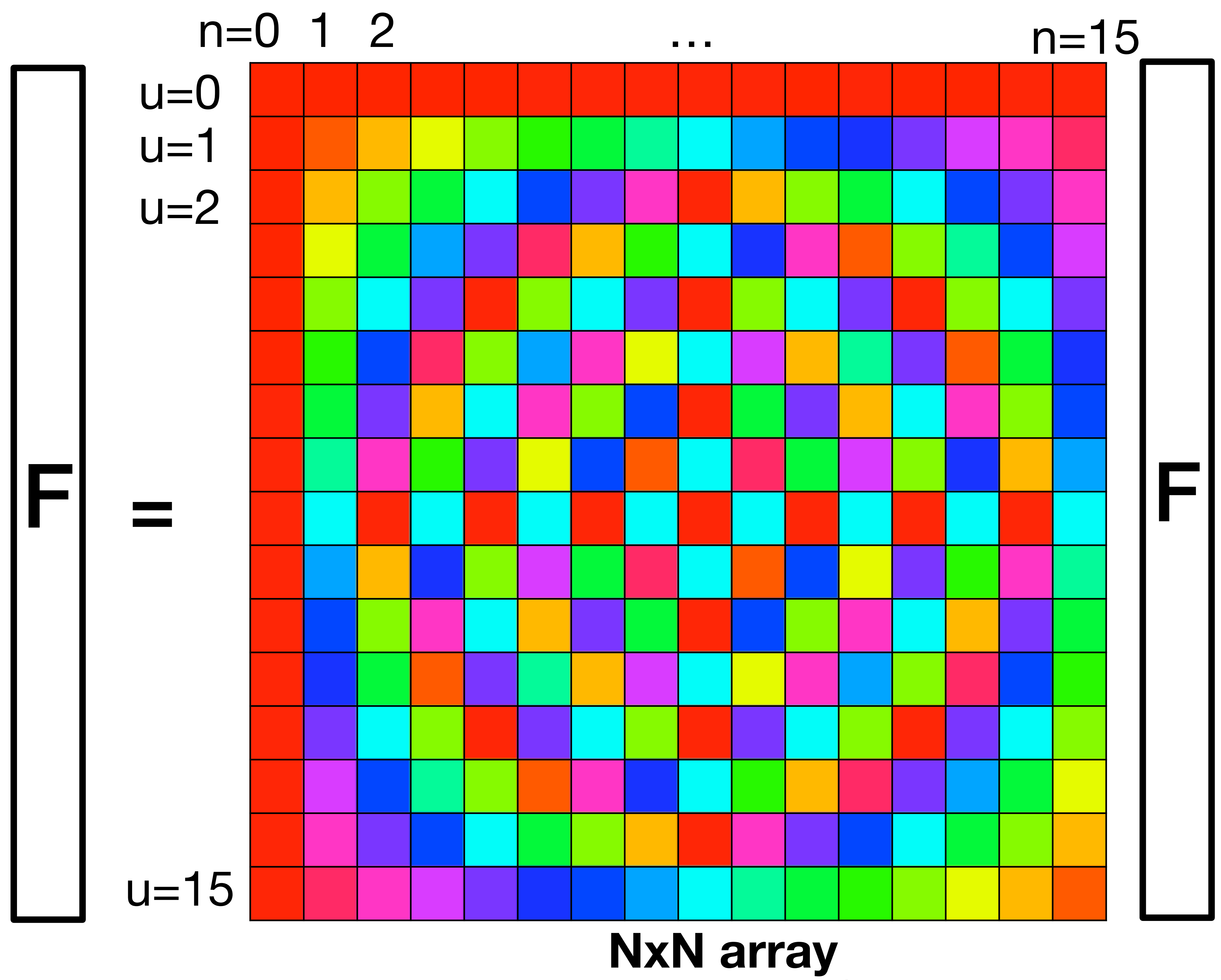
Discrete Fourier Transform (DFT):

$$F[u] = \sum_{n=0}^{N-1} f[n] \exp\left(-2\pi i \frac{un}{N}\right)$$



Its inverse:

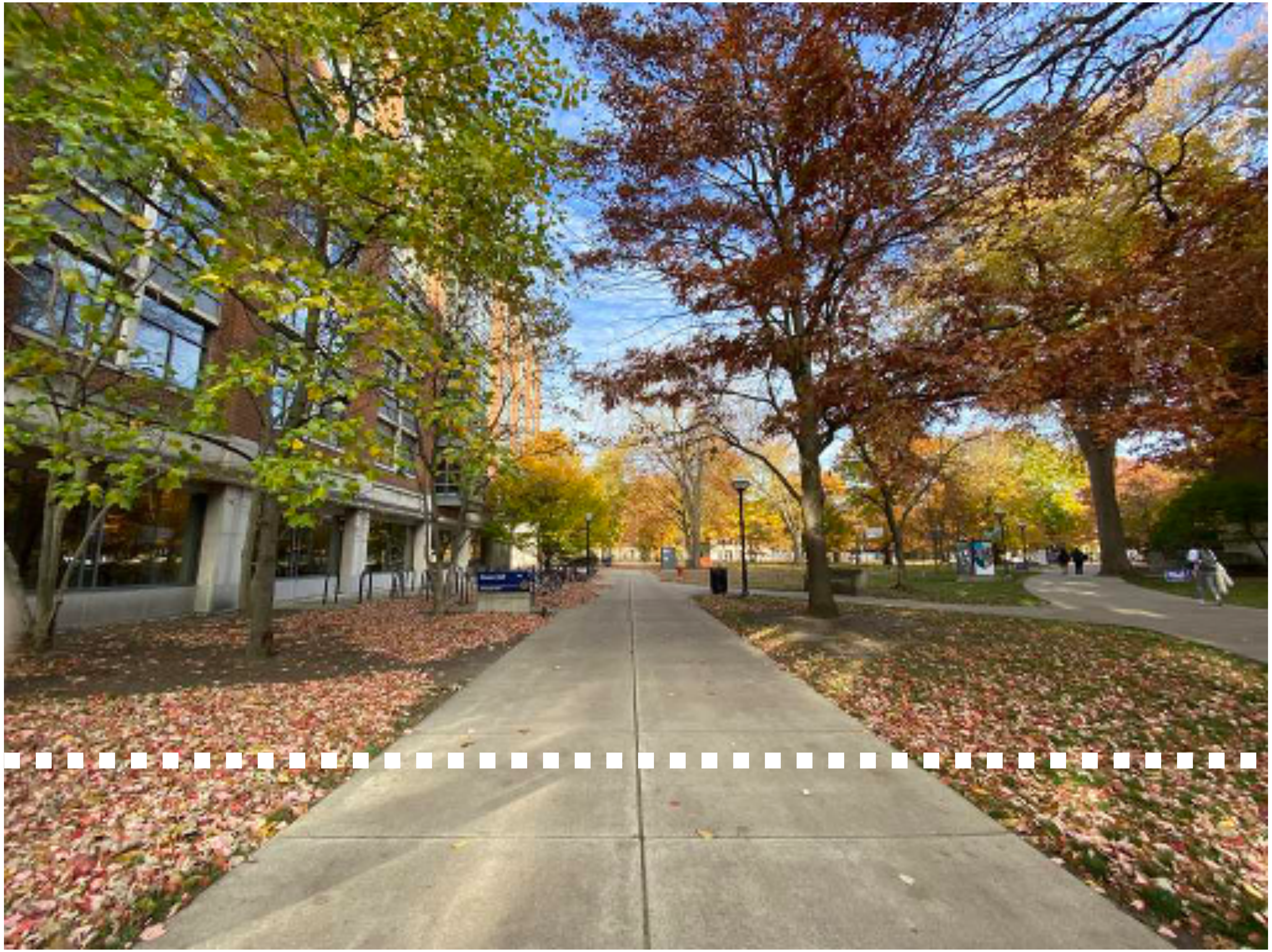
$$f[n] = \frac{1}{N} \sum_{u=0}^{N-1} F[u] \exp\left(2\pi i \frac{un}{N}\right)$$



Source: Torralba, Freeman, Isola

Source: Torralba, Freeman, Isola

The 1D Fourier transform and images



The 1D Fourier transform and images

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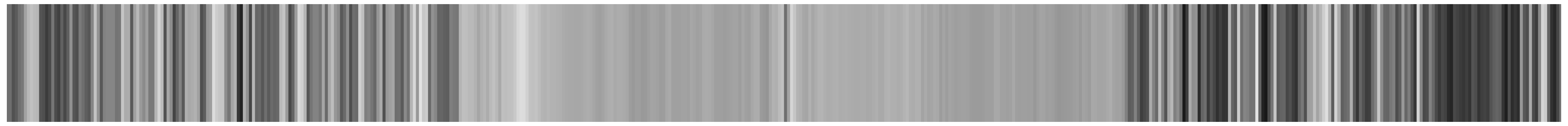
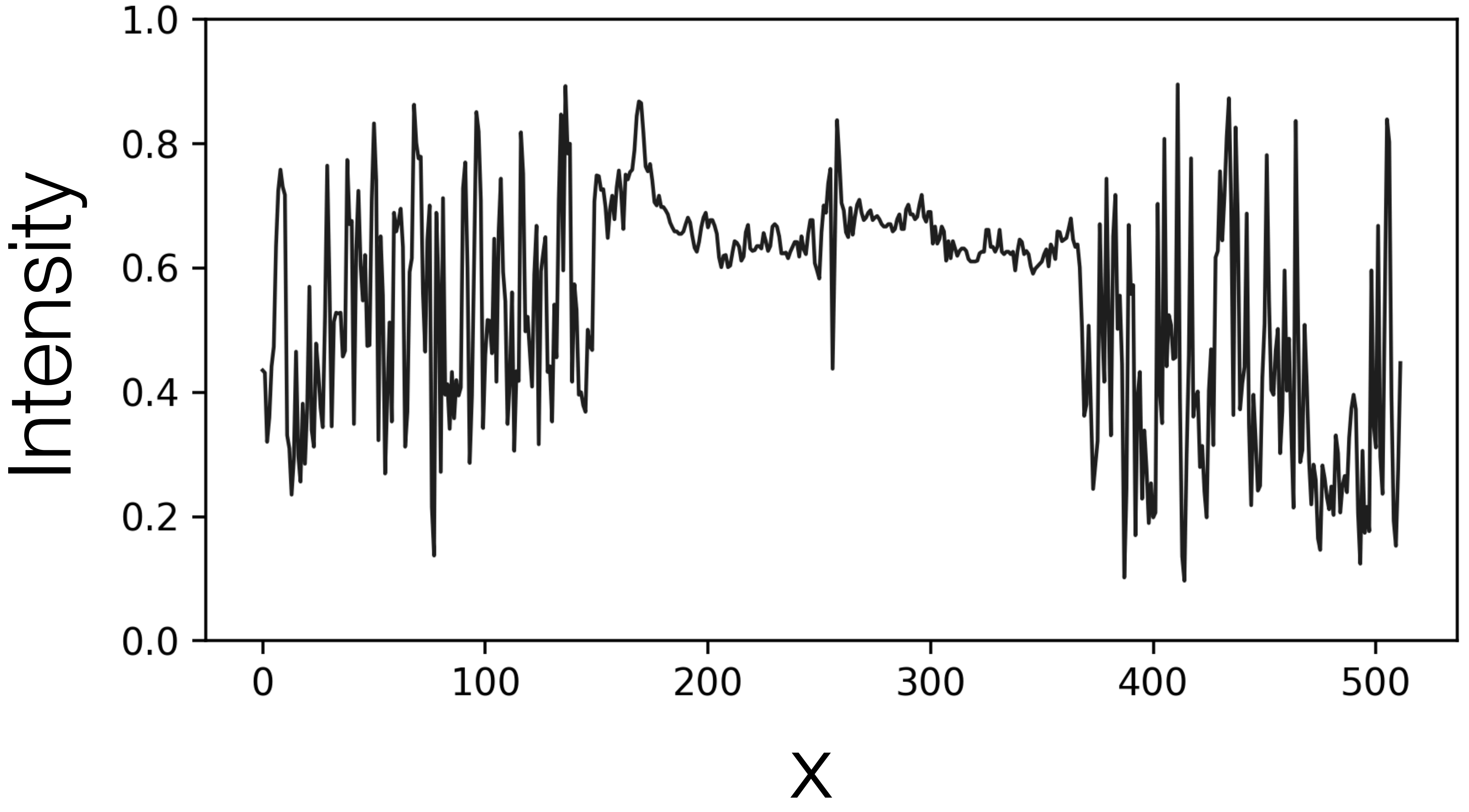
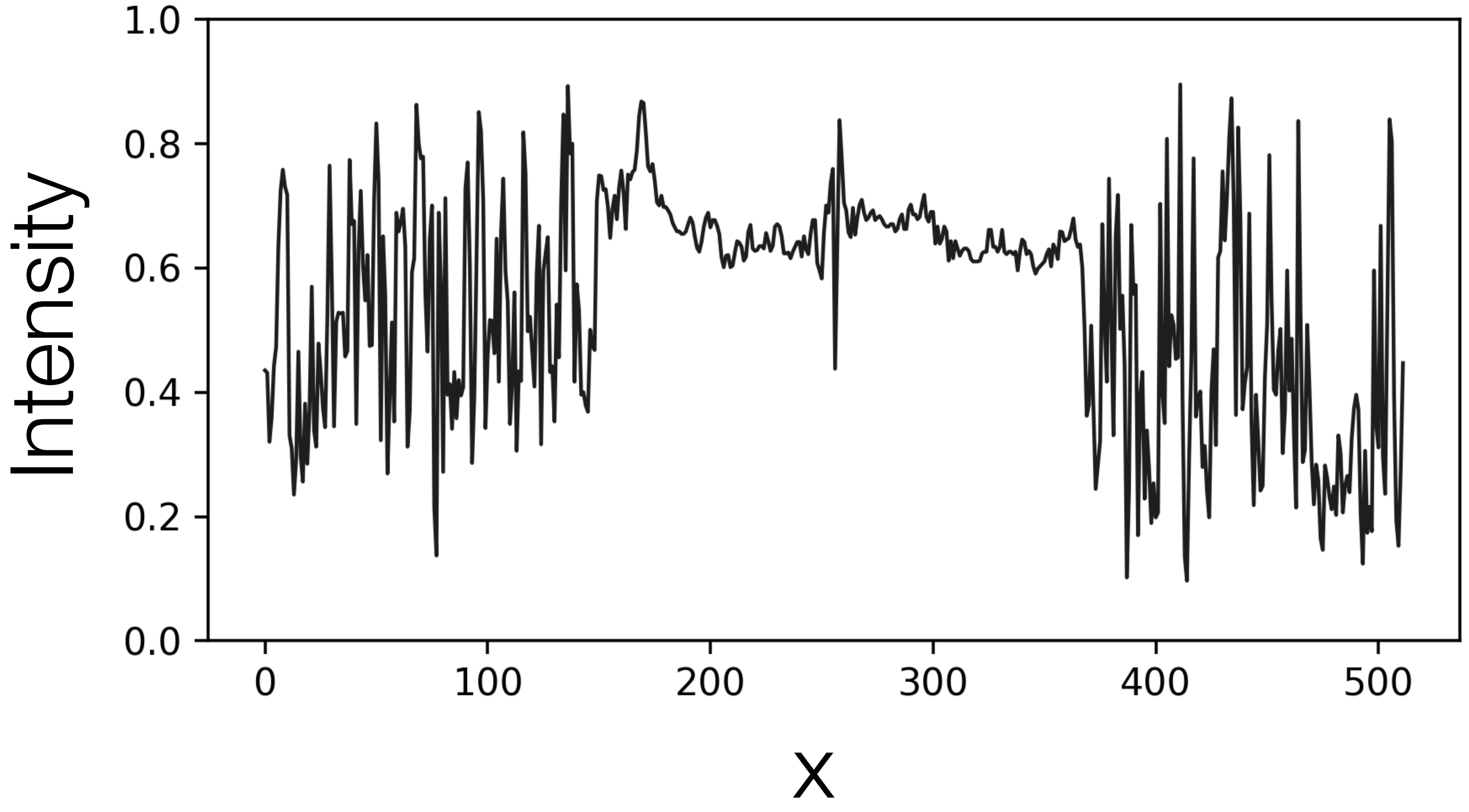


Image row

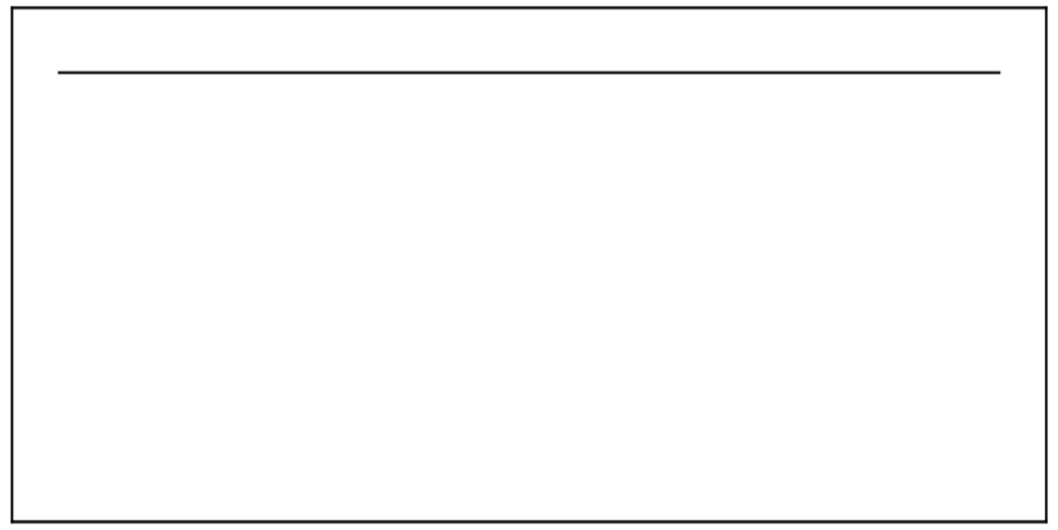


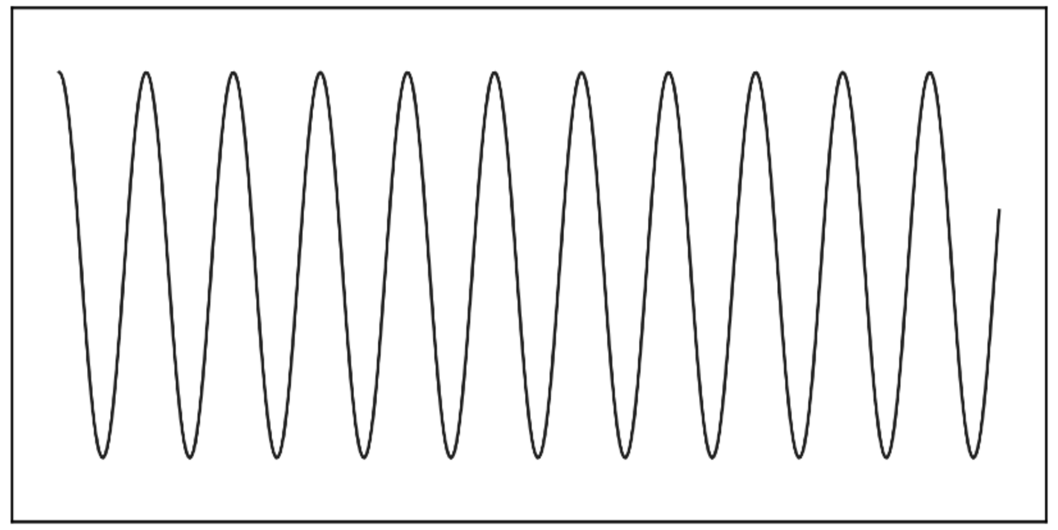
Represent this function in a Fourier basis.

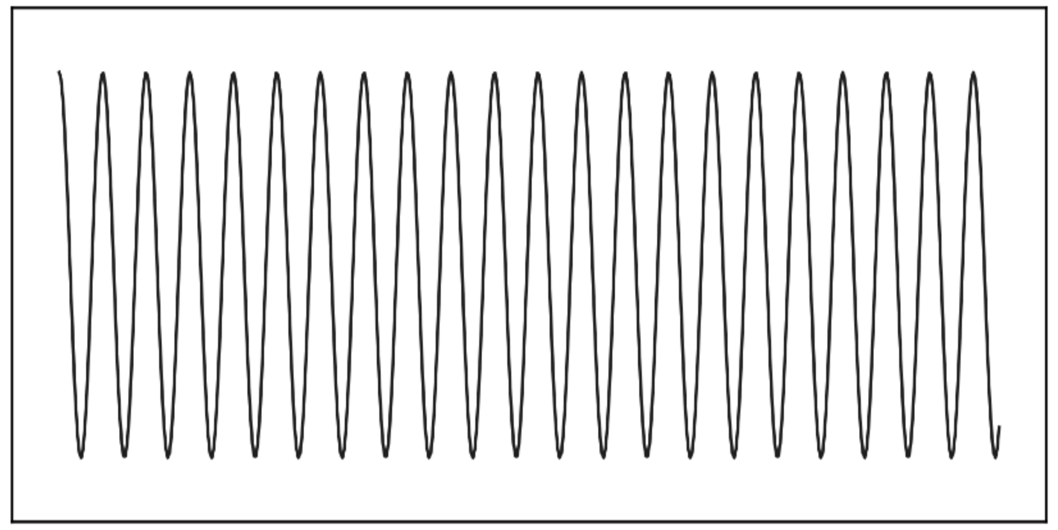
The 1D Fourier transform and images

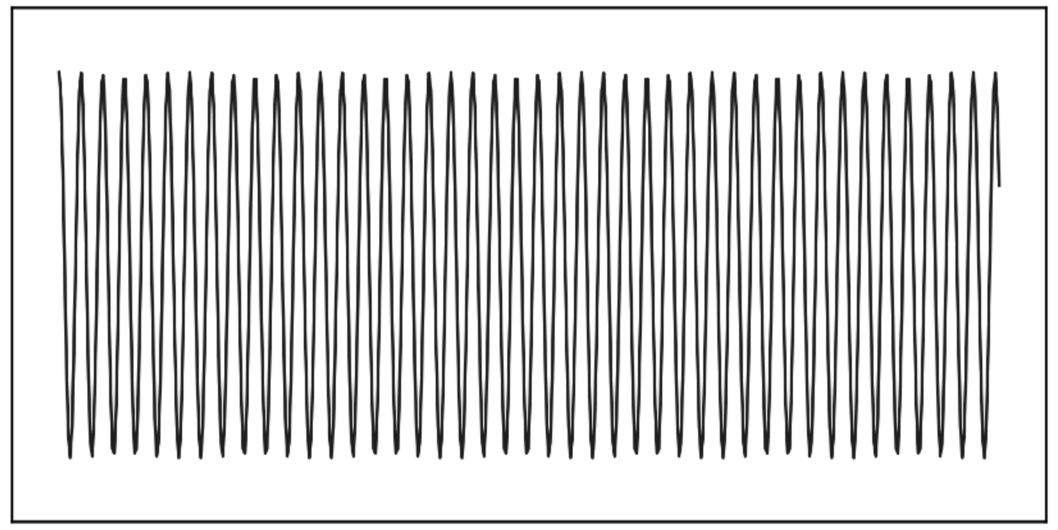


$=$

$F_1 \times$ 

$+ \ F_2 \times$ 

$+ \ F_3 \times$ 

$+ \ F_4 \times$ 

$\cdots$

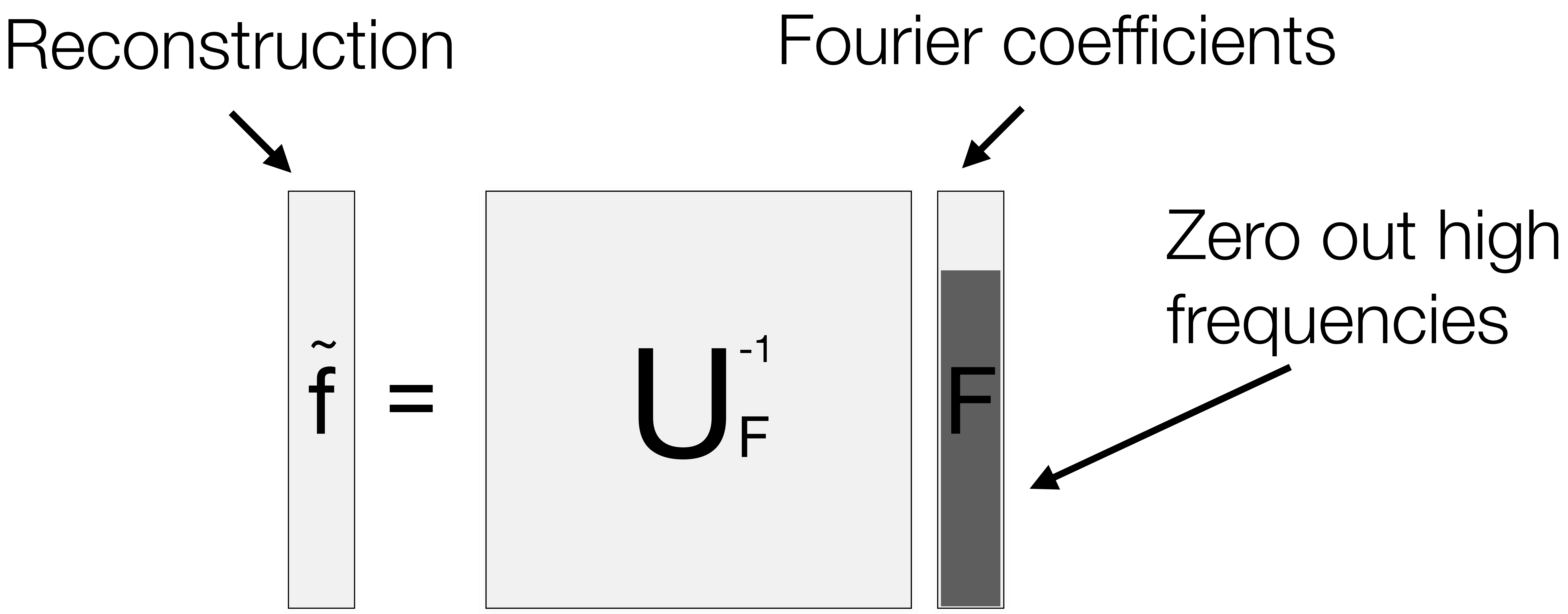
The 1D Fourier transform and images

Fourier coefficients

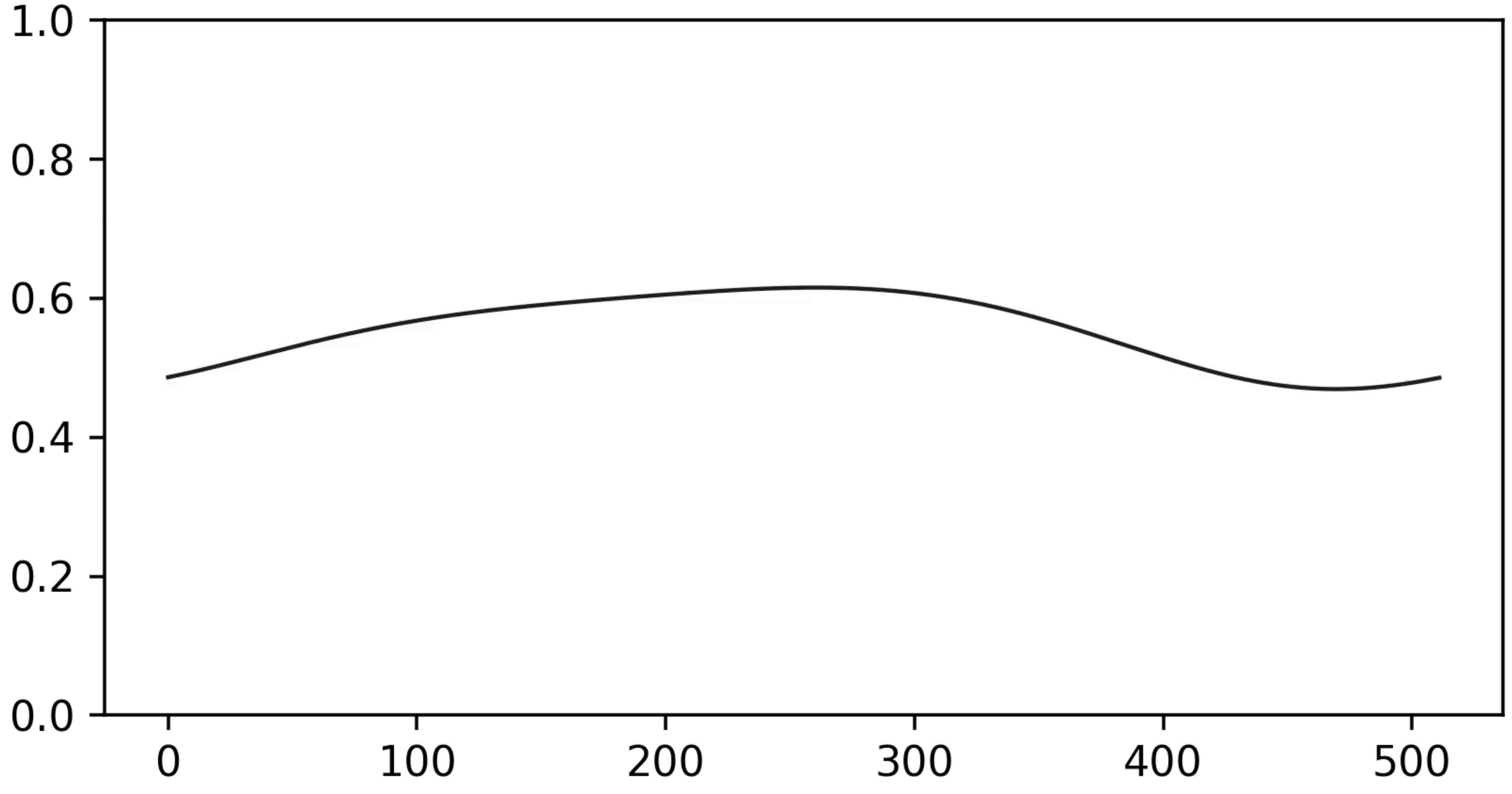
Row of image as vector

$$F = U_F f$$

The 1D Fourier transform and images



Reconstructing the image

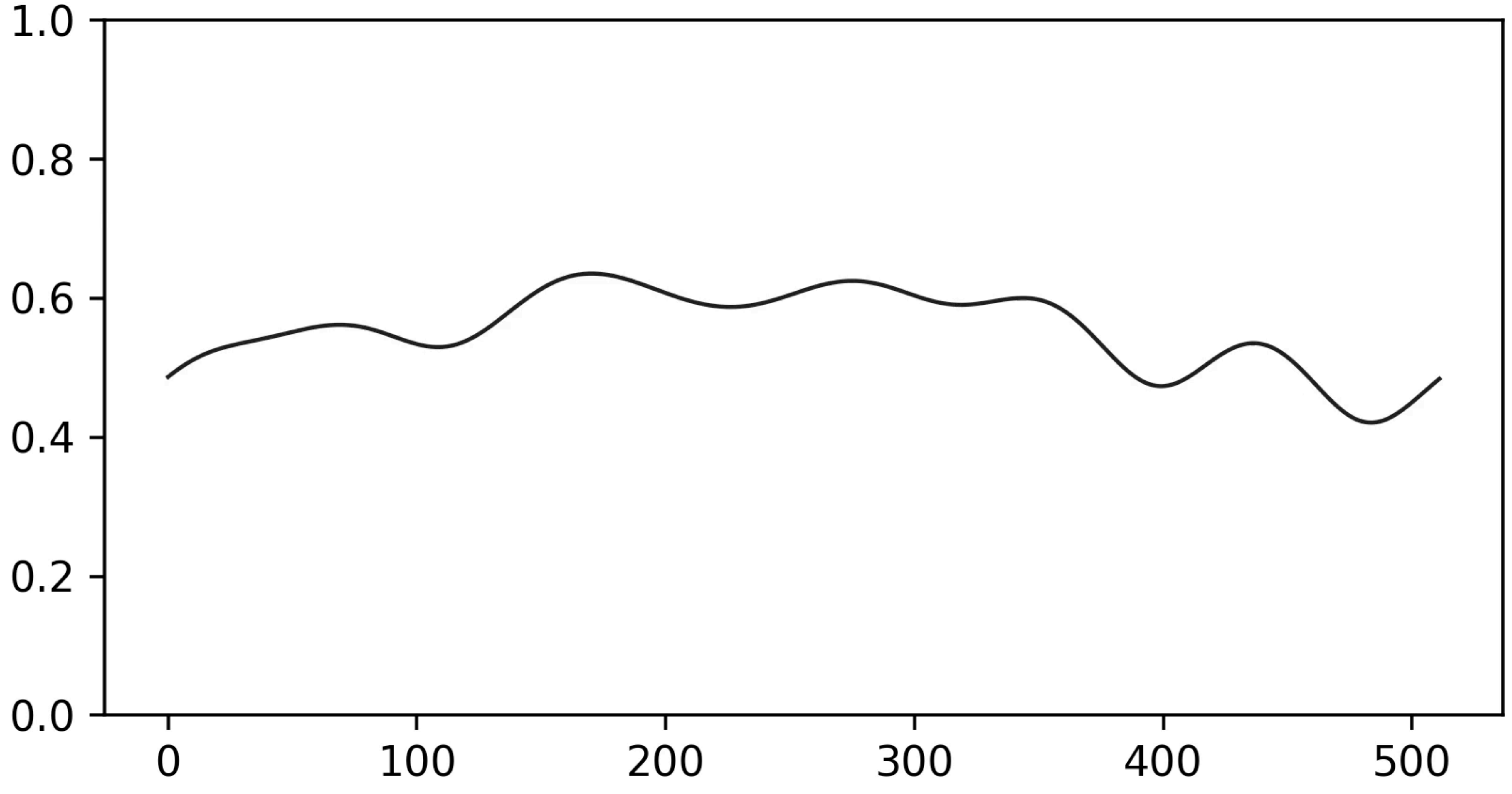


Reconstruction of one row

Reconstructed image

With coefficients from only the 3 lowest frequencies

Reconstructing the image

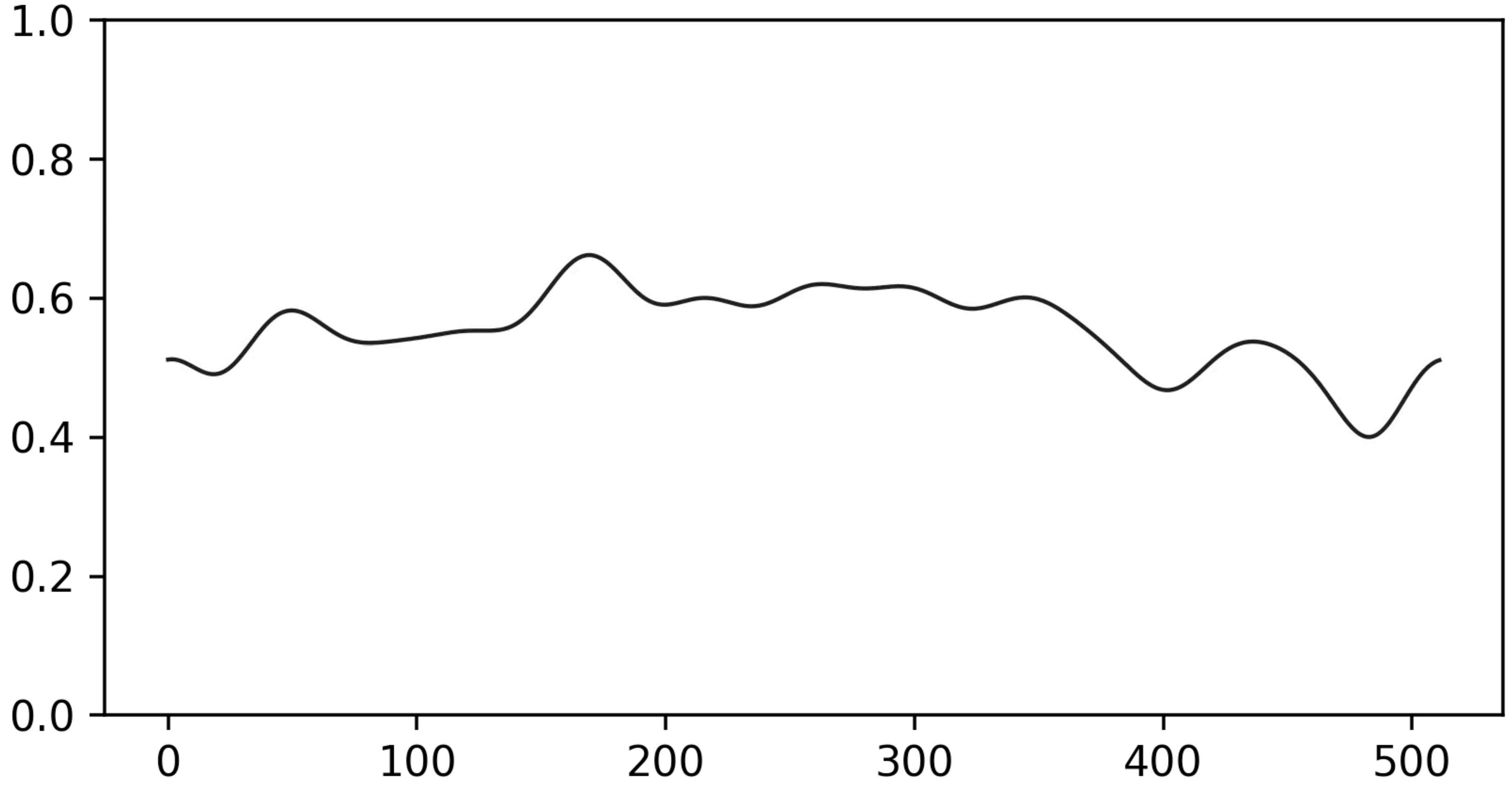


Reconstruction of one row

Reconstructed image

With coefficients from only the 8 lowest frequencies

Reconstructing the image

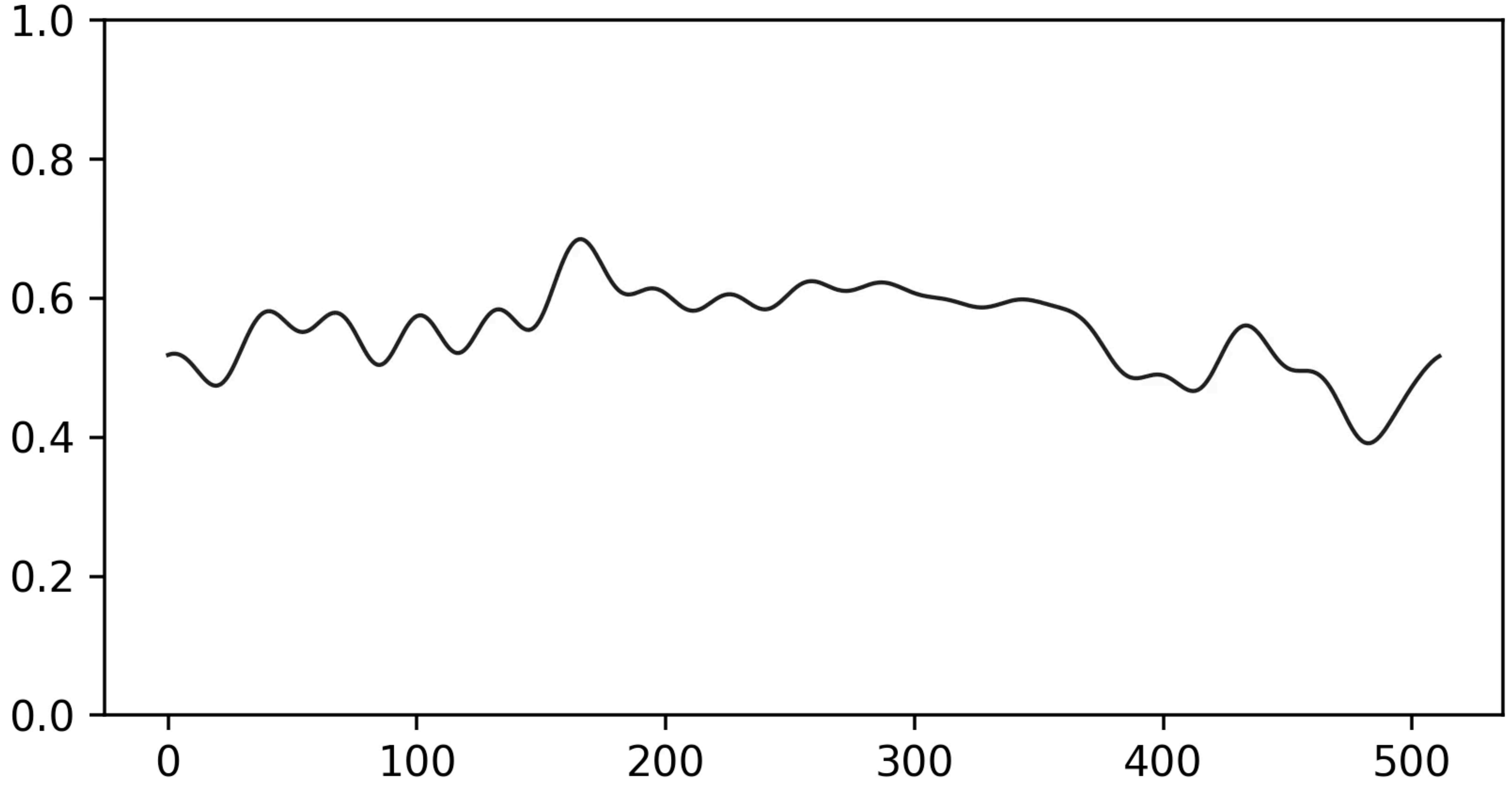


Reconstruction of one row

Reconstructed image

With coefficients from only the 13 lowest frequencies

Reconstructing the image

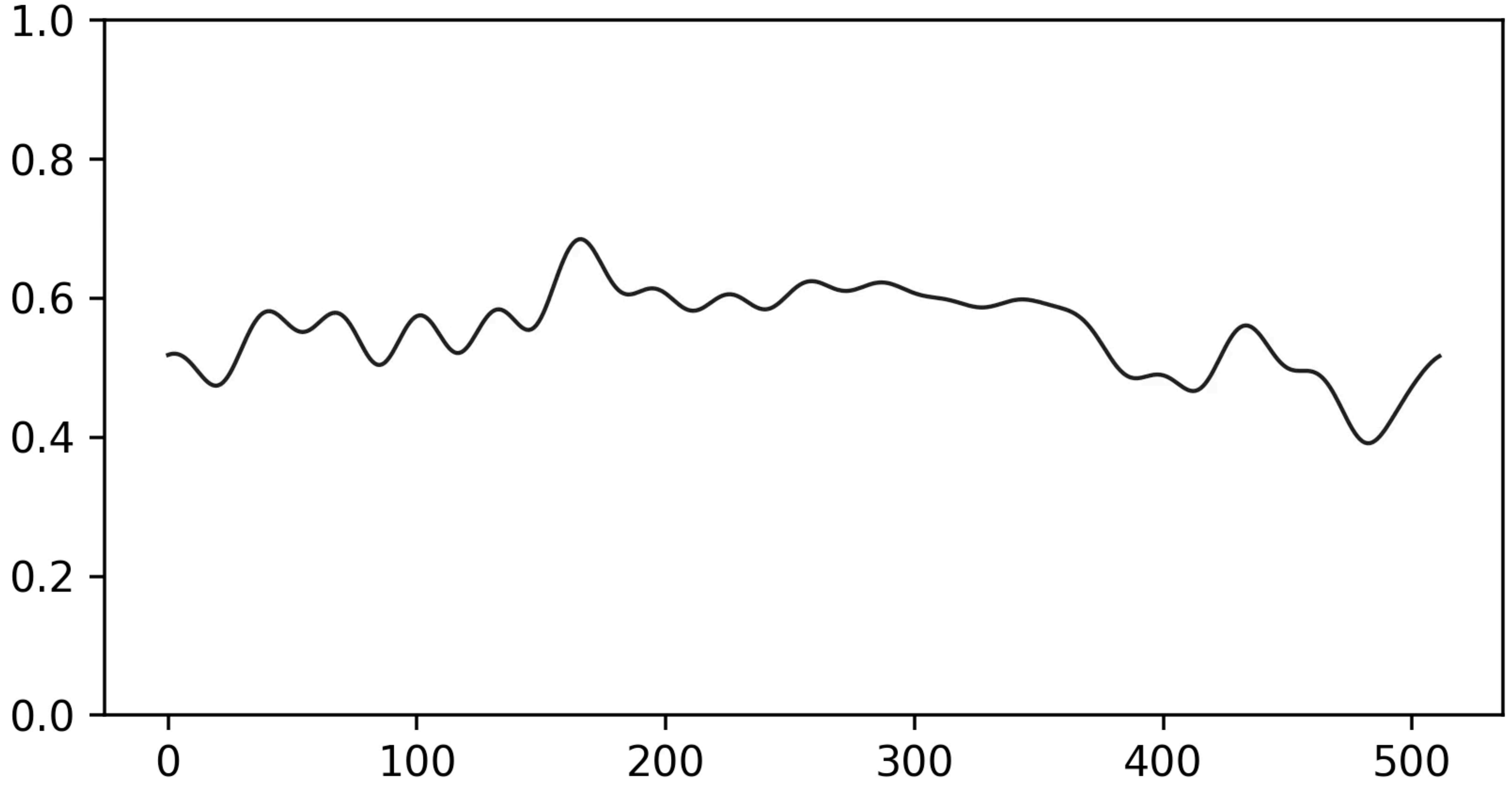


Reconstruction of one row

Reconstructed image

With coefficients from only the 18 lowest frequencies

Reconstructing the image

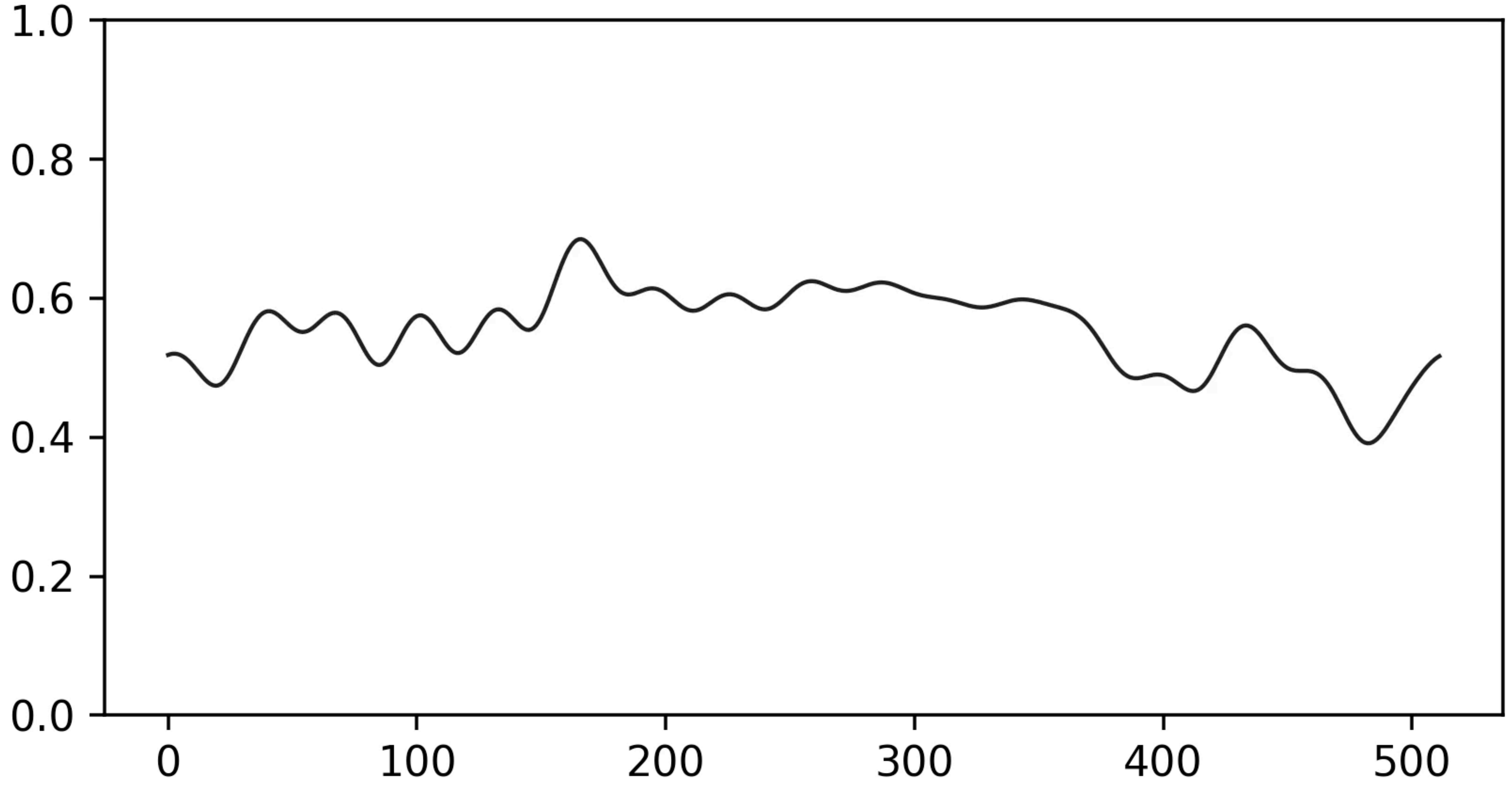


Reconstruction of one row

Reconstructed image

With coefficients from only the 23 lowest frequencies

Reconstructing the image



Reconstruction of one row

Reconstructed image

With more frequencies…

2D Discrete Fourier Transform

1D Discrete Fourier Transform (DFT) transforms a signal *f* [*n*] into *F* [*u*] as:

$$F[u] = \sum_{n=0}^{N-1} f[n] \exp\left(-2\pi i \frac{un}{N}\right)$$

2D Discrete Fourier Transform (DFT) transforms an image *f* [*n,m*] into *F* [*u,v*] as:

$$F[u, v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n, m] \exp\left(-2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

Source: Torralba, Freeman, Isola

Visualizing the image Fourier transform

$$F[u,v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m] \exp\left(-2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

The values of $F[u,v]$ are complex numbers.

Using the real and imaginary components:

$$F[u,v] = a + bi$$

Decompose into polar coordinates:

$$\sqrt{a^2 + b^2} \qquad \tan^{-1}\left(\frac{b}{a}\right)$$

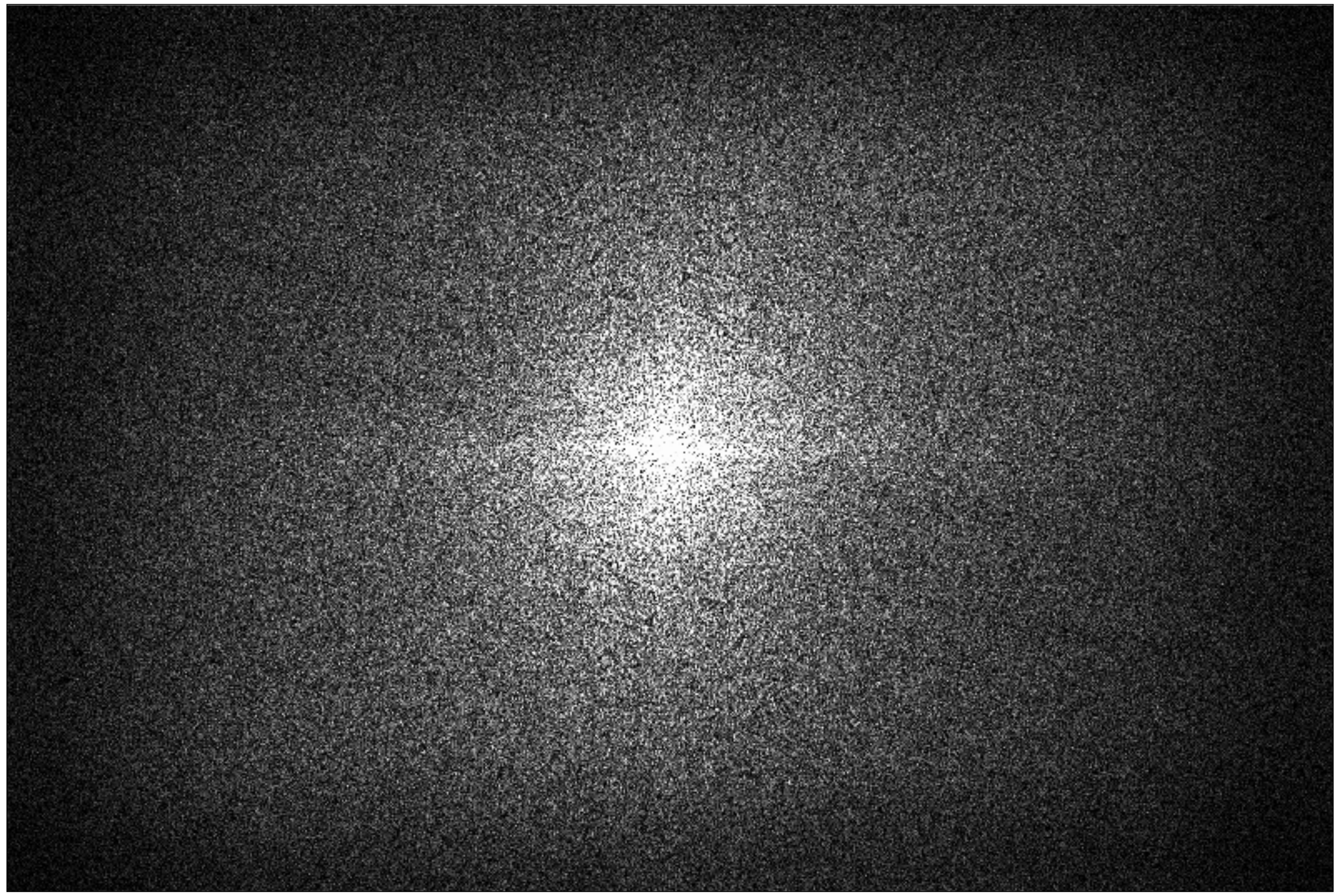
Magnitude $|F[u,v]|$

Phase

Source: Torralba, Freeman, Isola

2D Fourier transform example

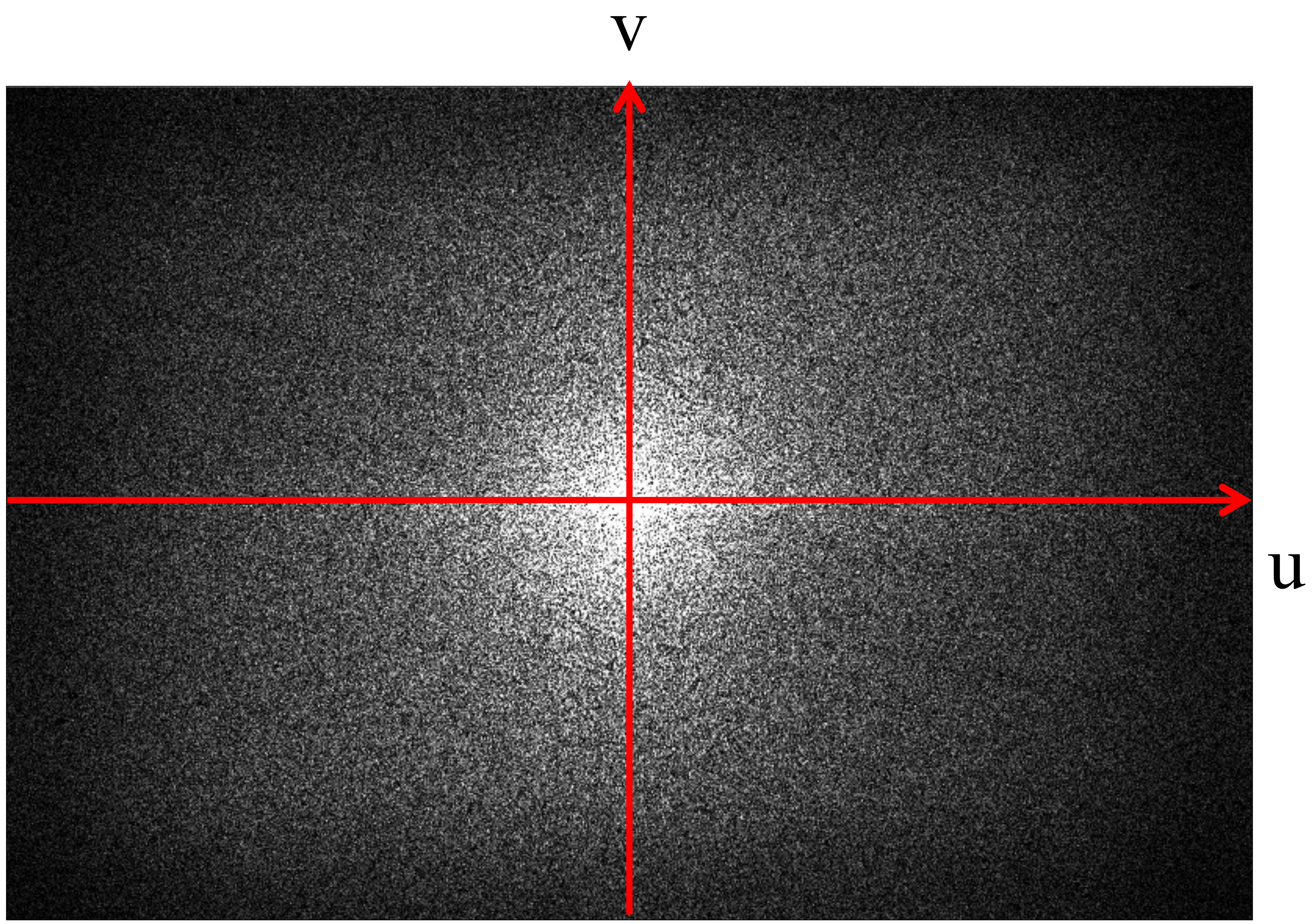
Image



DFT: F[u,v]

2D Fourier transform example

Image



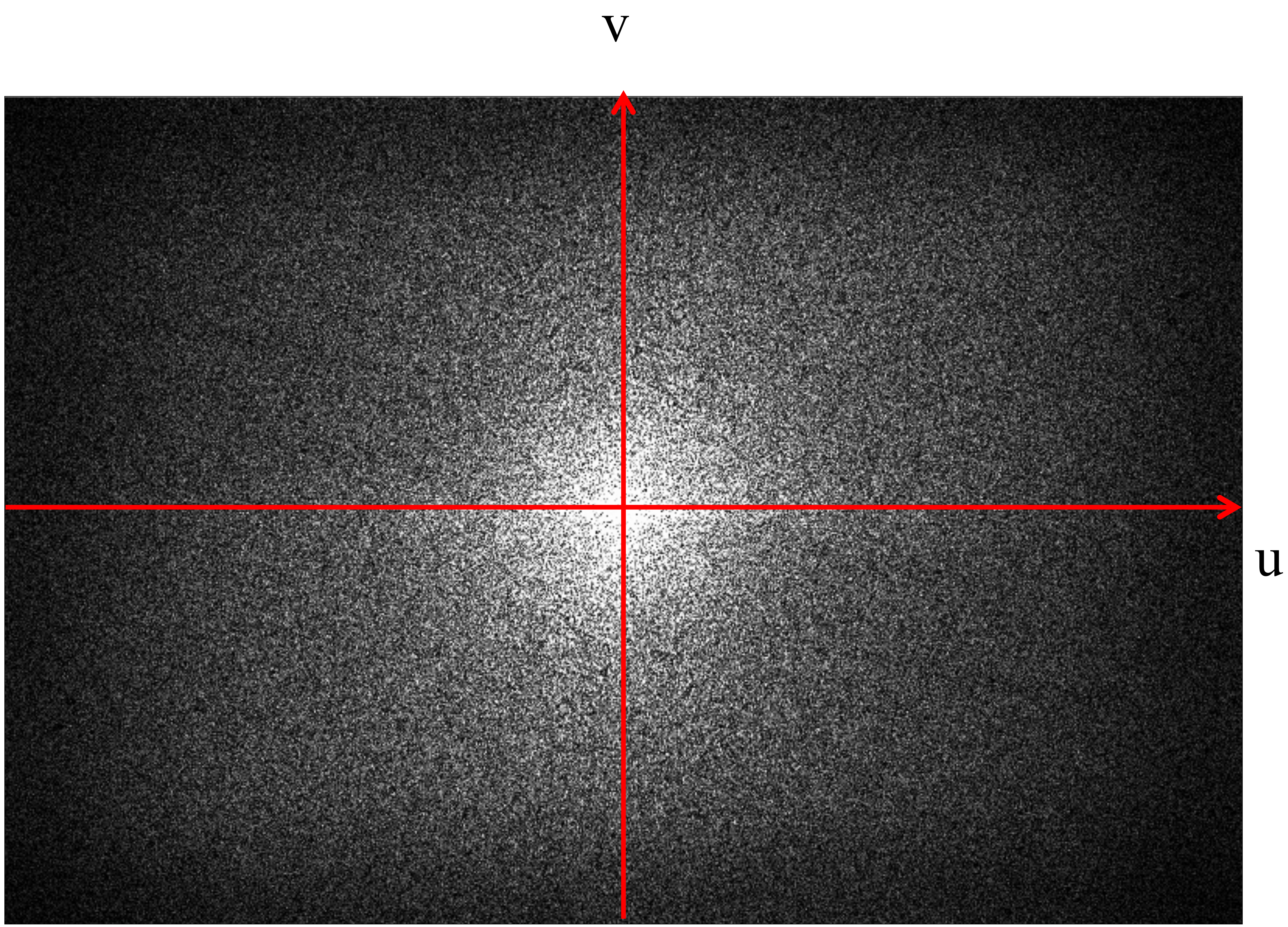
DFT: |F[u,v]|

$$F[u,v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m] \exp\left(-2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

2D Fourier transform example

Image

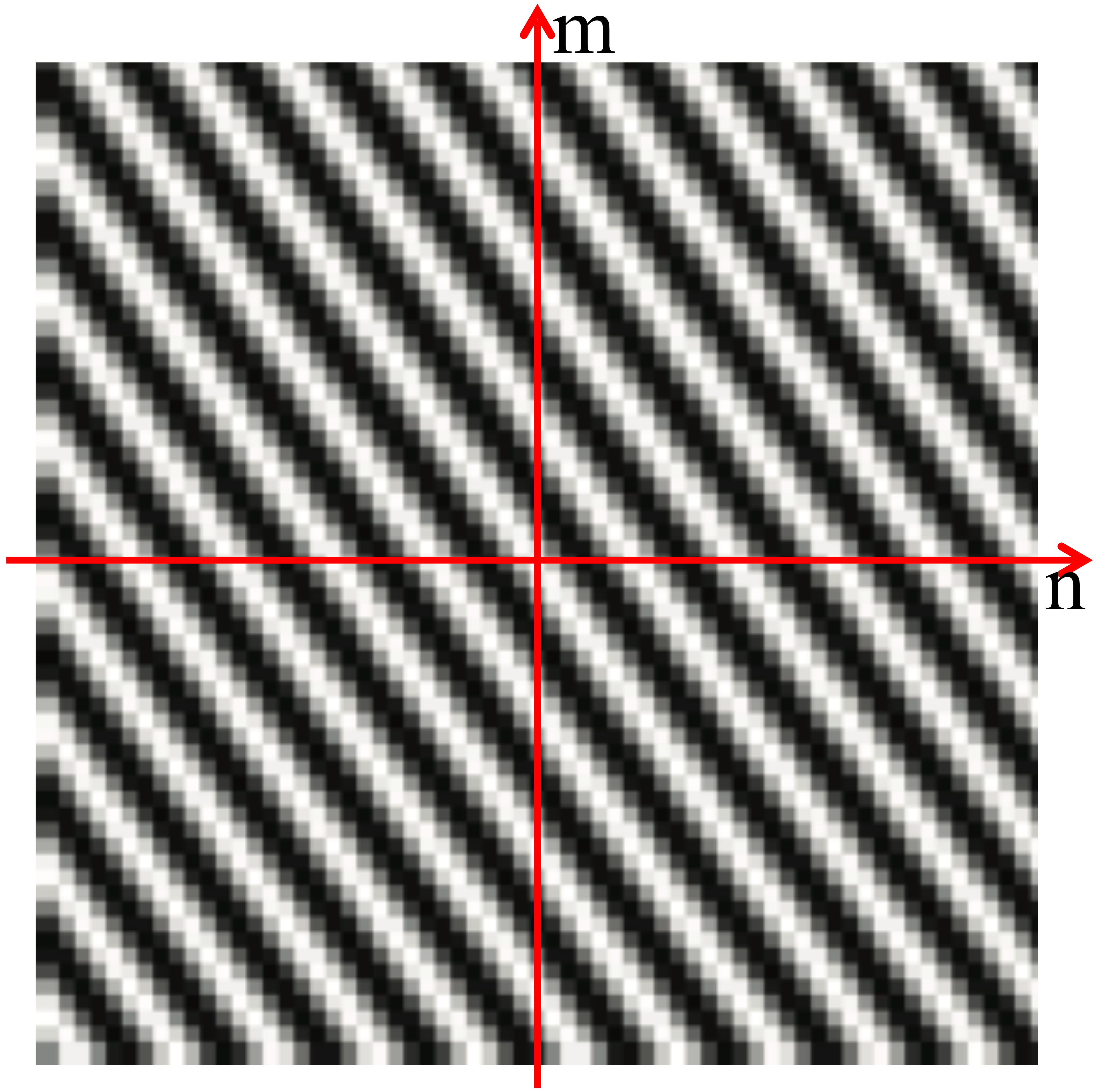
$$F[u,v] = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f[n,m] \exp\left(-2\pi i\left(\frac{un}{N}+\frac{vm}{M}\right)\right)$$



DFT: |F[u,v]|

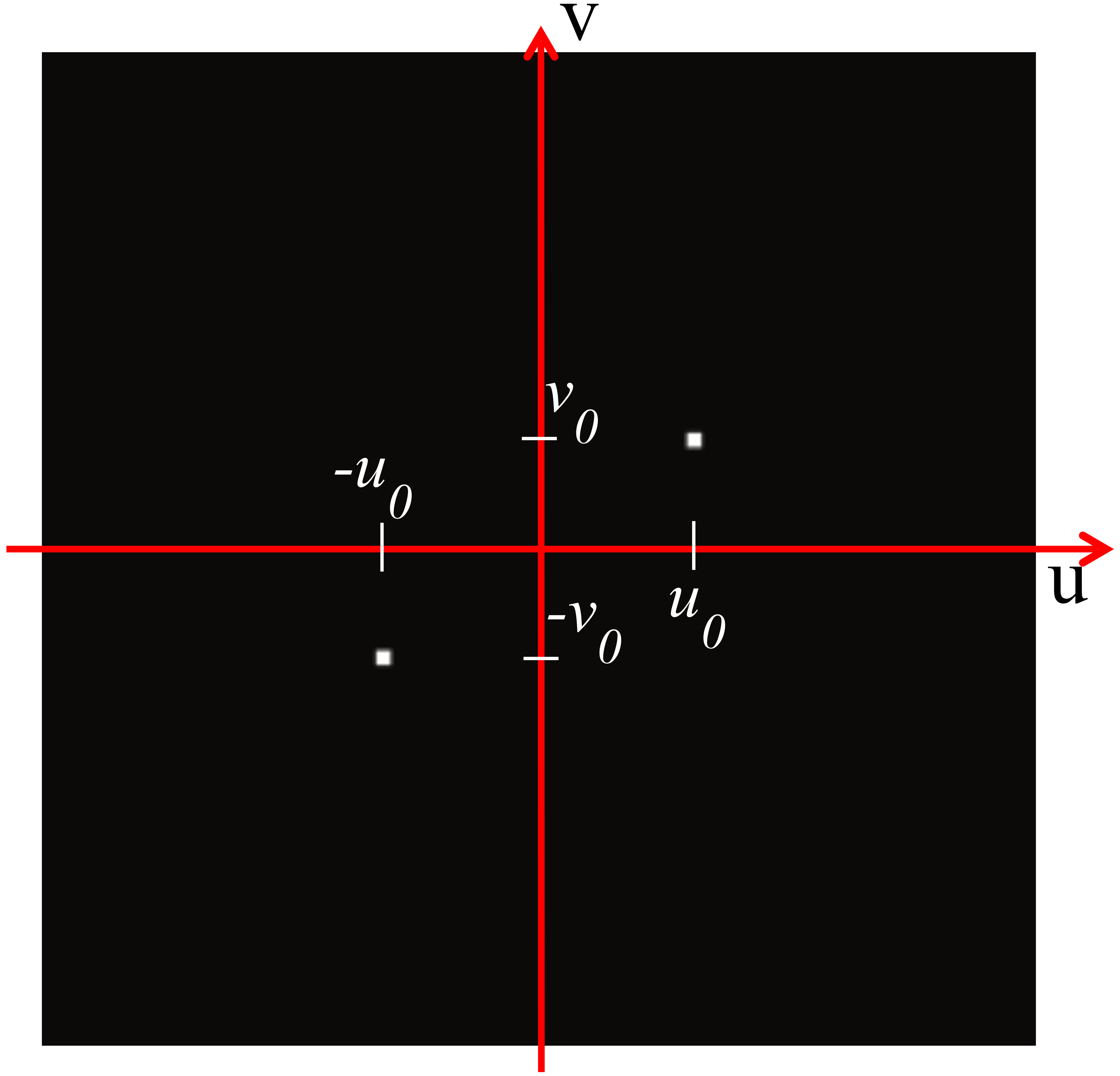
Simple Fourier transforms

Image



64x64 pixels

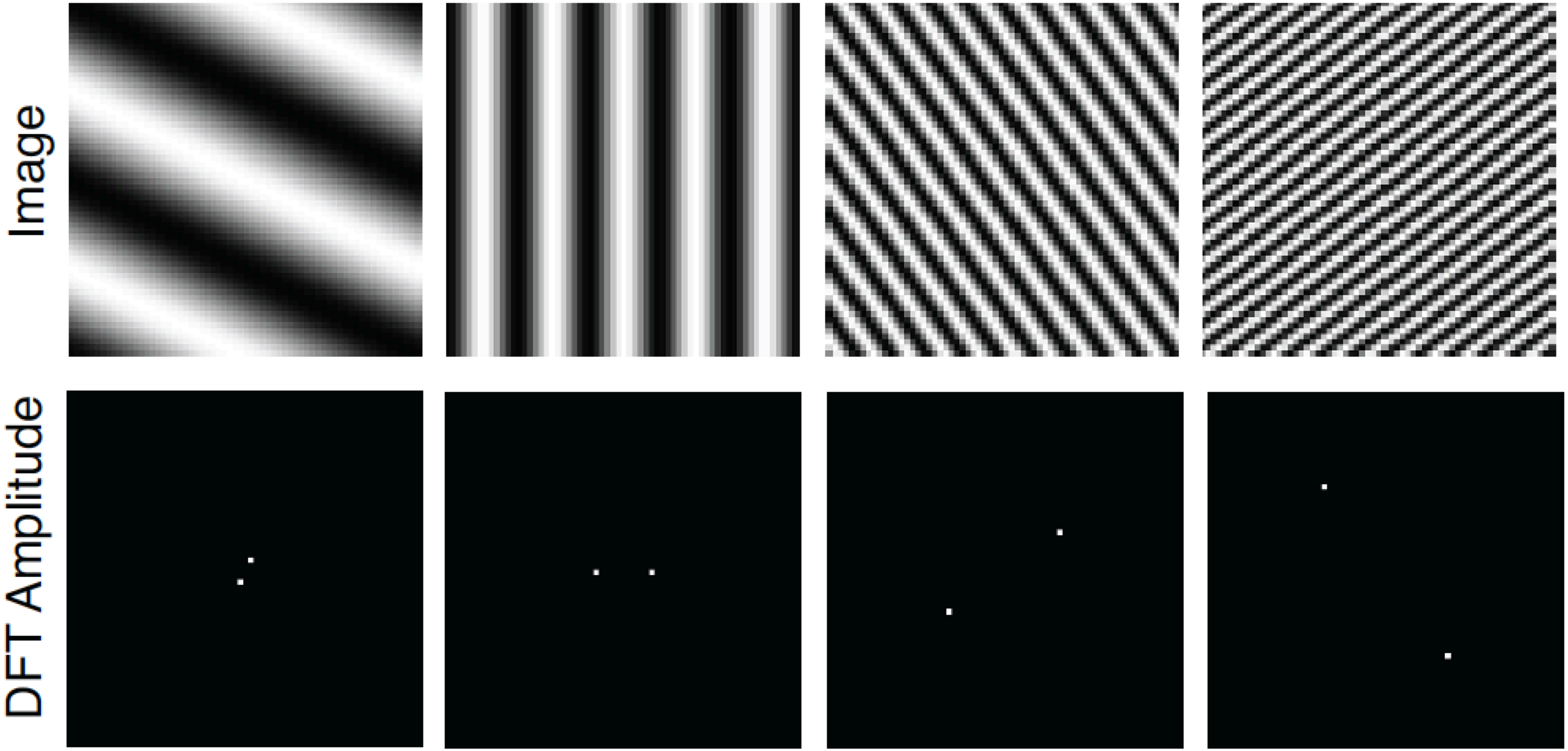
DFT (amplitude)



$$\cos\left(2\pi\left(\frac{u_0\, n}{N} + \frac{v_0\, m}{M}\right)\right) \leftrightarrow \frac{1}{2}\left(\delta\left[u - u_0, v - v_0\right] + \delta\left[u + u_0, v + v_0\right]\right)$$

Source: Torralba, Freeman, Isola

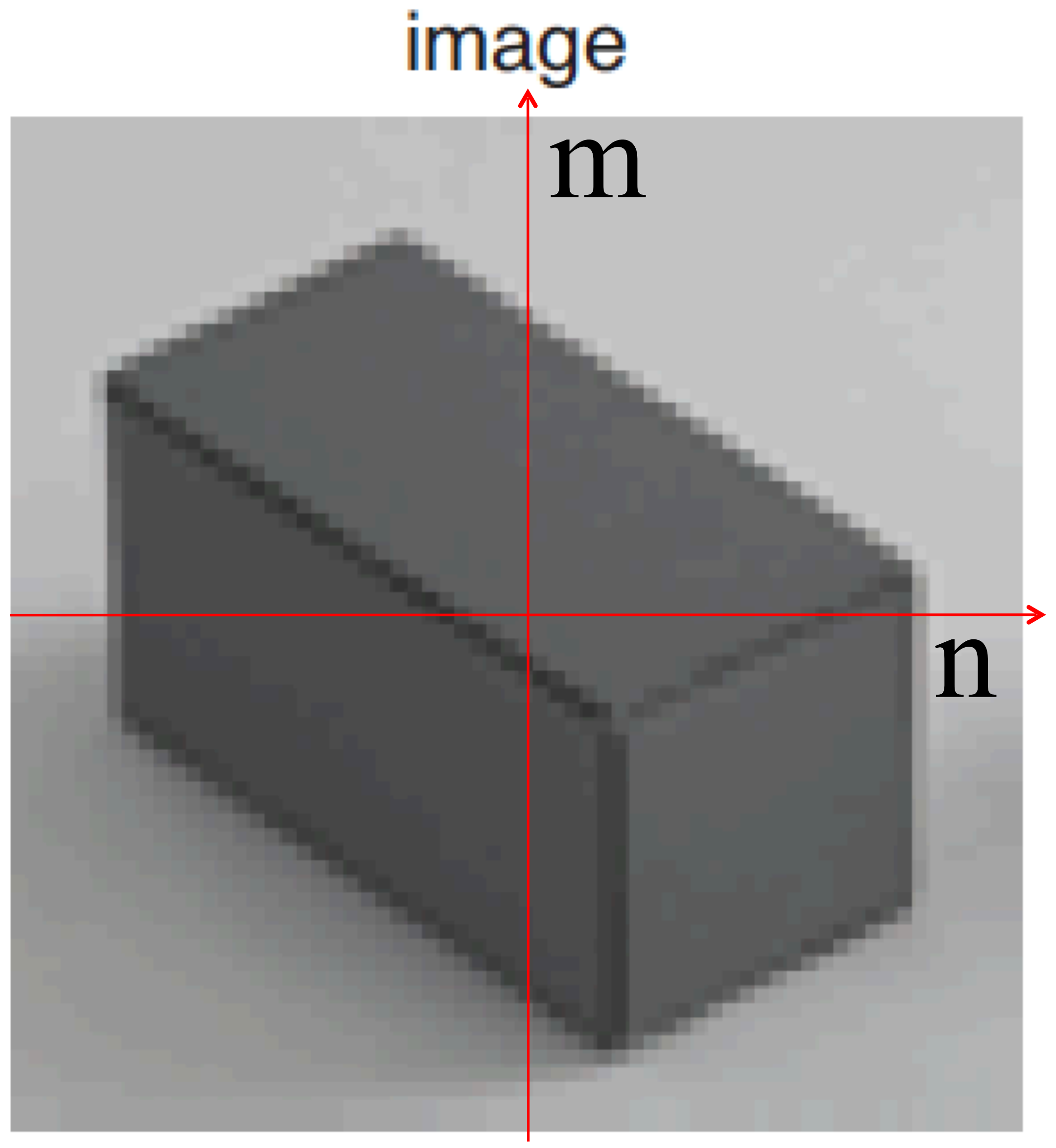
Simple Fourier transforms



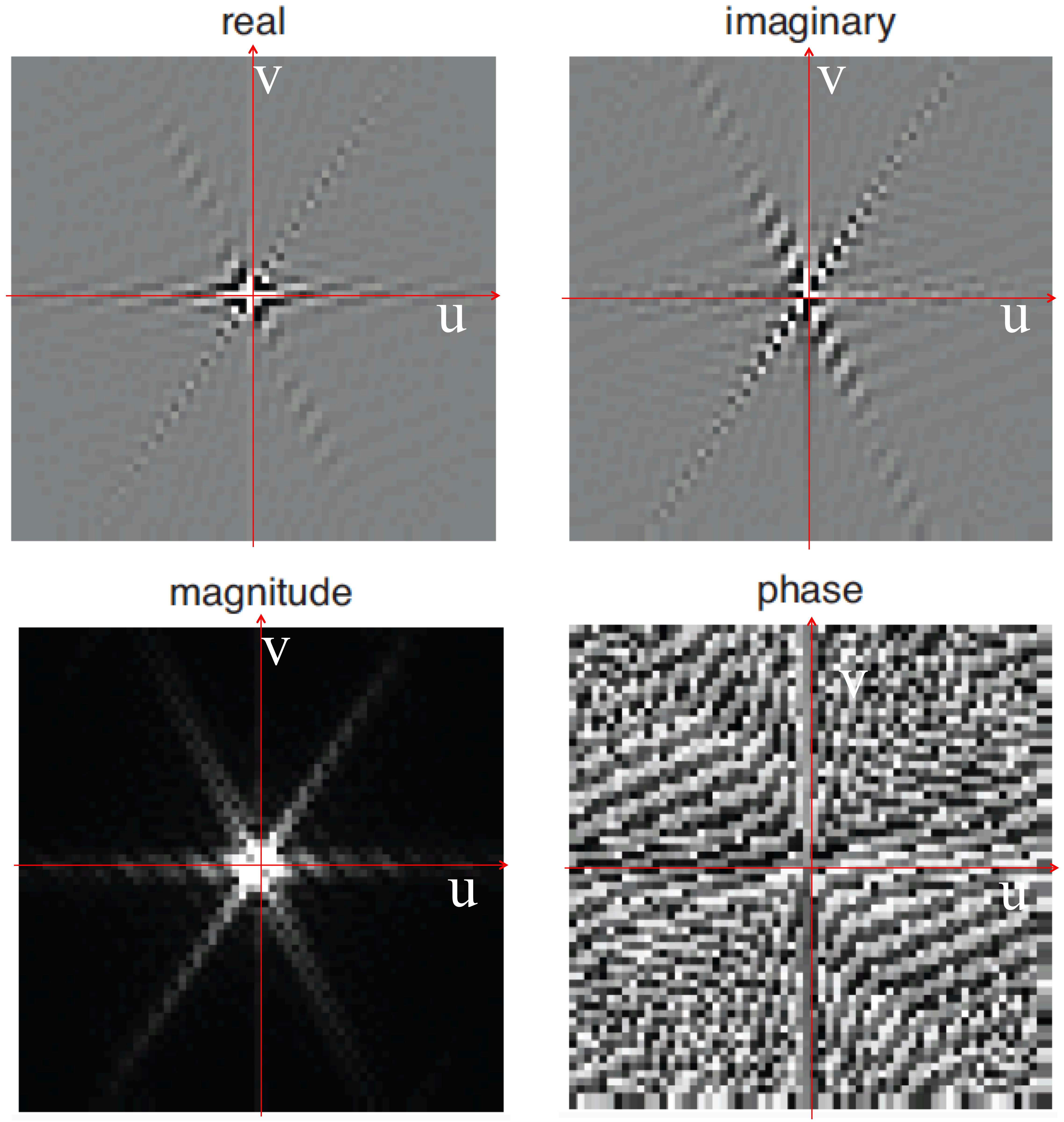
Source: Torralba, Freeman, Isola

Visualizing the image Fourier transform

$f[n, m]$

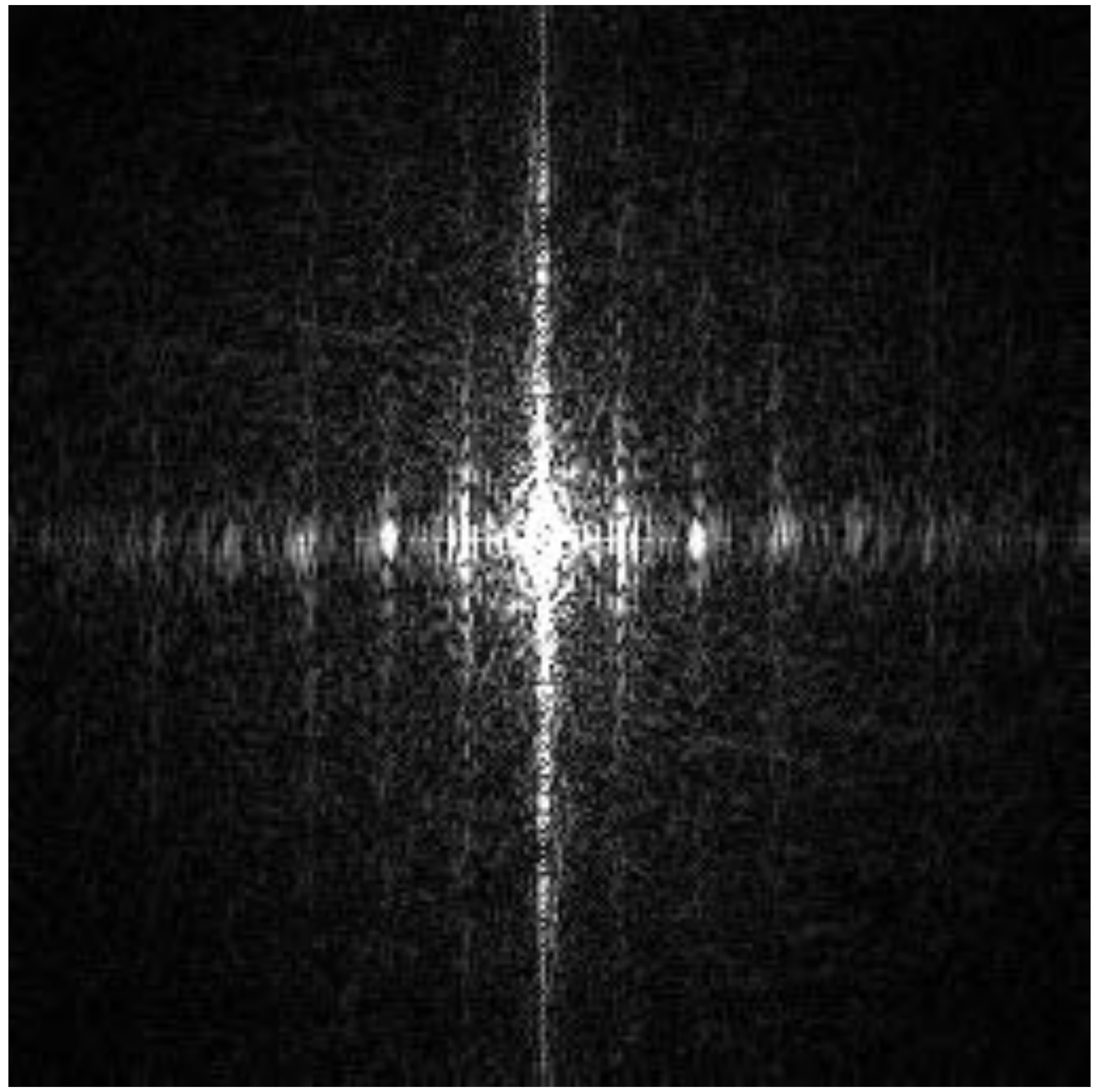


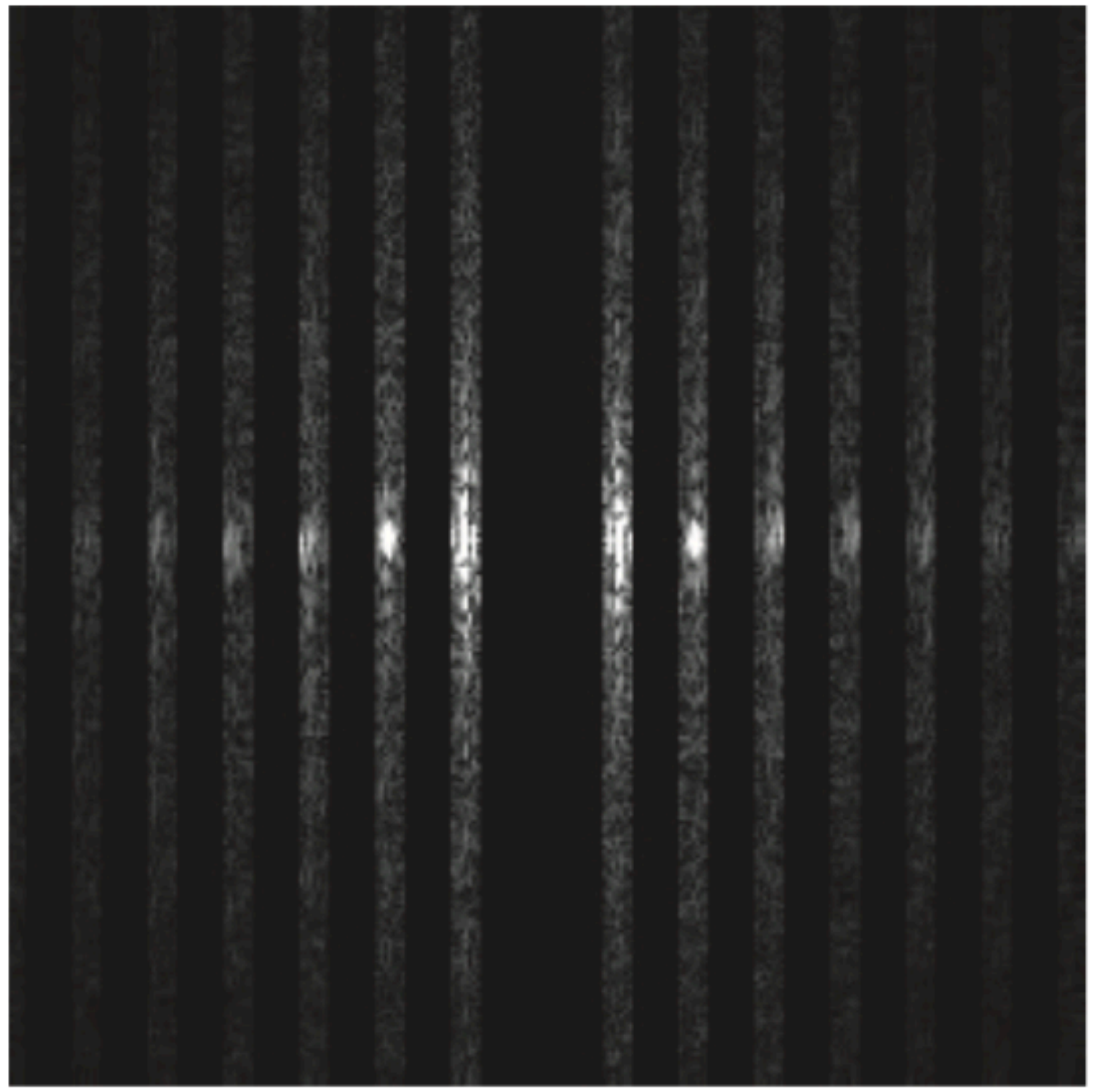
$F[u, v]$



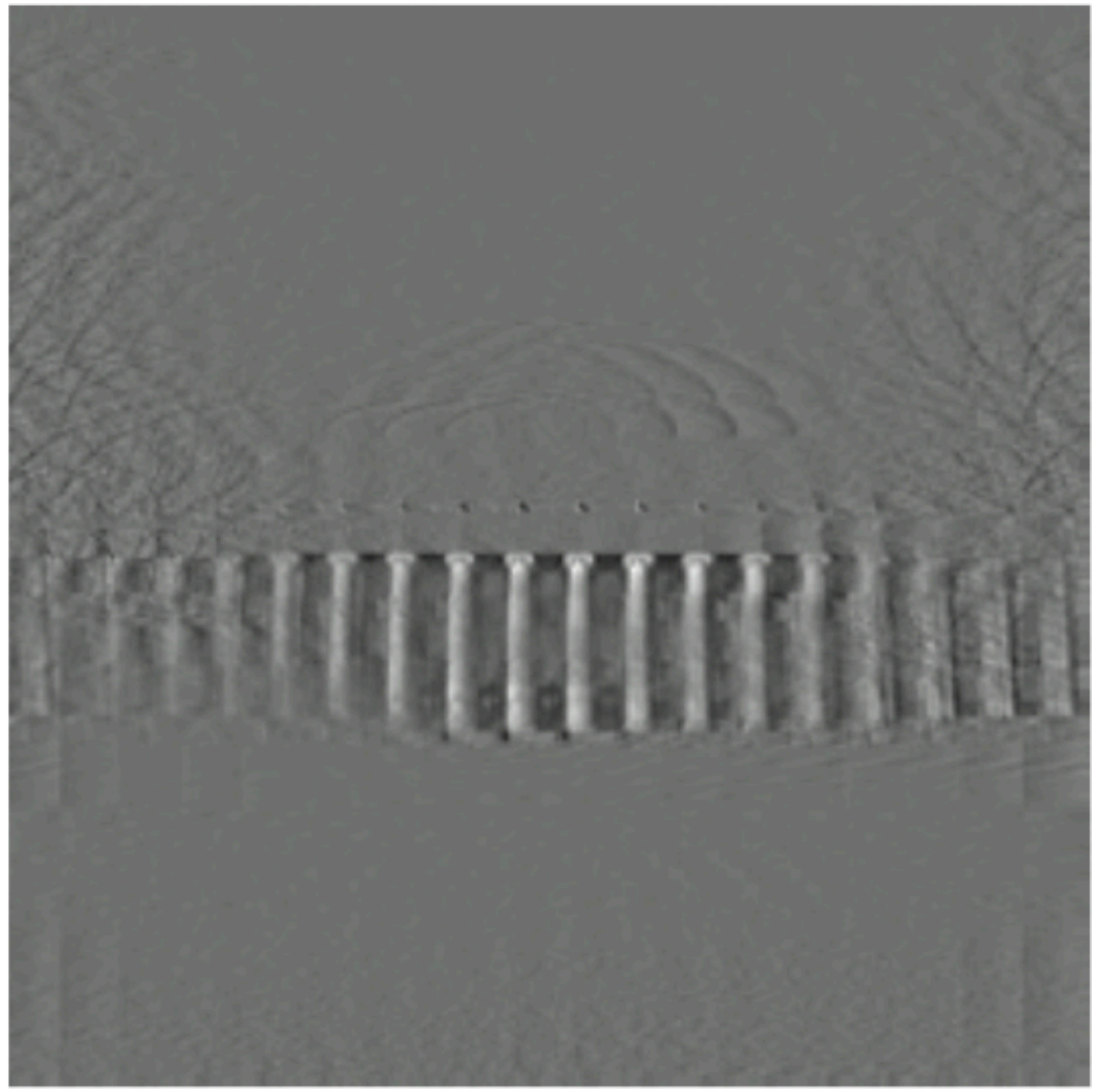
Source: Torralba, Freeman, Isola

DFT



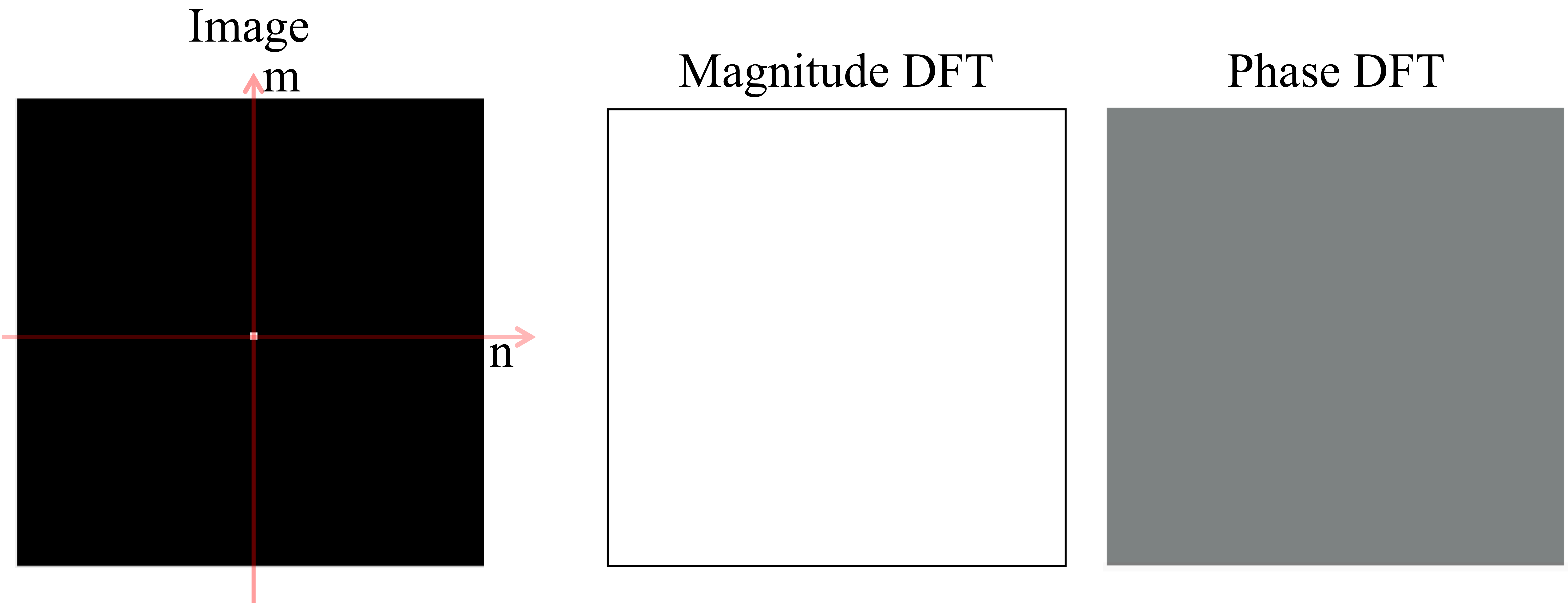


DFT^{-1}



Source: Torralba, Freeman, Isola

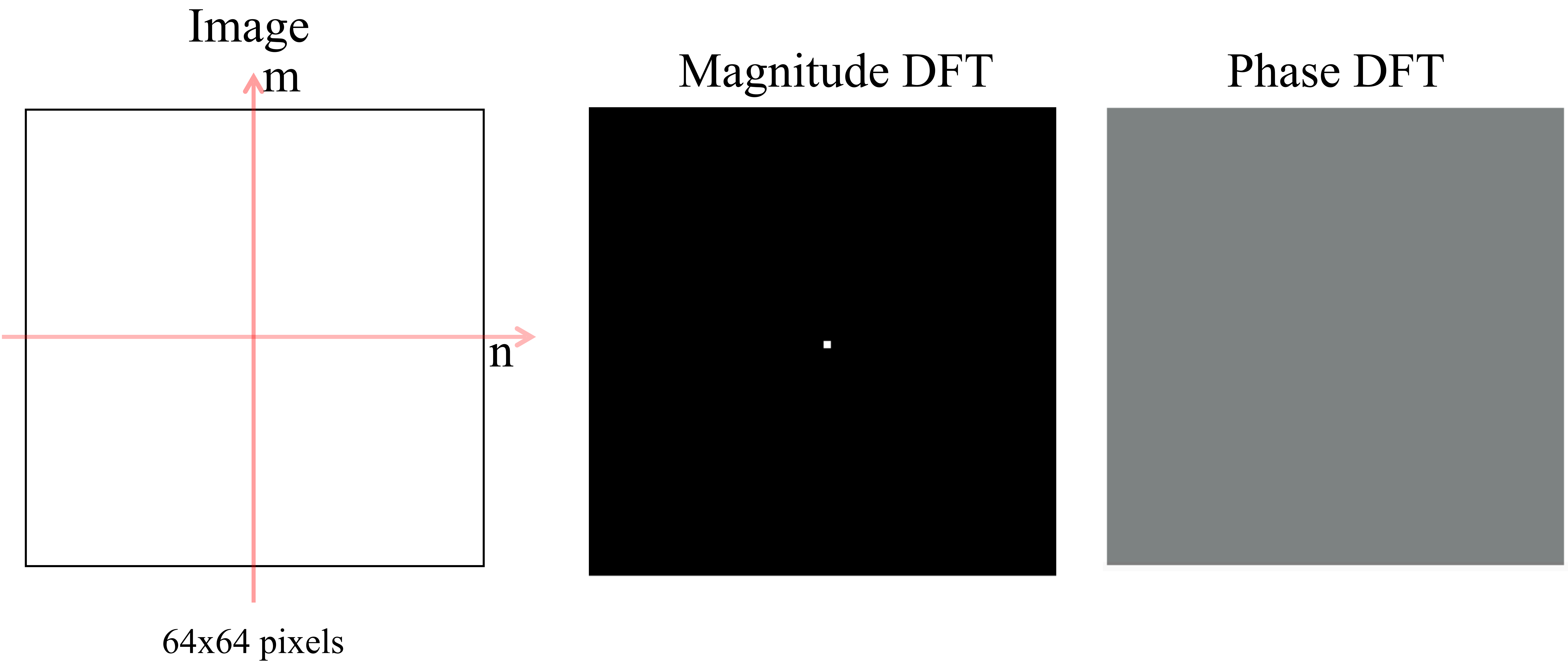
Some important Fourier transforms



64x64 pixels

Source: Torralba, Freeman, Isola

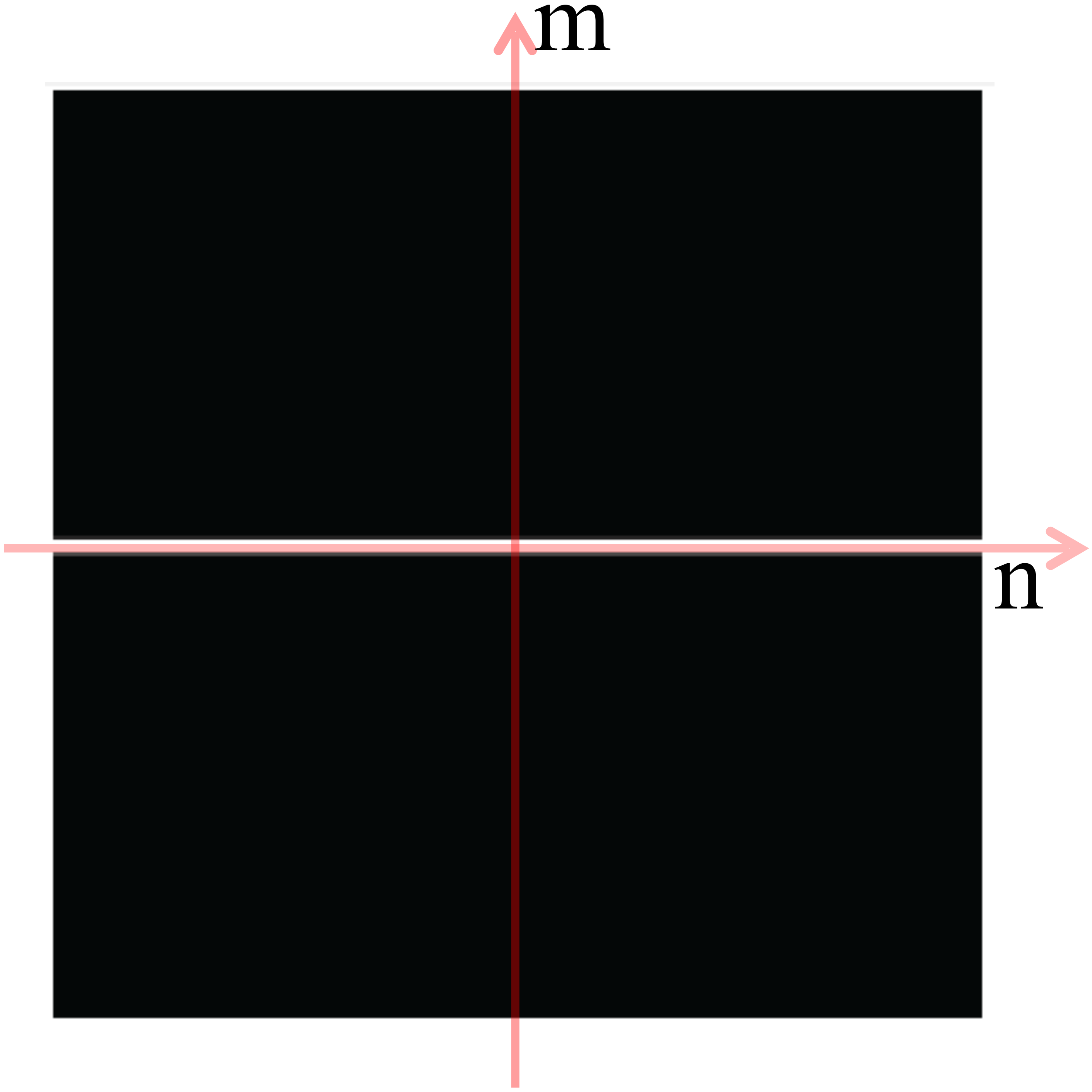
Some important Fourier transforms



Source: Torralba, Freeman, Isola

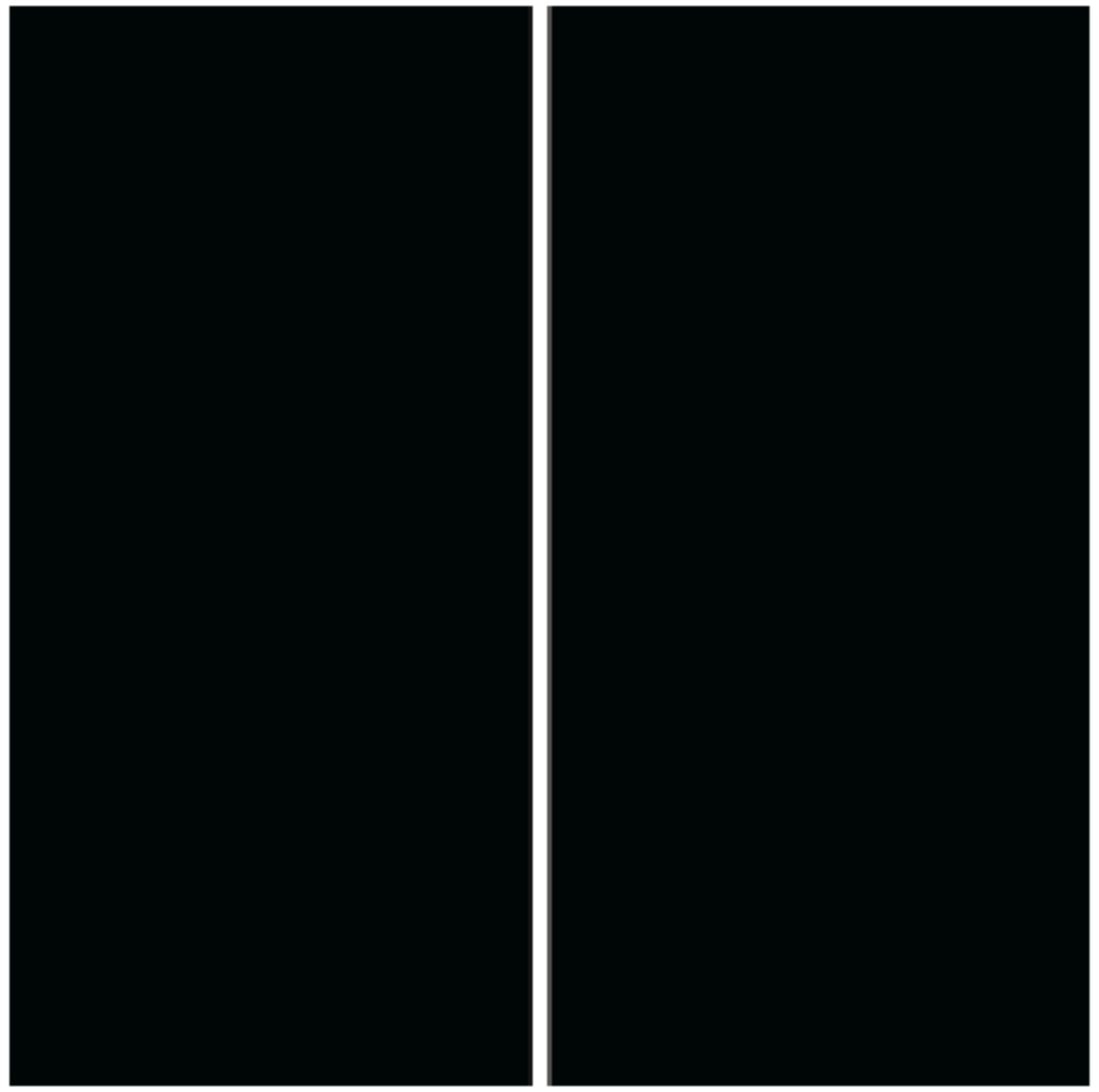
Some important Fourier transforms

Image



64x64 pixels

Magnitude DFT

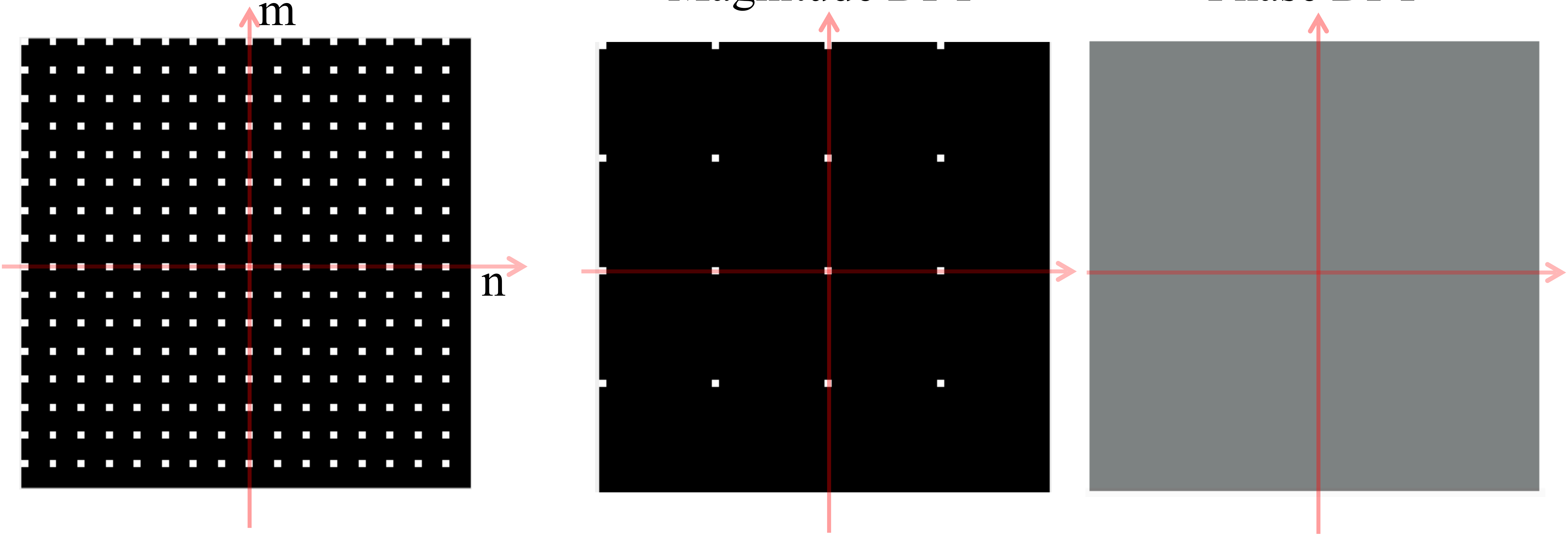


Phase DFT

Source: Torralba, Freeman, Isola

Some important Fourier transforms

Image | Magnitude DFT | Phase DFT



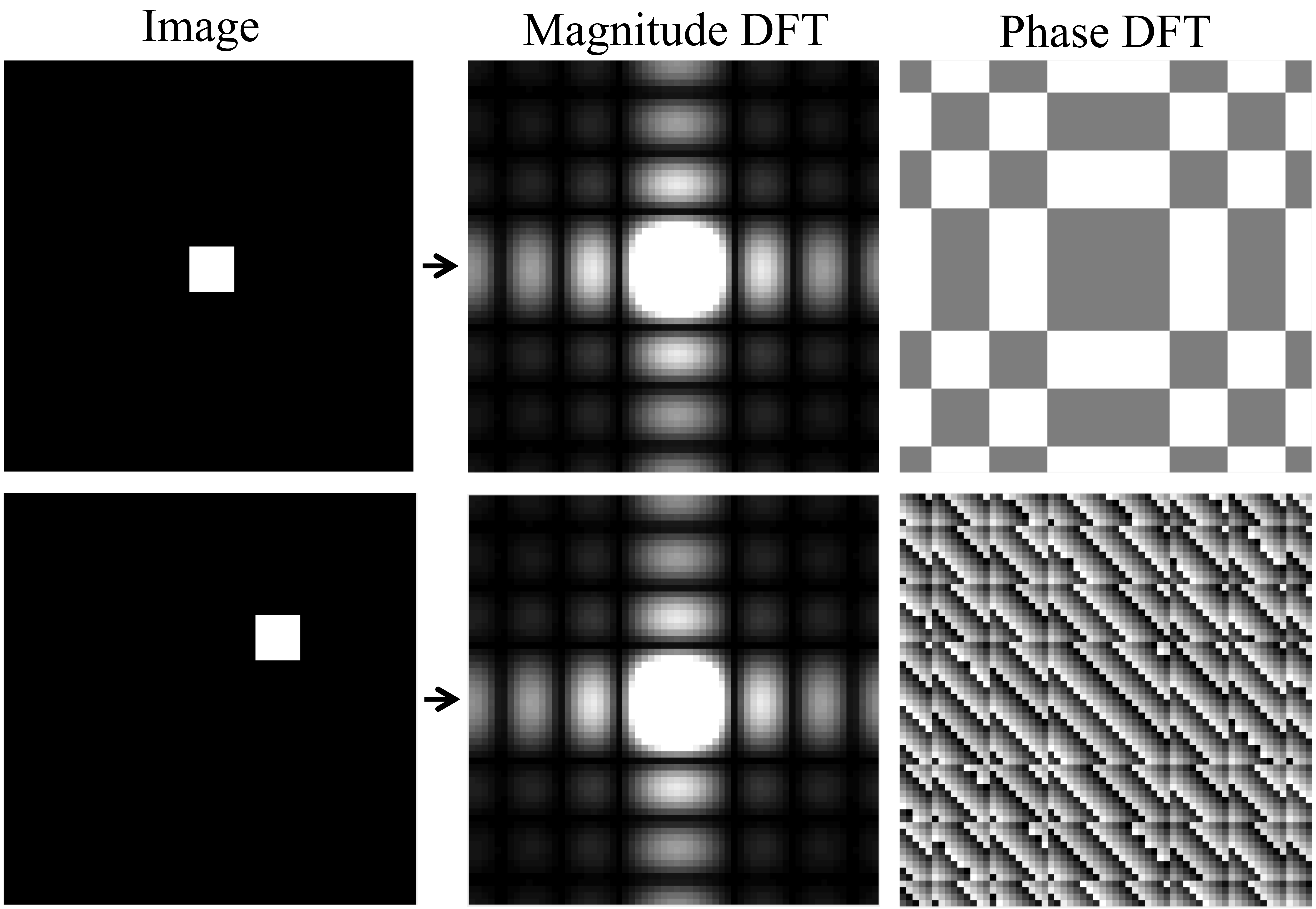
64x64 pixels

when M and N are divisible by k

$$\delta_k[n,m] = \sum_{s=0}^{N/k-1} \sum_{r=0}^{M/k-1} \delta[n - sk\,, m - rk] \quad \longleftrightarrow \quad \Delta_k[u,v] = \frac{NM}{k^2} \sum_{s=0}^{k-1} \sum_{r=0}^{k-1} \delta\left[u - s\frac{N}{k}\,, v - r\frac{M}{k}\right]$$

delta train, delta comb, impulse train

Some important Fourier transforms

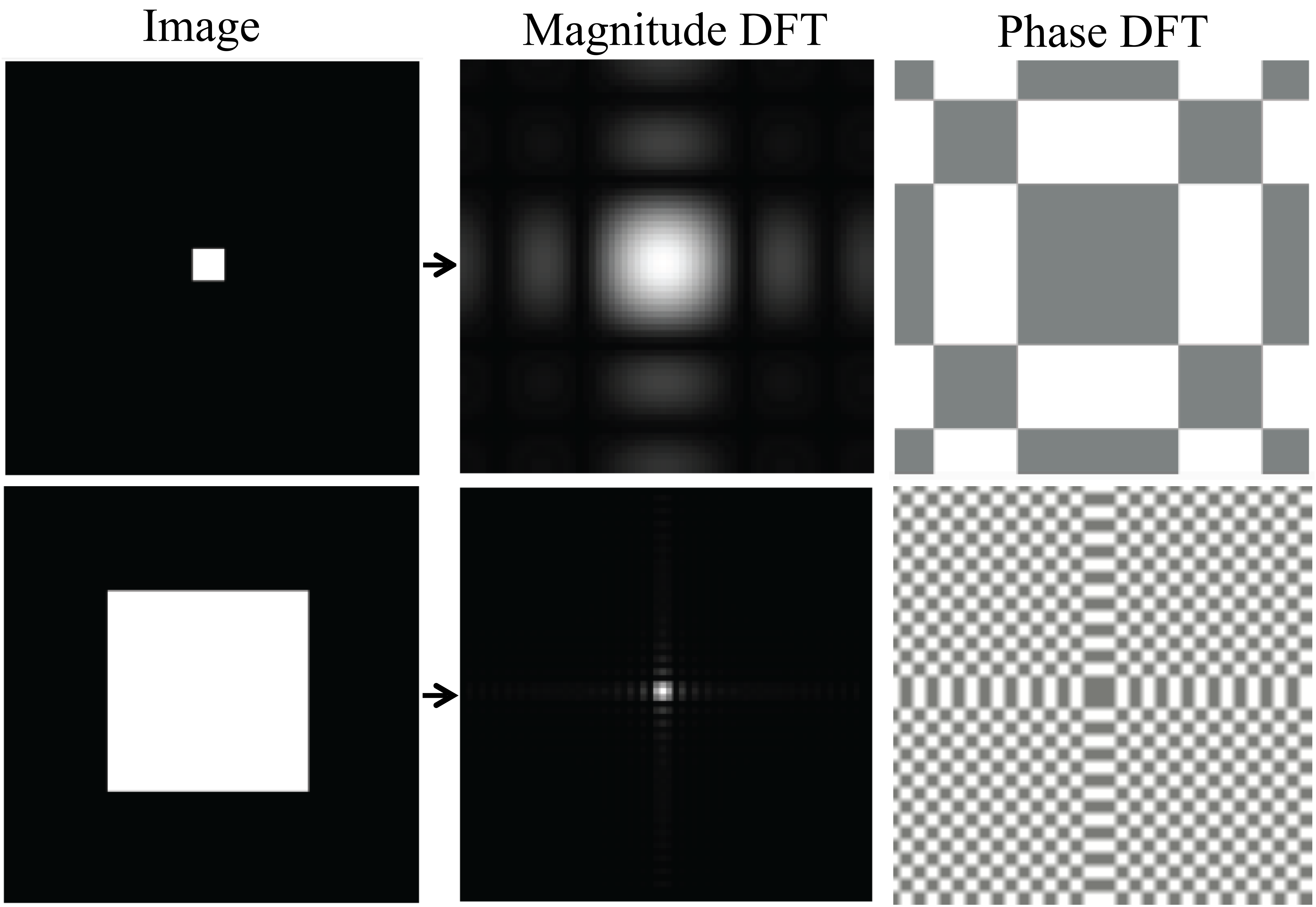


Translation

Shifts of an image only produce changes on the phase of the DFT.

Source: Torralba, Freeman, Isola

Some important Fourier transforms



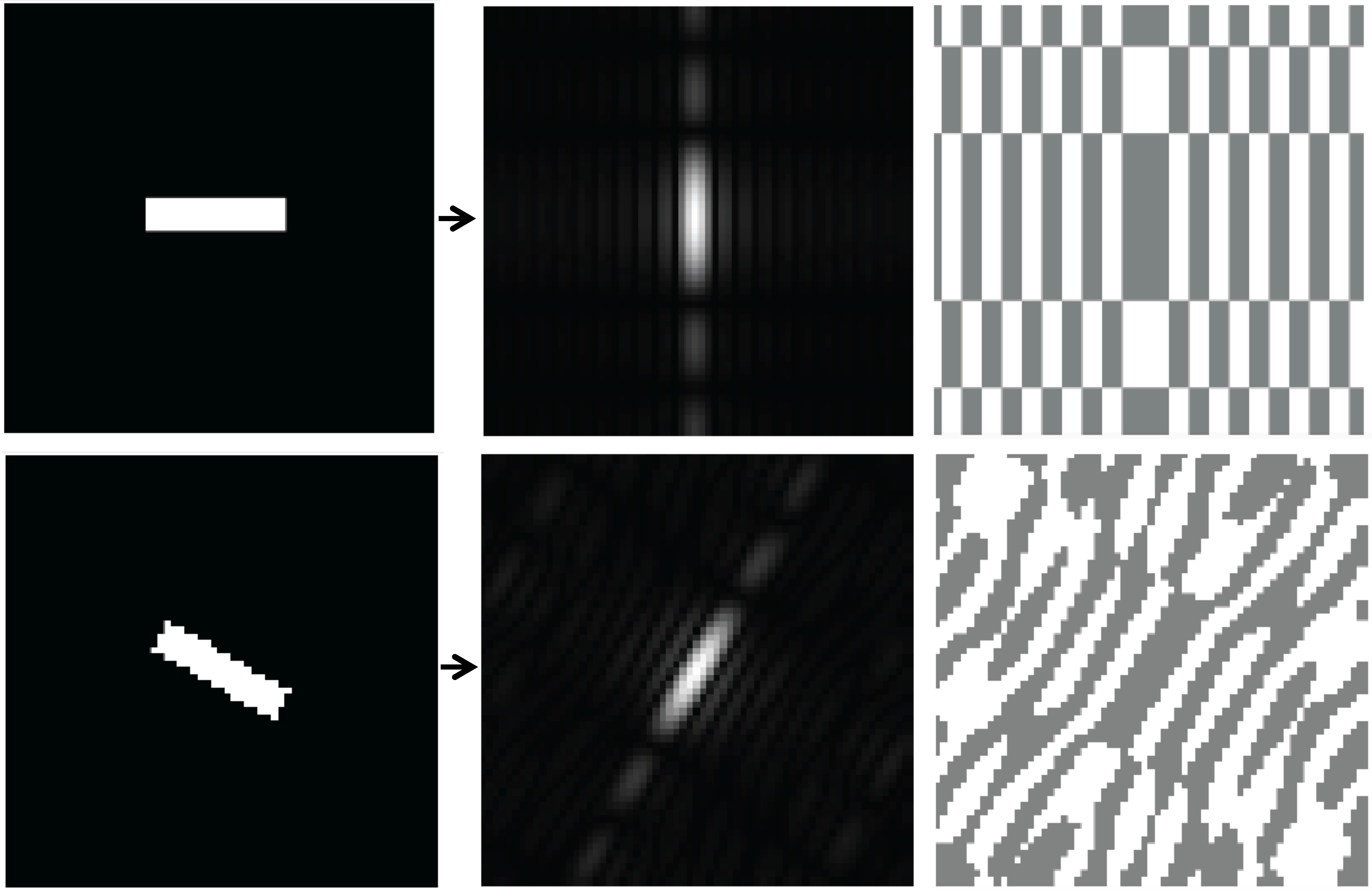
Scale

Small image details produce content in high spatial frequencies

Source: Torralba, Freeman, Isola

Some important Fourier transforms

Image Magnitude DFT Phase DFT

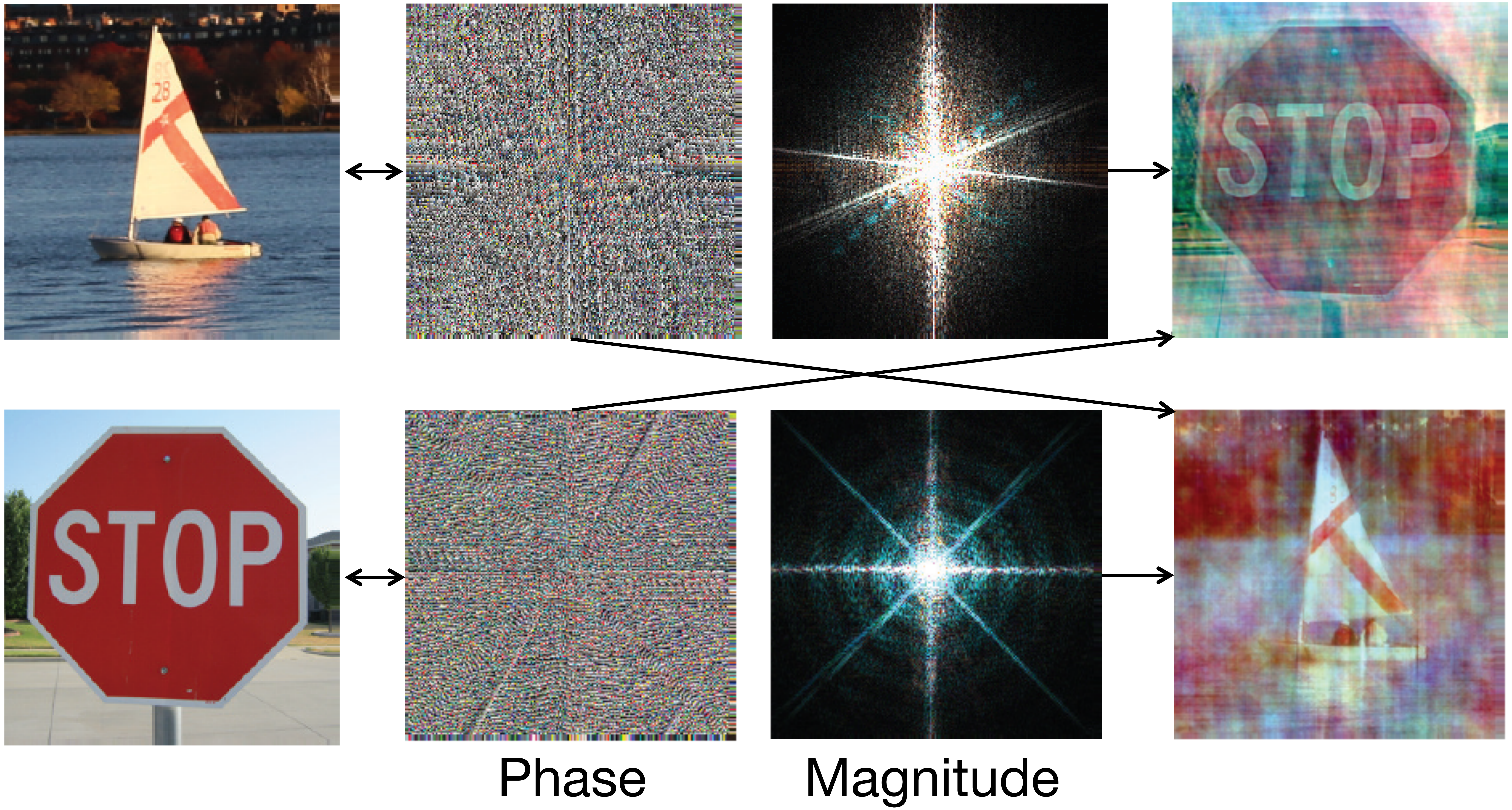


Orientation

A line transforms to a line oriented perpendicularly to the first.

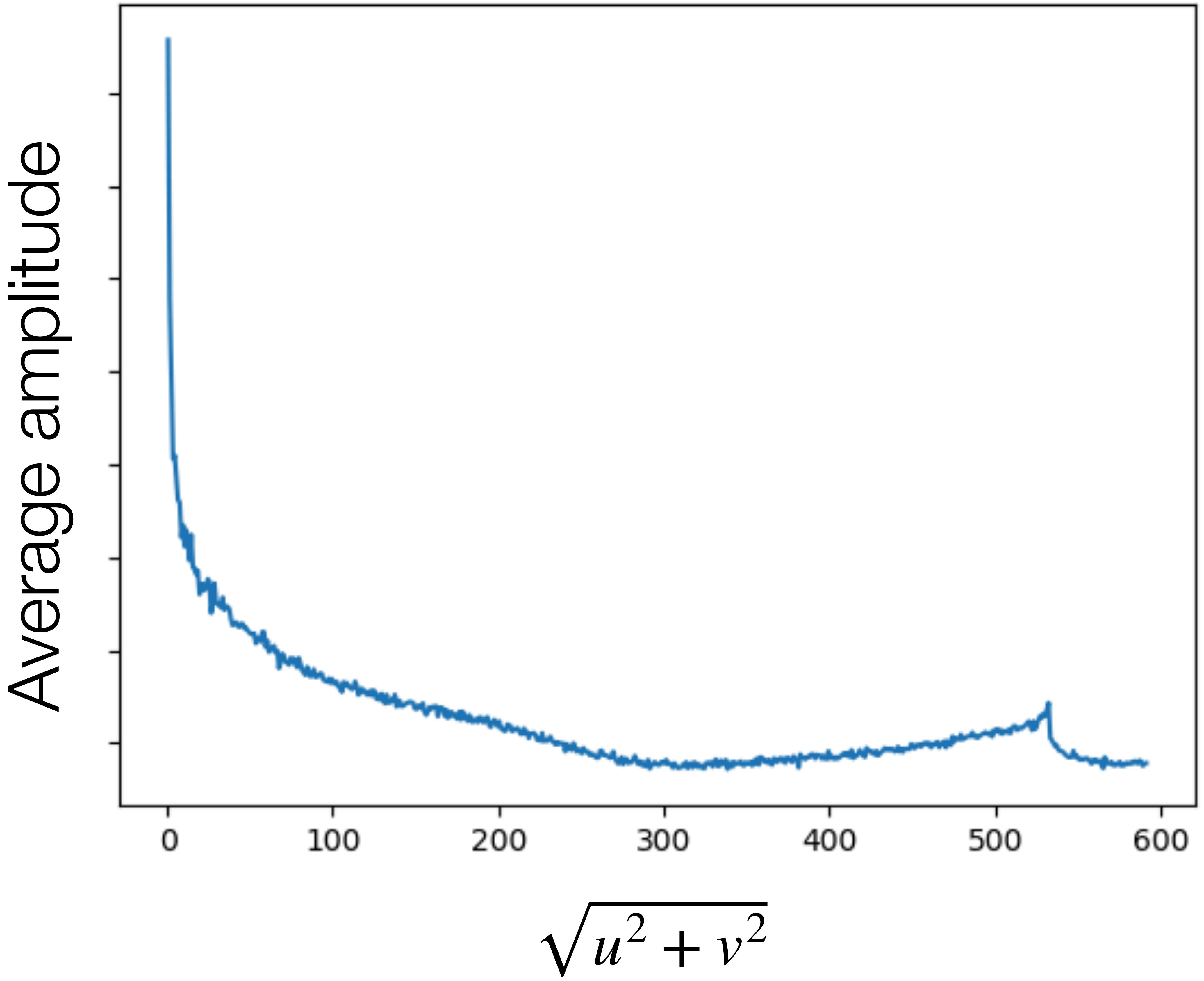
Source: Torralba, Freeman, Isola

Swapping phase and magnitude



Source: Torralba, Freeman, Isola

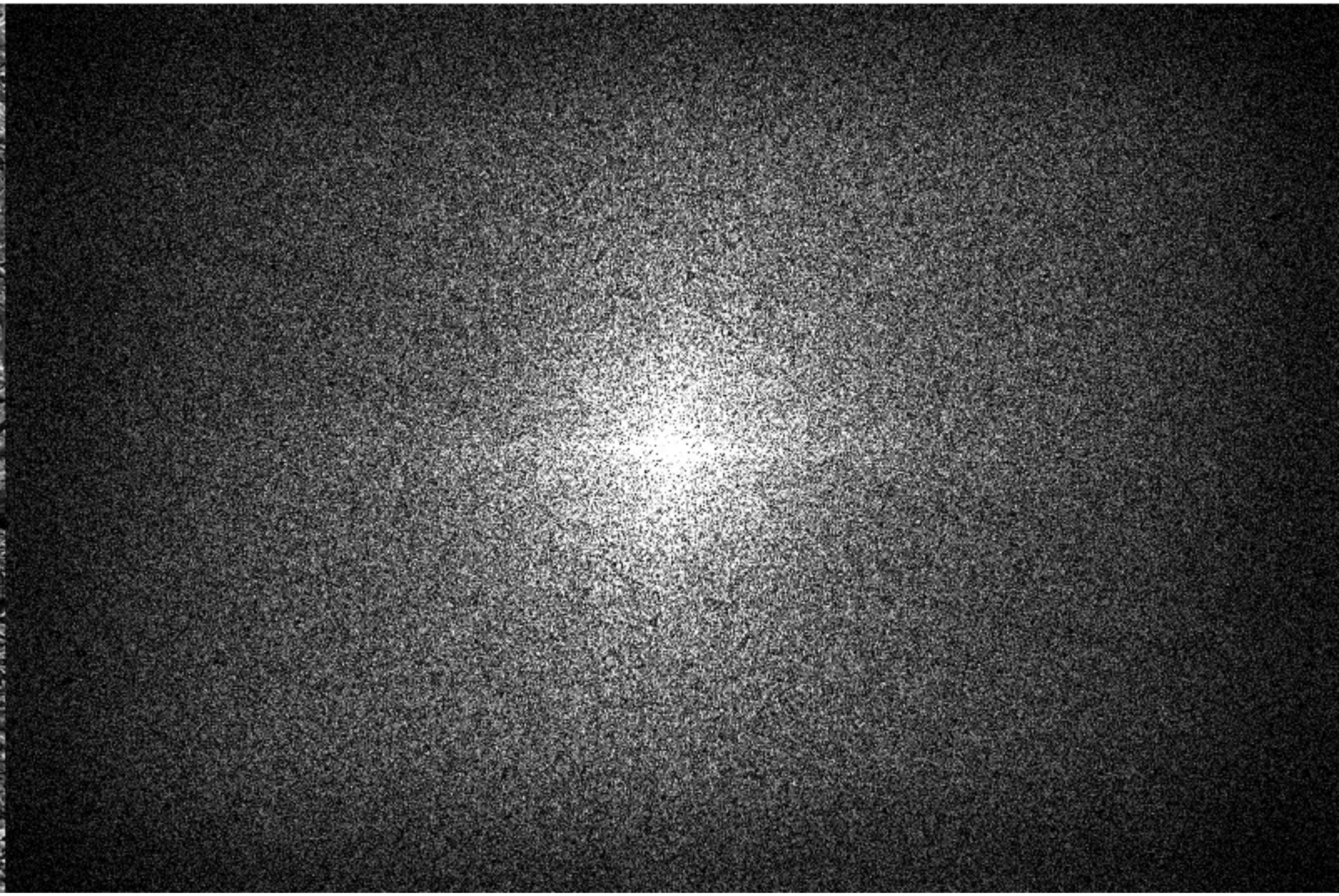
Natural image statistics



Most of the energy is in the low frequencies

Reconstruct an image, low frequency to high

Image



DFT

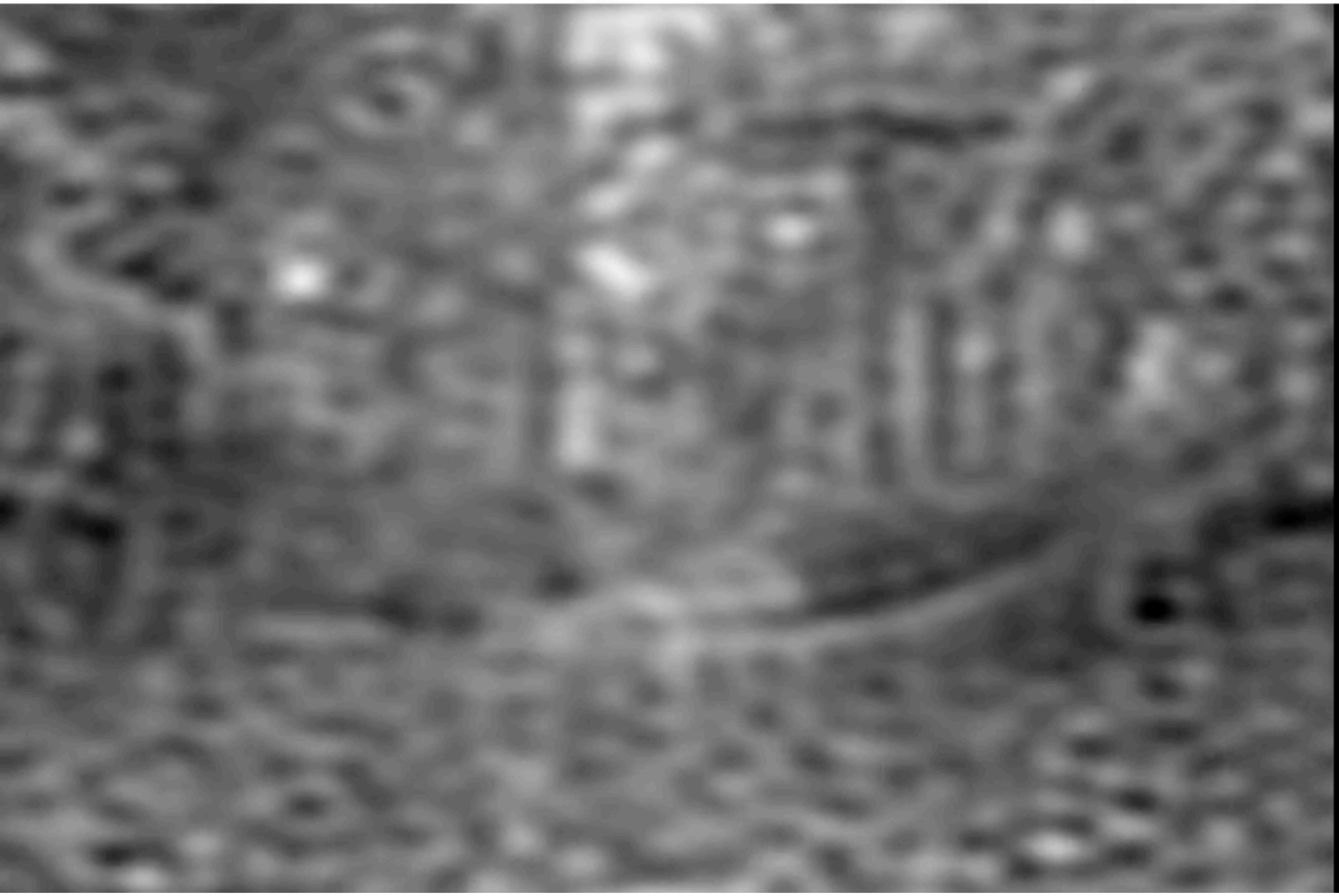
Reconstruct an image, low frequency to high

0.0%

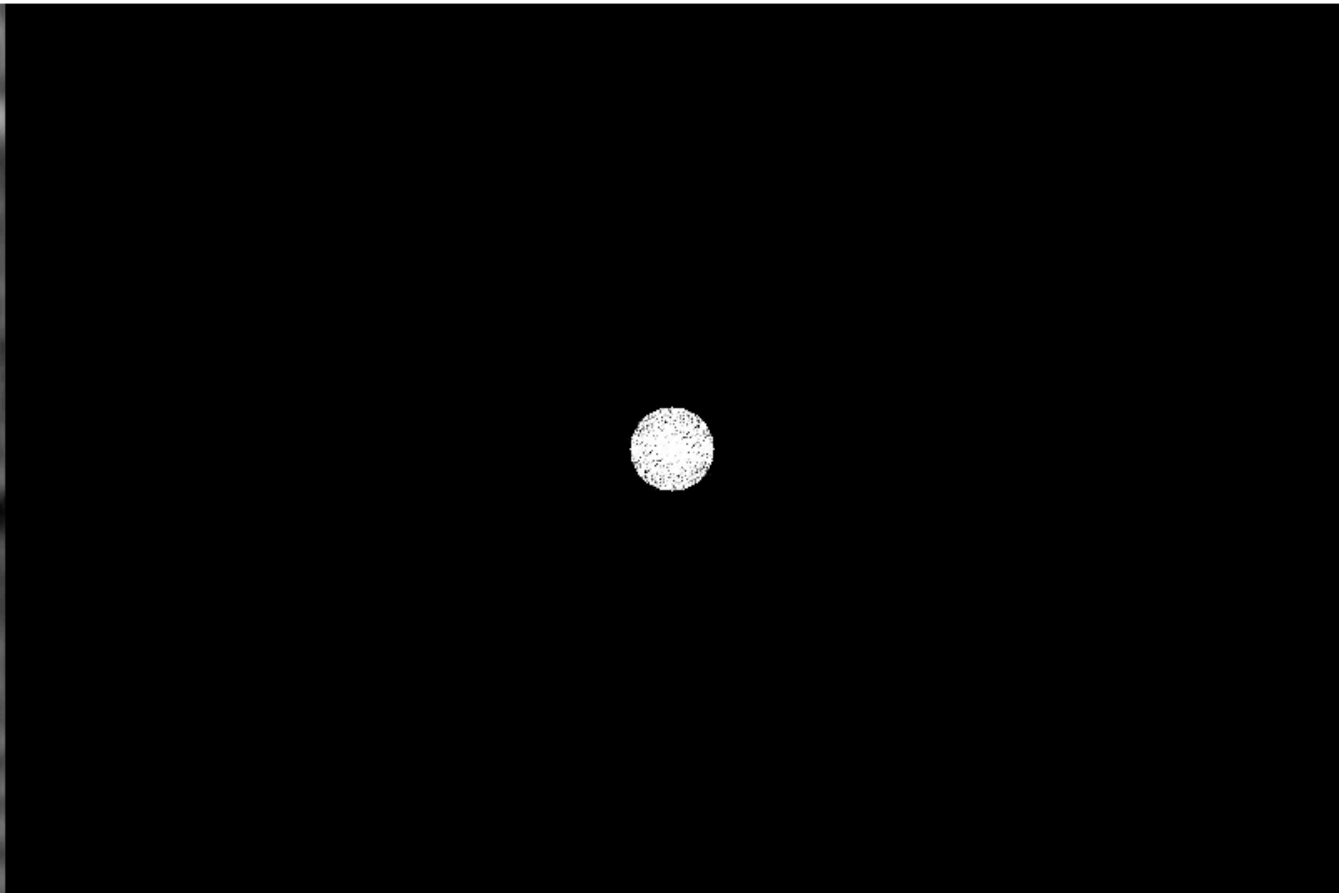
Image DFT

Reconstruct an image, low frequency to high

0.5%



Image

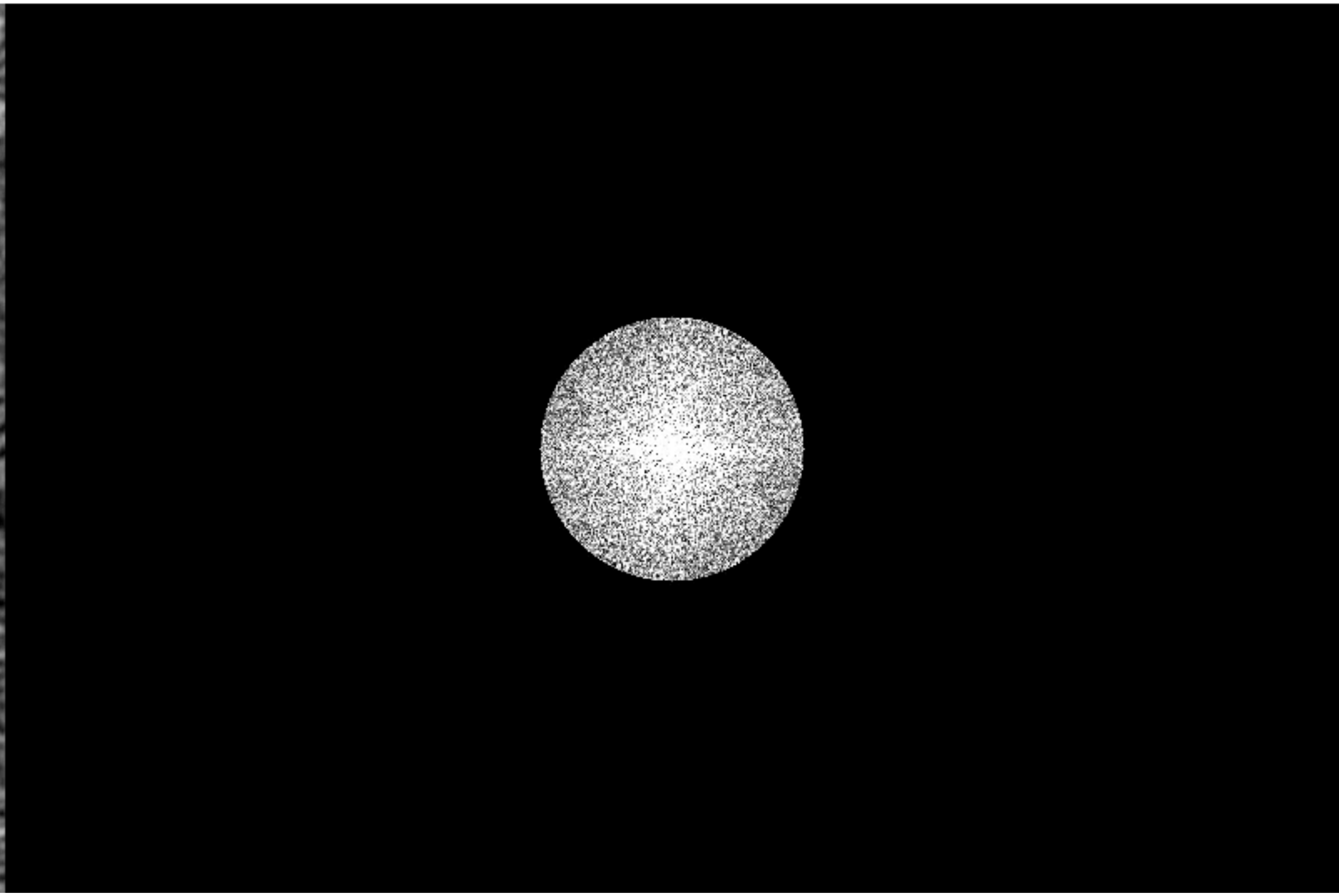


DFT

Reconstruct an image, low frequency to high

4.6%

Image

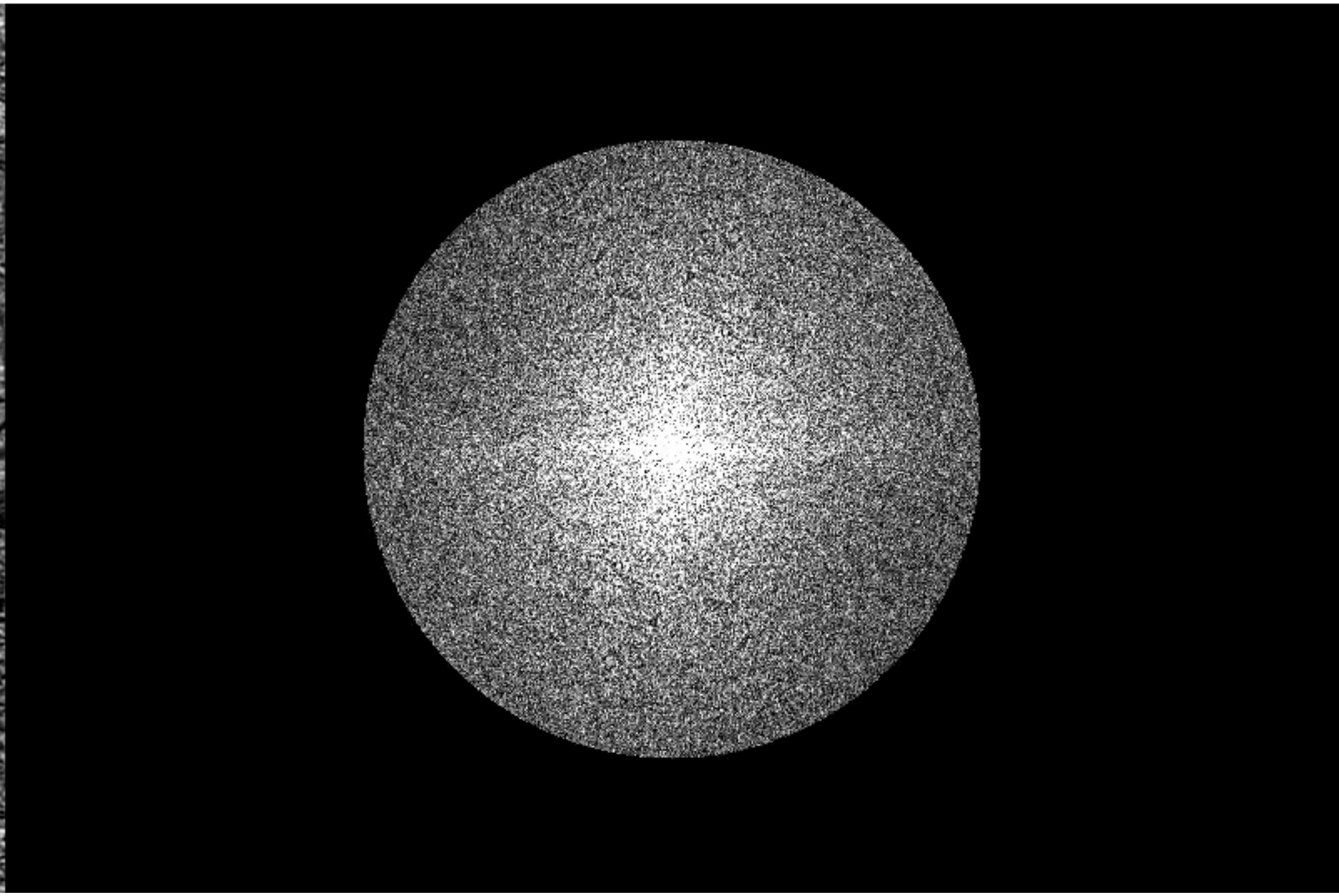


DFT

Reconstruct an image, low frequency to high

25.2%

Image

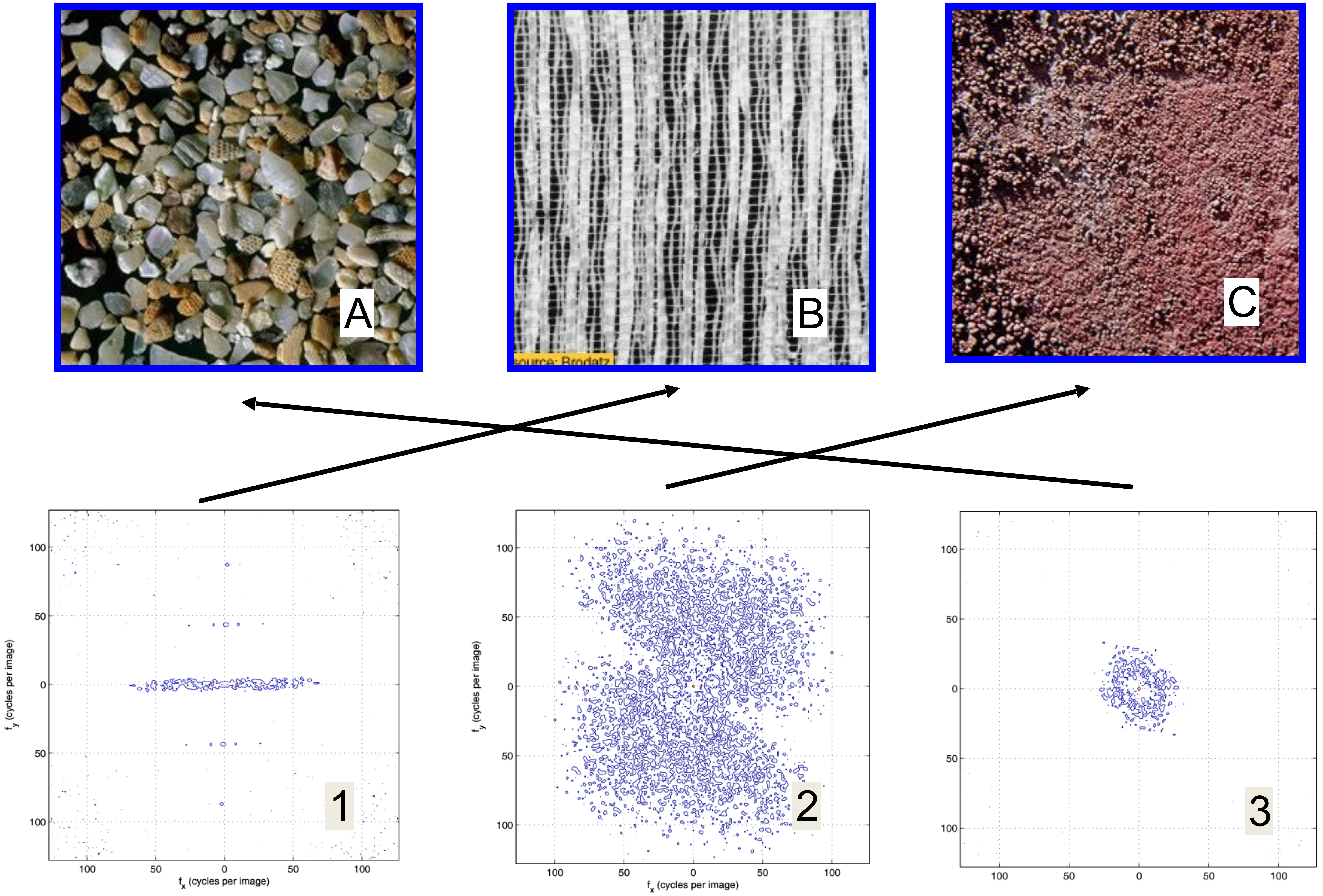


DFT

Fourier Transform Game: find the right pairs

Images

Fourier magnitude



fx(cycles/image pixel size)

fx(cycles/image pixel size)

fx(cycles/image pixel size)

Source: Torralba, Freeman, Isola

Fourier Transform Game: find the right pairs

We'll cover this in section this week.

Source: Torralba, Freeman, Isola

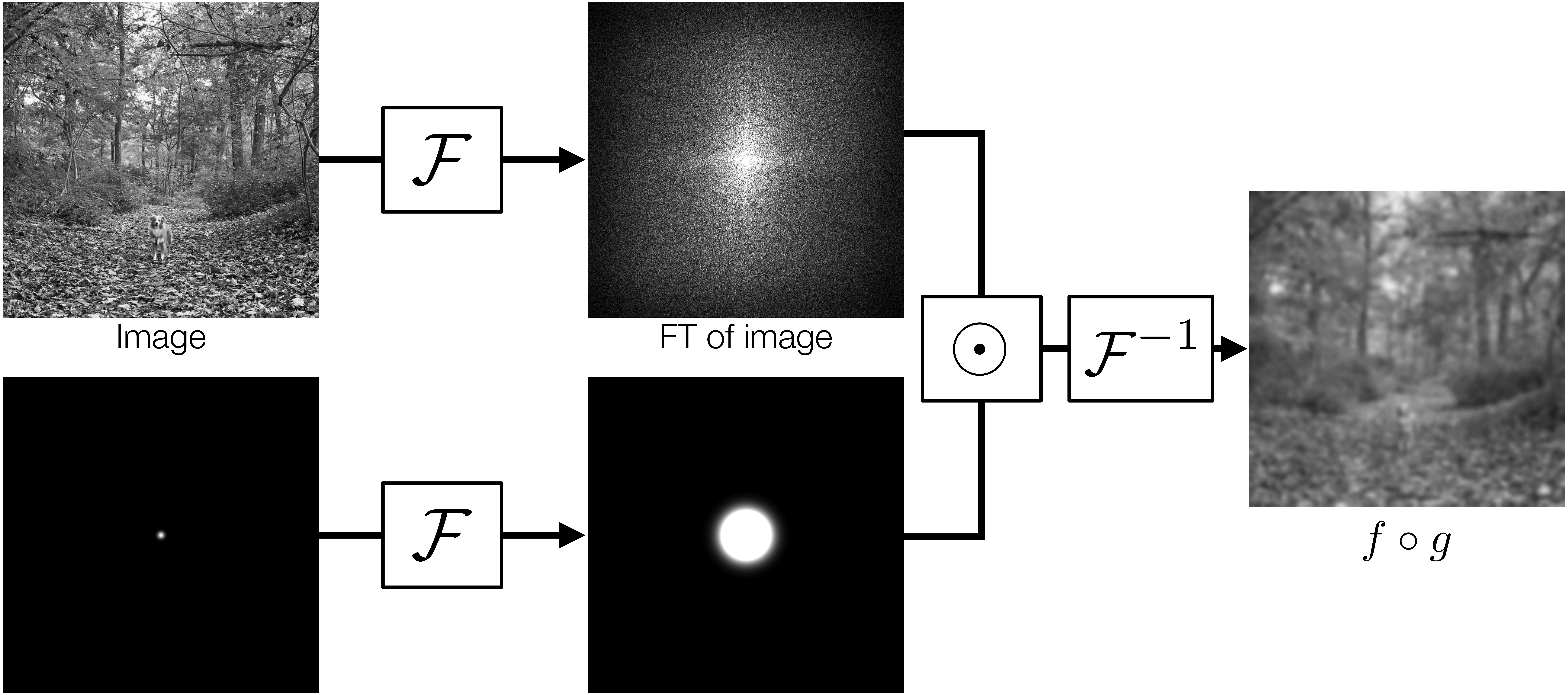
Useful properties of the Fourier transform

- 2D Fourier transform is separable (just like Gaussian)
- Computable in $O(n \log(n))$.
- **Convolution theorem:** convolution is pointwise multiplication in the Fourier domain!

$$\mathcal{F}\{f \circ g\} = \mathcal{F}\{f\} \odot \mathcal{F}\{g\}$$

- Useful trick for fast convolutions, especially for large filters. Sometimes used in convolutional neural networks

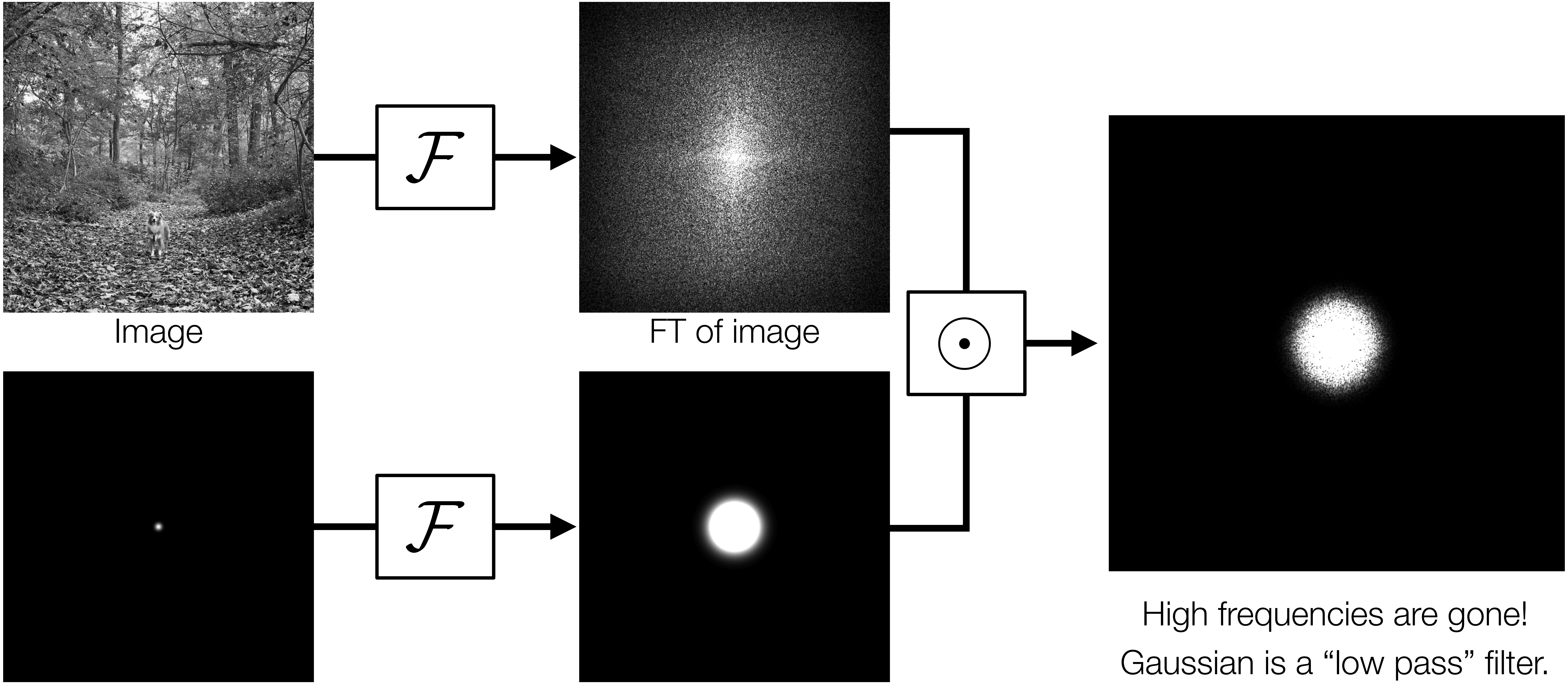
Convolution theorem example



Gaussian $\sigma = 4$

FT of Gaussian is another Gaussian

What does Gaussian blur do in the frequency domain?



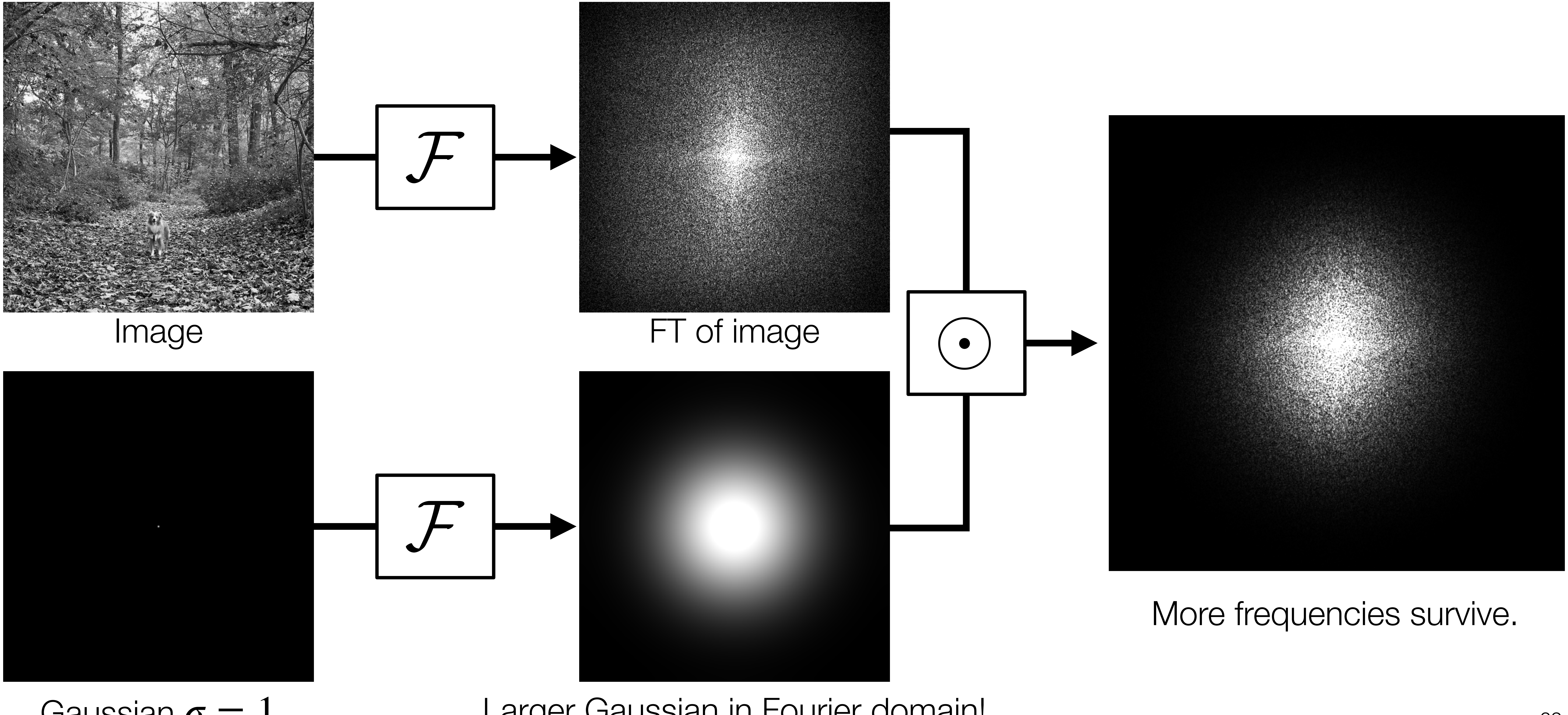
Gaussian $\sigma = 4$

FT of Gaussian is another Gaussian

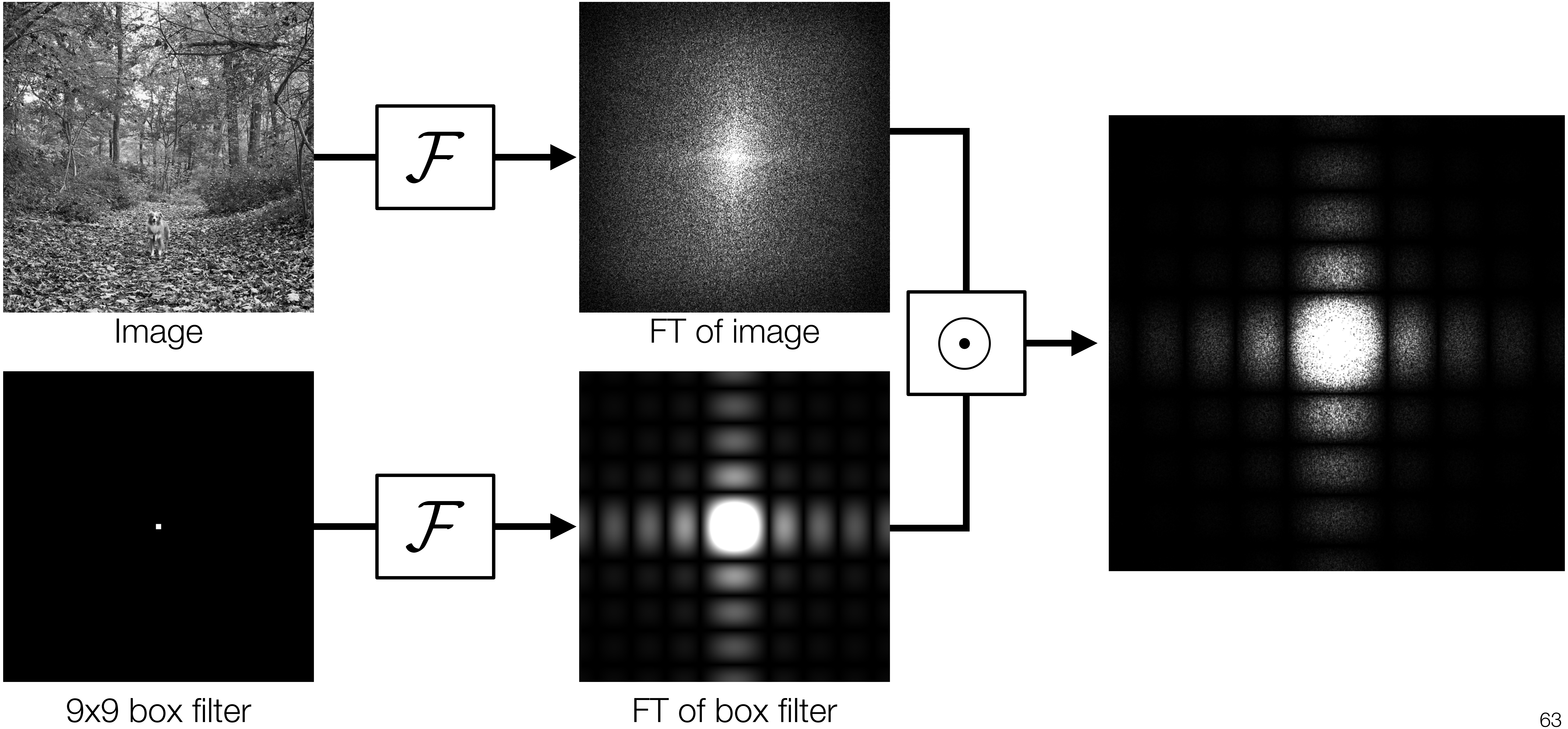
High frequencies are gone!

Gaussian is a “low pass” filter.

What happens when we decrease the blur?



Other kernels in frequency domain



Next: machine learning